Topological space

for

Fourth class

ماهو علم التوبولوجي ؟

التوبولوجي كلمة مترجمة من الكلمة الإنجليزية
Topology
و تنقسم كلمة التوبولوجي إلى مقطعين المقطع الأول
(Topo) و التي تعني مكان
(logy) (Study) و المقطع الثاني و التي تعني "دراسة

إذن يعرف علم التوبولوجي:هو أحد فروع علم الرياضيات و الذي يهتم في دراسة تراكيب و مكونات و خضائص جميع الفضاءات المختلفة ، بحيث تبقى هذه الخصائص متشابهه تحت عمليات التشكيل المتصلةدون أن يقوم بعملية تمزيق أو يترك فتحات عند الإنتقال من أحدهما إلى الآخر وبالعكس أيضاً .
و كأن التعريف يخبرنا أن الهندسة التي يتعامل بها التوبولوجي ليست الهندسة التي نعرفها ، بل كأنها هندسة مطاطية ، و لكي يتضح المفهوم بشكل جيد
من المعلوم لدينا أن المستوى الإقليدي في الهندسة الإعتيادية التي نعرفها ، أنه بإمكاننا أن نقوم بعملية نقل الأشكال من مكان إلى آخر عن طريق الإزاحة ، و بإمكاننا أيضاً أن نقوم بعملية دوران له و عكسه وقلبه ، و لكن لا نستطيع القيام بعملية ثني له أو القيام بعملية تمدد بشكل متصل .
مفهوم الهندسة المطاطية
بشكل موجز أن الأشكال عبارة عن قطع من المطاط قابلة للثني و التمدد ، و كل شكلين أوأكثر بإمكاننا أن نحصل على أحدهما من الآخر وبالعكس يكونا متشابهين
فمثلاالمثلث و الدائرة و المربع ، كلها أشكال موجودة في المستوى الإقليدي بخصائصها ، و نقول أن أحدهما كافيء الآخر إذا كان لهما نفس المساحة .
في الهندسة المطاطية جميع هذه الأشكال هي نفسها متشابهه ، فالدائرة هي نفسها المثلث ، و السبب يعود إلى أنه يمكن تشكل المثلث من الدائرة بثني محيط الدائرة و جعلها كزوايا للمثلث و بالعكس يمكن إعادة تشكل الدائرة من المثلث بعملية تمديد أضلاع المثلث إلى دائرة ، و هذا أيضاً ينطبق على المستطيل .
لأحدها Cut لاحظ أنه عندما قمنا بتشكل أحد هذه الأشكال من الآخر لم نقم بعملية قطع و لم نقم بعملية تمزيق للشكل من جهة أيترك أي نقطة انفصال .و بالتالي في الهدنسة المطاطية (التوبولوجي) يكون الأشكا ل متشابهه إذا استطعنا الحصول على أحدهما من الآخر بعمليات متصلة و بالعكس . وبالتالي الدائرة لا تشابه الشكل الذي يشبه الرقم بسبب أنه يمكن الحصول عليه من قبل الدائرة و لكن في العكس لا يمكن ، بل سنحتاج

إلى فصل منتصف رقم لم نحتاج إلى أي نقط انفصال من الدائرة إلى الرقم ، و قيس عل
ذلك بأمثله عديدة .
نستطيع القول بأن الأشكال التي تشترك بنفس العدد من الفتحات (نقاط الإنفصال) يكون
كلاهما متشابه في الهدنسة المطاطية ، أيكلاهما يشتركان في نفس التوبولوجي ، و التي
لا تحوي على أي فتحة تدعى مترابط بشكل بسيط Simply connected space.
التوبولوجي يدخل تقريباً في جميع فروع الرياضيات بلغته الخاصة و المميزة .

Definition: A function f from $X \to Y$ $(f: X \to Y)$ is a subset $f \subseteq X \times Y$ with the property :

$$\forall x \in X, \exists! \, y \in Y \text{ such that } (x, y) \in f, f(x) = y.$$

Example: Let X be any set define $I_x(x) = x \quad \forall x \in X, \;\; I_x$ called the identity map.

Example: Let Xbe any set and let $A \subset X$ define $i: A \to X$ as follows

$$i(a) = a \quad \forall a \in A$$

Example: Let X and Y be any two sets define $P_x: X \times Y \to X$ as follows:

$$P_x(x, y) = x, \qquad \forall (x, y) \in X \times Y$$

Similarly, we define

$$P_y(x, y) = y, \qquad \forall (x, y) \in X \times Y$$

Definition: Let $f: X \to Y$ be a map and let $A \subseteq X, B \subseteq Y$

$$f(A) = \{f(a); a \in A\}$$

$$f^{-1}(B) = \{x \in X\, ; f(x) \in B\}$$

Definition: Let $f: X \to Y$ be a map f is (1-1) or monomorphism if

$\forall x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$ we have $x_1 = x_2$

Or

$\forall x_1, x_2 \in X$ such that $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$.

Definition: Let $f: X \to Y$ be a map f is (onto) or epimorphism if

$\forall y \in Y\, , \exists x \in X$ such that $f(x) = y$.

Theorem: Let $f: X \to Y$ and let $\{A_\alpha : \alpha \in \curlywedge\}$ be a family of subset of X and let $\{C_\beta : \beta \in \zeta\}$ be a family of subset of Y , then

1- $f(\bigcup_{\alpha \in \curlywedge} A_\alpha) = \bigcup_{\alpha \in \curlywedge} f(A_\alpha)$
2- $f(\bigcap_{\alpha \in \curlywedge} A_\alpha) \subseteq \bigcap_{\alpha \in \curlywedge} f(A_\alpha)$
3- If $A \subseteq B \subseteq X$, then $f(A) \subseteq f(B)$
   * $f(B - A) \supseteq f(B) - f(A)$
4- $f^{-1}(\bigcup_{\beta \in \zeta} C_\beta) = \bigcup_{\beta \in \zeta} f^{-1}\left(C_\beta\right)$

5- $f^{-1}(\bigcap_{\beta \in \zeta} C_\beta) = \bigcap_{\beta \in \zeta} f^{-1}(C_\beta)$

6- If $S \subseteq T \subseteq Y$, then $f^{-1}(S) \subseteq f^{-1}(T)$ and $f^{-1}(T - S) = f^{-1}(T) - f^{-1}(S)$.

Definition: Two sets A and B said to have the same cardinality (same number of elements) , if there is a 1-1 and onto function $f: A \to B$.

Definition: A set E is called finite if E has cardinality as $\{1,2,\ldots,n\}$ for some $n \in N$, otherwise E is infinite.

Definition: A set E is called countable if it has the cardinality as subset of N, otherwise E is uncountable.

Remark:

1- If S is finite and $S_1 \subset S$, then S_1 is finite.

2- If S is countable and $S_1 \subset S$, then S_1 is countable.

3- If $\{A_n\}_{n\in N}$ is a countable collection of countable sets , then $\bigcup_{n\in N} A_n$ is countable.

4- If X and Y are countable sets, then $X \times Y$ is countable.

5- Z and Q (countable)

6- R (uncountable)

7- If a set has the same cardinality as a proper subset of itself, then it is infinite.

8-

Definition: Let X be any set and let $d: X \times X \to R$ be a map such that :

1- $d(x,y) > 0 \qquad \forall x,y \in X$

2- $d(x,y) = 0$ iff $x = y$

3- $d(x,y) = d(y,x) \qquad \forall x,y \in X$

4- $d(x,z) \leq d(x,y) + d(y,z) \qquad \forall x,y,z \in X$

Then d is called matric on X and (X, d) is called matric space.

Example: Let R be the set of real numbers define $d: R \times R \to R$ as follows:

$$(x, y) \in R \times R \quad , \quad d(x, y) = |x - y|$$

(R, d) is matric space.

Example: R^2 define $d: R^2 \times R^2 \to R$ as follows:

$$(x_1, y_1), (x_2, y_2) \in R^2$$

$$d\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

(R^2, d) is matric space.

Definition: Let (X, d) be a matric space and let $x \in X$ and $r > 0$,

$$N_r(x) = \{y \in X : d(x, y) < r\}$$

Is called neighborhood of x.

Definition: Let (X, d) be a matric space and let $U \subseteq X$, U is called open set in X if

$\forall\, x \in U$, $\exists\, r > 0$ such that $B_r(x) \subseteq U$.

Definition: Let $\subseteq X$, F is called closed if $X - F$ is open in X.

Proposition: Let (X, d) be a matric space ,then

1- $X, \emptyset$ open sets.

2- Let $\{A_\alpha : \alpha \in \wedge\}$ be a family of open subsets in X, then $\bigcup_{\alpha \in \wedge} A_\alpha$ is open set.

3- Let $A_1, A_2, \dots, A_n$ are open subsets in X, then $A_1 \cap A_2 \cap \dots \cap A_n$ is open set.

Proposition: Let (X, d) be a matric space ,then

4- $X, \emptyset$ closed sets.

5- Let $\{F_\beta : \beta \in \wedge\}$ be a family of closed subsets in X, then $\bigcap_{\beta \in \wedge} F_\beta$ is closed set.

6- Let $F_1, F_2, \dots, F_n$ are closed subsets in X, then $A_1 \cup A_2 \cup \dots \cup A_n$ is closed set.

Definition: Let $(X, d), (y, d_1)$ are matric spaces , $x_0 \in X$ and $f : (X, d) \to (Y, d_1)$, then f is continuous at x_0 if ,

$\forall \epsilon > 0$, $\exists\ \delta_\epsilon > 0$ such that $\forall\ y \in X$, $d(x_0, y) < \delta$ we have $d_1(f(x_0), f(y)) < \epsilon$.

i.e $y \in B_\delta(x_0) \to f(y) \in B_\epsilon(f(x_0))$.

Definition: Let $f : (X, d) \to (Y, d_1)$ be a map, f is called continuous map if it is continuous at x , $\forall\ x \in X$.

Theorem: $f : (X, d) \to (Y, d_1)$, then f is continuous if and only if , $\forall\ U, U$ open in Y we have $f^{-1}(U)$ open in X.

Definition: Let (X, d) be a matric space and let $\{x_n\}_{n\in N}$ be a sequence in X and let $x_0 \in X$, we say that the sequence $\{x_n\}_{n\in N}$ converge to x_0 if :

$\forall\, r > 0$, $\exists\, k_r$ positive integer such that $x_n \in B_r(x_0) \quad \forall\, n \geq k$.

Or $\quad d(x_n, x_0) < r \quad \forall\, n \geq k$.

Topological space

<u>Definition 1-1:</u> Let X be any set and let T be a family of subset of X such that:

1- $X, \emptyset \in T$
2- Each union of members of T a member of T
3- Each finite intersection members of T is also a member of T.

Then T is called a topological on X and (X, T) is called topological space.

<u>Remark1-2:</u>

1- Let (X, T) be a topological space , then the elements of T are called open set.
2- Let x be a point in a topological space X. A subset S of X is a neighborhood of x iff
$\exists$ an open set G containing x such that $x \in G \subset N$.

<u>Example1-3:</u> Let $X = \{a, b\}$, $T = \{\emptyset, X, \{a\}\}$
Is (X, T) topological space ?

Or $T_1 = \{\emptyset, X, \{b\}\}$
Is (X, T_1) topological space ?

Exam ple1-4: $X = \{a, b\}$, $T_2 = \{X, \emptyset\}$

Then (x, T_2) is topological space.

Example1-5: $X = \{a, b\}$, $T_3 = \{X, \emptyset, \{a\}, \{b\}\}$

Then (x, T_3) is topological space.

Definition1-6: Let (X, T_X) and (Y, T_Y) are two topological spaces we say that the topological space (X, T_X) is equivalent to the topological space (Y, T_Y) if $X = Y$ and $T_X = T_Y$.

Definition1-7: Let X be any set and let $D = P(X)$, then $(X, D)$is topological space called **discrete topological space**.

Definition1-8: Let X be any set and let $I = \{X, \emptyset\}$, then (X, I) is topological space called **indiscrete topological space**.

Example1-9: Let $X = \{1,2, \dots, n, \dots\}$ be the set of natural numbers and let

$$A_1 = \{1\}, A_2 = \{1,2\}, \dots, A_n = \{1,2, \dots, n\}, \dots$$

Let $T = \{X, \emptyset, A_1, A_2, \dots, A_n, \dots\}$

Then (X, T) is topological space (check)

Example1-10: Let X be any infinite set and let $T = \{A \subseteq X : X - A$ is finite$\} \cup \emptyset$, then (X, T) is topological space. T is called the **co-finite topology on X.**

Sol:

1- $\emptyset \in T$ (من الفرض)

$X \in T$ since $X - X = \emptyset$ (finite)

2- Let $A_\alpha \in T \ \forall \alpha \in \wedge$

We have to show that $\bigcup_{\alpha \in \wedge} A_\alpha \in T$

Since $A_\alpha \in T \to A_\alpha \subseteq X$ and $X - A_\alpha$ finite $\forall \alpha \in \wedge$.

We want to show that $\bigcup_{\alpha \in \wedge} A_\alpha \subseteq X$

$X - \bigcup_{\alpha \in \wedge} A_\alpha$ is finite

But $(X - \bigcup_{\alpha \in \wedge} A_\alpha) = \underbrace{\bigcap_{\alpha \in \wedge} \underbrace{(X - A_\alpha)}_{finite}}_{finite}$

3- Let $A_1, A_2, \dots, A_n \in T$

We have to show that $A_1 \cap A_2 \cap \dots \cap A_n \in T$ i.e ($X - (A_1 \cap A_2 \cap \dots \cap A_n)$is finite)

Since $A_1 \in T \to A_1 \subseteq X$ and $X - A_1$ finite

$\vdots$

$A_n \in T \to A_n \subseteq X$ and $X - A_n$ finite

But

$X - (A_1 \cap A_2 \cap \dots \cap A_n) =$

$\underbrace{\underbrace{(X - A_1)}_{finite} \cup \underbrace{(X - A_2)}_{finite} \cup \dots \cup \underbrace{(X - A_n)}_{finite}}_{finite}$

So T is topological space.

Definition1-11: Let (X,T) be a topological space and let $B \subseteq X$, B is called closed in X if $X - B$ open in X.

Remark1-12: Let (X, T) be a topological space , then

1- $X, \emptyset$ are closed set.

2- The intersection of any family of closed set is closed.

3- The union of a finite family of closed set is closed.

Proof:

1- $X - X = \emptyset \in T$

$\therefore X$ is closed

$X - \emptyset = X \in T$

$\therefore \emptyset$ is closed

2- Let A_α is closed $\forall \alpha \in \wedge$

We want to show that $\bigcap_{\alpha \in \wedge} A_\alpha$ is closed ?

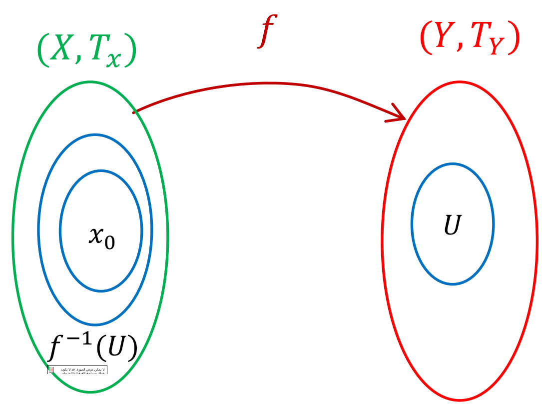
$$X - \bigcap_{\alpha \in \wedge} A_\alpha = \bigcup_{\alpha \in \wedge} \underbrace{(X - A_\alpha)}_{open \ \forall \alpha \in \wedge}$$

Then $X - \bigcap_{\alpha \in \wedge} A_\alpha$ is open set

$\therefore \bigcap_{\alpha \in \wedge} A_\alpha$ is closed

3- (check) ?

Definition1-13: Let (X, T_X) and (Y, T_Y) two topological spaces. A map $f: (X, T_X) \to (Y, T_Y)$ is called continuous if for each U open in Y, then $f^{-1}(U)$ open in X.



Definition1-14: Let $f: (X, T_X) \to (Y, T_Y)$ be a map and $x_0 \in X$, we say that f is **continuous at x_0** if for each open set U in Y such that $f(x_0) \in U$, there exist an open set V in X such that

$$x_0 \in V \subseteq f^{-1}(U)$$

Example1-15: Let $X = \{a, b, c\}$, $T_x = \{X, \emptyset\{a\}\}$ $Y = \{b, c\}$, $T_y = \{Y, \emptyset, \{c\}\}$

Define $f: (X, T_X) \to (Y, T_Y)$ as follows:

$$f(a) = b \quad , \quad f(b) = c \quad , \quad f(c) = c$$

Is f continuous ?

Sol:

1- $Y \in T_Y \to \ f^{-1}(Y) = \{x \in X\,; f(x) \in Y\} = X \in T_x$ (open)

2- $\emptyset \in T_Y \to \ f^{-1}(\emptyset) = \{x \in X\,; f(x) \in \emptyset\} = \emptyset \in T_x$ (open)

3- $\{c\} \in T_y \to f^{-1}(\{c\}) = \{x \in X\,; f(x) = c\} =$ {b,c} (not open)

Then f is not continuous.

Example1-17: Let $X = \{a, b, c\}$, $T_x = \{X, \emptyset, \{a\}\}$ $Y = \{b, c\}$, $T_y = \{Y, \emptyset, \{c\}\}$

Define $g: (X, T_X) \to (Y, T_Y)$ as follows:

$$g(a) = c \quad , \quad g(b) = b \quad , \quad g(c) = b$$

Is g continuous ?

1- $Y \in T_Y \to \ g^{-1}(Y) = \{x \in X\,; g(x) \in Y\} = X \in T_x$ (open)

2- $\emptyset \in T_Y \to \ g^{-1}(\emptyset) = \{x \in X\,; g(x) \in \emptyset\} = \emptyset \in T_x$ (open)

3- $\{c\} \in T_y \to f^{-1}(\{c\}) = \{x \in X\,; f(x) = c\} =$ {a} (open)

Then , g is continuous.

Questions-18:

1- List all possible topology on $= \{a, b, c\}$?

2- Let N be the set of the natural numbers and $T = \{N, \emptyset, A_n\}$, where $A_n = \{n\,, n+1\,, \ldots\}$ show that T is topology on N.

3- Let N be the set of the natural numbers and $T = \{N, \emptyset, A_n\}$, where A_n is the set of all finite subset of N. Is (N, T) topological space

Example1-19: Let

$X = \{a, b, c, d\}$, $T_x = \{X, \emptyset\{a\}, \{b\}, \{a, b\}\}$, $T_1 = \{Y, \emptyset, \{c\}\}$

Define $f: (X, T_X) \to (X, T_1)$ as follows:

$f(a) = a$, $f(b) = b$, $f(c) = c$, $f(d) = d$

Is f continuous ? (check)

Remark1-20: Let (X, T_X) and (Y, T_Y) two topological spaces.

Then any constant map $k: (X, T_x) \to (Y, T_y)$ is continuous.

Proof: since k is constant

Then , $\exists\, y_0 \in Y$ such that $k(x) = y_0$, $\forall\, x \in X$

Let U open set in , we have to show that $K^{-1}(U)$ open set in X ?

Case 1: if $y_0 \in U$,

$$\Rightarrow K^{-1}(U) = \left\{ x \in X \,;\, \underbrace{k(x)}_{y_0} \in U \right\} = X \qquad \text{(open in } X\text{)}$$

Case 2: if $y_0 \notin U$

$$\Rightarrow K^{-1}(U) = \{x \in X \,; k(x) \in U\} = \emptyset \qquad \text{(open in } X\text{)}$$

Then , K is continuous.

Remark1-21: Let D be the discrete topology on a set , then any map $f: (X, D) \to (Y, T_Y)$ where (Y, T_Y) any topological space , is continuous.

Proof: $D(X) = P(X) = \{A \,; a \subseteq X\}$

Let U be any open set in Y

$$f^{-1}(U) = \{x \in X\,; f(x) \in U\} \subseteq X$$

$\because$ the topology on X is discrete topology

$\therefore$ $f^{-1}(U)$ open in $X \Rightarrow f$ is continuous.

Remark1-22: Let I be the indiscrete topology on a set , then any map $f:(X, T_X) \to (Y, I)$ where (X, T_X) any topological space , is continuous.

Proof: $I = \{\emptyset, Y\}$

$$f^{-1}(\emptyset) = \emptyset \in T_x$$

$$f^{-1}(Y) = X \in T_x$$

$\therefore f$ is continuous.

Remark1-23: Let $f:(X, T_x) \to (Y, T_y)$ and let $g:(Y, T_Y) \to (Z, T_Z)$ be two continuous map. Then

$g \circ f:(X, T_x) \to (Z, T_Z)$ is also continuous.

Proof: Let U be open set in (Z, T_Z)

We want to show that $(g \circ f)^{-1}(U)$ open in X ?

$\because$ g continuous

$\therefore g^{-1}(U)$ open in Y

$\because$ f continuous

$\therefore f^{-1}(g^{-1}(U))$ open in X

$\Rightarrow (f^{-1} \circ g^{-1})(U)$ open in X

$\Rightarrow (g \circ f)^{-1}(U)$ open in X

Then , $(g \circ f)$ is continuous.

Theorem1-24: A map $f: (X, T_x) \to (Y, T_y)$ is continuous if and only if $\forall\, V$ closed set in Y , we have $f^{-1}(V)$ closed in X.

Proof: $(\Rightarrow$

Let f be a continuous map, let V be closed set in Y

$\Rightarrow Y - V$ open in Y

$\because f$ continuous

$\therefore f^{-1}(y - v)$ open in X

$\Rightarrow f^{-1}(Y) - f^{-1}(V)$ open in X

$\Rightarrow X - f^{-1}(V)$ open in X

$f^{-1}(V)$ closed in X.

$\Leftarrow$)

Let U be open set in Y, we want to show that $f^{-1}(U)$ open in X ?

Since U open in Y, then

$Y - U$ is closed in Y

By our assumption

$\Rightarrow f^{-1}(Y - U)$ closed in X

$\Rightarrow f^{-1}(Y) - f^{-1}(U) = X - f^{-1}(U)$ closed in X

$\Rightarrow f^{-1}(U)$ open in X.

Definition1-25: Let $f:(X,T_x) \to (Y,T_y)$ be a map, then f is called homeomorphism if f is continuous , 1-1 , onto and f^{-1} is continuous.

Definition1-26: Let (X,T_X) and (Y,T_Y) be two topological spaces, then we say that X is homeomorphic to Y, or is topological equivalent to Y if there exist a homeomorphism $f:X \to Y$.

Remark1-27:

1- Let (X,T_X) be a topological space , then X is homeomorphic to X

2- If (X,T_X) is homeomorphic to (Y,T_Y), then (Y,T_Y) is homeomorphic to (X,T_X)

3- Let $(X,T_X),(Y,T_Y)$ and (Z,T_Z) be topological spaces,
If (X,T_X) is homeomorphic to (Y,T_Y), and (Y,T_Y) is homeomorphic to (Z,T_Z), then (X,T_X) is homeomorphic to (Z,T_Z).

Proof:

1- $I_x:(X,T_X) \to (X,T_X)$ be the identity map

$$I_x(x) = x \quad \forall x \in X$$

I_x is continuous , 1-1 , onto , and $(I_x)^{-1} = I_x$ continuous
Then X homeomorphic to X.

2- Since X homeomorphic to Y
Then $\exists\ f:(X,T_x) \to (Y,T_y)$ such that f,f^{-1} are continuous, f is onto and 1-1

Now $f^{-1}: (Y, T_Y) \to (X, T_X)$ is 1-1 onto and continuous (since f is homeomorphism)

$(f^{-1})^{-1} = f$ is continuous

Then Y is homeomorphic to X.

3- X homeomorphic to Y and Y homeomorphic to Z

We have to show that X homeomorphic to Z.

$\exists\ f: (X, T_x) \to (Y, T_y)$ such that f, f^{-1} are continuous, f is onto and 1-1

And $\exists\ g: (Y, T_Y) \to (Z, T_Z)$ such that g, g^{-1} are continuous, g is onto and 1-1

Consider $g \circ f:: (X, T_x) \to (Z, T_Z)$

a- Since f and g are continuous, then $g \circ f$ is continuous.

b- $g \circ f$ is 1-1 and onto since f, g are 1-1 and onto.
(check)

c- $(g \circ f)^{-1}$ is continuous since
$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, but f^{-1} and g^{-1} are continuous
Then $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$ continuous.

Definition1-28: Let $f: (X, T_x) \to (Y, T_y)$ be a map, then f is called open map if, $\forall\ U$ open set in X, we have $f(U)$ open in Y.

Definition1-29: Let $f: (X, T_x) \to (Y, T_y)$ be a map, then f is called closed map if , $\forall\ V$ closed set in X, we have $f(V)$ closed in Y.

Remark1-30: A continuous map need not be open map as the following example:

Example1-31: Let $X = \{a, b\}$ $\quad D = P(X) = \{X, \emptyset, \{a\}, \{b\}\}$

$I = \{X, \emptyset\}$

Let $I_x : (X, D) \rightarrow (X, I)$ be the identity map (it is continuous) why?

$\{a\} \in D$ open in (X, D)

$I_x(\{a\}) = \{a\} \notin I$ not open in (X, I)

Then I_x is not open map.

Definition2-1: Let (X,T) be a topological space and let $p \in X$, any open set U in X such that $p \in U$ is called an open neighborhood of p and denoted by N_p.

Definition2-2: Let (X,T) be a topological space and let $E \subseteq X$ and $p \in E$, p is called interior point of E if there exists an open neighborhood N_p such that $p \in N_p \subseteq E$.

the interior of $E = i(E) = \{p \in E; p \text{ is an interior point of } E\}$

Remark2-3: Let (X,T) be an topological space and let $E \subseteq X$, then E is open if and only if $i(E) = E$.

Proof: $(\Rightarrow$

Let E open set, we want to show that $i(E) = E$?

Clearly that $i(E) \subseteq E$ (by definition)

Let $p \in E$

Since E open , then $N_p = E \subseteq E$

Then $p \in i(E) \Rightarrow i(E) = E$.

$\Leftarrow)$ Let $i(E) = E$, we want to show that E is open ?

Let $p \in E, \because i(E) = E \Rightarrow p \in i(E)$

$\therefore \exists$ open set N_p such that $p \in N_p \subseteq E$

$$\therefore E = \underbrace{\bigcup_{p \in E} N_p}_{open \in T}$$

Thus E is open set.

Definition2-4: let (X, T) be a topological space, let $E \subseteq X$ and $p \in X$. p is called limit point of E if every open neighbourhood of p has at least one point of E different from p.

i.e) $\forall$ open neighborhood N_p, we have

$$(N_p - \{p\}) \cap E \neq \emptyset$$

The set of all limit point of E is called the derived set of E and denoted by $d(E)$.

$d(E) = \{p \in X; p$ is a limit point of $E\}$.

* closure of $E = \bar{E} = E \cup d(E)$.

Definition2-5:- let (X, T) be a topological space and $E \subseteq X$ and $P \in X$, p is called a boundary point of E if every open neighbourhood , $N_p \; of \; p$ has a nonempty intersection with both of E and $X - E$.

i.e. $\forall N_p \, , N_p \cap E \neq \phi \; and \; N_p \cap (X - E) \neq \phi$

the set of boundary points of E is called the boundary of E and denoted by $b(E)$

$b(E) = [\, P \in X; P \; is \; boundary \; point \; of \; E]$

Example2-6:- let $X = \{a, b, c, d\}$

$T = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$

$E = \{a, c\}$

Find $i(E), d(E), \widetilde{E}, b(E)$

Sol

1- $i(E) = \{p \in E; p\ is\ interior\ point\ of\ E\}, \exists\ open\ set\ U\ such\ that\ p \in U \subseteq E$
   $a \in E \quad N_a = \{a\} \to open\ a \in N_a \subseteq E\} \Rightarrow a \in i(E)$
   $c \in E$, $there\ is\ no\ open\ set\ U\ s.t.\ c \in U \subseteq E, \therefore c \notin i(E) \Rightarrow i(E) = \{a\}$

2- $d(E) = \{p \in X; p\ is\ a\ limit\ point\ of\ E\}$ $\forall$ neighbourhood

$$N_p\ \left(N_p - \{p\}\right) \cap E \neq \emptyset$$

$a \in X$, $N_a = X\ open\ set$
$(N_a - \{a\}) \cap E =$
$(X - \{a\}) \cap E =$
$\{b, c, d\} \cap \{a, c\} = \{c\} \neq \emptyset$
$N_a = \{a\}\ open\ set$
$(N_a - \{a\}) \cap E =$
$\{a\} - \{a\} \cap E =$
$\emptyset \cap E = \emptyset$
$thus\ a\ is\ not\ limit\ point\ of\ E.$
Now $b \in X$, $N_b = X\ open\ set$
$(N_b - \{b\}) \cap E =$
$(X - \{b\}) \cap E =$
$\{a, c, d\} \cap \{a, c\} = E \neq \emptyset$
$N_b = \{b\}\ open\ set$
$(N_b - \{b\}) \cap E = \emptyset \to b \notin d(E)$

$b\ is\ not\ limit\ point\ of\ E.$

Now $c \in X, N_c = X\ open\ set$

$(N_c - \{c\}) \cap E =$
$(X - \{c\}) \cap E =$
$\{a, b, d\} \cap \{a, c\} = \{a\} \neq \emptyset$

$\therefore$ *c is a limit point of E*.

Now $d \in X$, $N_d = X$ *open set*
$(N_d - \{d\}) \cap E =$
$\{a, c\} = E \neq \emptyset$

$\therefore$ *d is a limit point of E*.
$\therefore d(E) = \{c, d\}$

3- $\bar{E} = E \cap d(E) = \{a, c\} \cup \{c, d\} = \{a, c, d\}$
4- $b(E) = \{p \in X; p \text{ is a boundary point of } E\} \forall N_p$ *neigh. of p*
$N_p \cap E \neq \emptyset$ *and* $N_p \cap (x - E) \neq \emptyset$

$a \in X$, $N_a = X$ *open set*
$N_a \cap E = X \cap E = E \neq \emptyset$

$N_a \cap (X - E) = X \cap (X - E) = X - E \neq \emptyset$
$N_a = \{a\}$ neighbourhood *of a*
$N_a \cap E = \{a\} \cap \{a, c\} \neq \emptyset$

$N_a \cap (X - E) = \{a\} \cap \{b, d\} = \emptyset$
$\therefore$ *a is not boundary point of E*

Now
$b \in X$, $N_b = X$ *open set*
$N_b \cap E = X \cap E = E \neq \emptyset$

$N_b \cap (X - E) = X \cap (X - E) = X - E \neq \emptyset$

$N_b = \{b\}$neighbourhood *of b*
$N_b \cap E = \{b\} \cap \{a, c\} = \emptyset$
$\therefore$ *b is not boundary point of E*.

$c \in X$, $N_c = X$ *open set*
$N_c \cap E = X \cap E = E \neq \emptyset$

$X \cap (X - E) = X - E \neq \emptyset$

$c \in b(E)$
$also\ d \in b(E)$
$\rightarrow b(E) = \{c, d\}$

H.W 2-7 let $x = \{a, b, c, d\}$

$T = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$
$s = \{a, b\}$
$find \quad i(s), d(s), \bar{s}, b(s)$

Example2-8:- let $X = \{a, b, c, d\}$ with the indiscrete $topology\ I$ and let $E = \{a, c\}$, find $i(E), d(E), \bar{E}, b(E)$
$I = \{X, \emptyset\}$

$i(E) = \{p \in E; p\ is\ an\ interior\ point\ of\ E\} \exists\ open\ set\ U\ s.t$
$p \in U \subseteq E$

1- $a \in E$, the only open set that contain a is X but $X \nsubseteq E$. Thus $a \notin i(E)$. By the same way $c \notin i(E) \Longrightarrow i(E) = \emptyset$

2- $d(E) = \{p \in X; p\ is\ limit\ point\ of\ E\} \forall N_p of\ p$
$(N_p - \{p\}) \cap E \neq \emptyset$
$a \in X$
$take\ N_a = X$

$(N_a - \{a\}) \cap E = (X - \{a\}) \cap E = \{b, c, d\} \cap \{a, c\} = c \neq \emptyset$
$\Longrightarrow a \in d(E)$
similarly we have $b \in d(E), c \in d(E), d \in d(E) \Longrightarrow \therefore d(E) = X$

3- $\bar{E} = d(E) \cup E = X \cup E = X$

4-$b(E) = \{p \in X; p \text{ is a boundary point of } E\}$
$\forall N_p$

$N_p \cap E \neq \emptyset \text{ and } N_p \cap (X - E) \neq \emptyset$
$a \in X$
$N_a = X$
$N_a \cap E = X \cap E = E \neq \emptyset$

$N_a \cap (X - E) = X \cap (X - E) = X - E \neq \emptyset$
$\therefore a \in b(E)$

Similarly $b \in b(E)$
$c \in b(E) \rightarrow b(E) = X.$
$d \in b(E)$

Example2-10:- let $N = \{1,2, \ldots, N, \ldots\}$ be the set of natural numbers with the cofinite top. , and let $E = \{2,4,6, \ldots\}$, find $i(E), d(E), \bar{E}, b(E)$.

Sol. $T = \{A \subseteq N; N - A \text{ is finite}\} \cup \{\emptyset\}$

1- $i(E) = \{p \in E; p \text{ is interiorm point of } E\}$
$2 \in E$ is $2 \in i(E)$
$\exists \text{ open set } U \ \ s.t. 2 \in U \subseteq E$
$\because U \text{ open set}$
$U \in T$
$N - U \text{ is finite} \rightarrow U^c \text{is finite}, E^C \text{ is infinite} =$
$\{1,3,5, \ldots\}$

$\because U \subseteq E$

$E^C \subseteq U^c$

$$\begin{cases} 2 \notin i(E) \\ 4 \notin i(E) \ldots\ldots. \end{cases} \rightarrow i(E) = \emptyset$$

2- $d(E) = \{p \in x; is\ a\ limit\ point\ of\ E\}$

$\forall N_p \left(N_p - \{p\}\right) \cap E \neq \emptyset$

$let\ p \in N\ any\ point$

$assume\ p\ is\ not\ a\ limit\ point\ of\ E \Rightarrow$

$i.e\ \exists\ neigh. U\ s.t$

$(U - \{p\}) \cap E = \emptyset \ldots\ldots\ldots..(1)$

$U\ open\ set$

$\Rightarrow N - U\ is\ finite$

$\Longrightarrow N - (U - \{p\})\ is\ finite\ but\ from\ (1)$

$U - \{p\} \subseteq E^c \Longrightarrow E \subseteq (U - \{p\})^c$

$but\ E\ is\ infinite\ ,\ (U - \{p\})^c finite\ , contradiction$

3- Thus for each $p \in N$ p is limit point

$i.e.\ d(E) = N$

4- $\bar{E} = d(E) \cup E = N$

5- Check(find $b(E)$)?

Suppose p is not boundary point A

$\Longrightarrow \exists\ U \in T\ s.t$

$U \cap E = \emptyset \Rightarrow E \subseteq U^c \rightarrow but\ E\ is\ infinite\ , is\ (U)^c finite$

$, contradiction$

Or $U \cap (N - E) = \emptyset \Longrightarrow (N - E) \subseteq U^c \rightarrow$
$but\ N - E\ is\ infinite\ \ , is\ (U)^c finite\ , contradiction$
$\Rightarrow b(E) = N$

<u>Definition</u>2-11:-let (X, T) be a topological space and let $E \subseteq X$ E is called dence in X if $\bar{E} = E$.

<u>Theorem2-12:-</u> let (X, T) be a topological space and let $E \subseteq X$, then is closed iff $d(E) \subseteq E$.

Proof: let E be a closed set, $\therefore X - E$ open set we want to show that $d(E) \subseteq E$ let $p \in d(E) \rightarrow p\ is\ a\ limit\ point\ of\ E\ to\ show\ p \in E$
$assume\ not\ \{p \notin E\}$
$\therefore p \in X - E$
$but\ X - E\ open\ set$
$put\ N_p = X - E$
$but\ p\ is\ a\ limit\ point\ of\ E$
$\therefore \left(N_p - \{p\}\right) \cap E \neq \emptyset$
$\left((X - E) - \{p\}\right) \cap E \neq \emptyset\ \ contradiction$
$\Longrightarrow p \in E$

$\Longleftarrow)\ let\ d(E) \subseteq E$
We want to show E is closed $(X - E)$ open
Let $P \in X - E \Rightarrow P \in X\ \wedge P \notin E$
$\because d(E) \subseteq E$
$P\ is\ not\ limit\ point\ of\ E$
$\exists$ neighbourhood N_P s.t
$(N_P - \{P\}) \cap E = \emptyset \Rightarrow$
$N_P \cap E = \emptyset \Rightarrow$
$N_P \subseteq X - E$

$$\mathrm{X}-\mathrm{E}=\bigcup_{P\in X-E} N_P \in T \quad (\text{why},,,,,?)$$

$\Rightarrow$ E is closed.

Corollary2.13.: - let (X,T) be a topological space and let $E\subseteq X$, then E is closed iff $\bar{E}=E$.

Remark2.14.:- let (X,T) be a topological space and let $E\subseteq X$

$i(E)=$ The largest open set that contained in E

$\bar{E}$ = the smallest closed set that contained E

Remark2.15.:- $i\big(i(E)\big)=i(E) \Longrightarrow \bar{\bar{E}}=\bar{E}$

Remark2.16:- Let (X,T) be a topological space and let $E\subseteq X$, then $b(E)\ is\ closed.$

<u>**Proof**</u>:- We want to show that $X-b(E)$ is open

Let $p\in \mathrm{X}-b(E)\Rightarrow p\notin b(E)$

$\exists\ N_p of\ p\ \ s.t.$

$either\ N_p\cap E=\emptyset\ or\ N_p\cap(X-E)=\emptyset$

$claim\ that\ N_p\subseteq \mathrm{X}-b(E)$

$let\ y\in N_p\ \ we\ want\ to\ show\ y\notin b(E)$

$if\ not$

$y\in b(E)\Longrightarrow y\ is\ a\ boundary\ point\ of\ E\ and\ N_p\ open\ set\ \ s.t\ y\in N_p$

$\Longrightarrow N_p\cap E\neq\emptyset\ and\ N_p\cap(X-E)\neq\emptyset\ contradiction$

$thus\ y\notin b(E)\Longrightarrow y\in X-b(E)$

$\therefore N_p\subseteq X-b(E)$

$X-b(E)=\bigcup_{p\in X-b(E)} N_p\in T$

$\therefore X - b(E)$ *open set* $\Rightarrow b(E)$ *is closed*

$let\ (R, T_u)\ , A = (0,1]\ find\ i(A), d(A), \bar{A}\ , b(A)?$

$\{p \in A, \exists\ N_p\ s.t\ \ p \in N_p \subseteq A\}$

$\forall p \in (0,1), \exists\ a, b\ \ s.t\ a < b\ , p \in (a, b) \subseteq (0,1]$

$\therefore i(A) = (0,1)$

2-$d(A)$

$\forall p \in R\ s.t, \forall N_p\ , p \in N_p\ , (N_p - \{p\}) \cap (0,1] \neq \emptyset$

$\therefore (0,1) \in d(A)$

$0 \in R, \forall a < 0, ((a, b) - \{0\}) \cap (0,1] \neq \emptyset$

$\therefore 1 \in d(A)$

$d(A) = [0,1]$

3- $\bar{A} = A \cup [0,1] = [0,1]$

4-$b(A)$

$\forall p \in R, \forall N_p\ , N_p\ \cap A \neq \emptyset$

$N_p\ \cap (R - A) \neq \emptyset$

$b(A) = \{0,1\}$

A *is not dence since* $\bar{A} \neq A$.

Theorem2.17:- let $(X, T_X) \to (Y, T_Y)$ be a cont. map and let $E \subseteq X$ and let $p \in X$, if p is a limit point of E, then $f(p) \in f(E)$. Thus $F(\bar{E}) \subseteq (\overline{f(E)})$

<u>Proof</u> Let p be a limit point of E we have to show that

$f(p) \in \overline{f(E)} = f(E) \cup d(f(E))$

<u>Now</u> $p \in X$

If $p \in E \to f(p) \in f(E) \subseteq f(E) \cup d(f(E)) \longrightarrow f(p) \in (f(E))$

$p \notin E \to f(p) \notin f(E) - - - (1)$

We want to show $f(p) \in d\big(f(E)\big)$

i.g $f(p)$ limit point of $f(E)$

let $N_{f(p)}$be a neigh of $f(p)$ we have to show

$(N_{f(p)} - \{f(p)\}) \cap f(E) \neq \emptyset$?

$\because N_{f(p)}$open in y and f continuous

Thus $f^{-1}(N_{f(p)})$ open in X

But $f(p) \in N_{f(p)} \Rightarrow p \in f^{-1}(N_{f(p)})$

And p is limit point of $E \Rightarrow \left(f^{-1}\left(N_{f(p)}\right) - \{p\}\right) \cap E \neq \emptyset$

$\therefore \exists\, w \in f^{-1}\left(N_{f(p)}\right)$ and $w \neq p$ and $w \in E$

$f(w) \in N_{f(p)}$ and $f(w) \in f(E)$

If $f(w) = f(p)$

$f(p) = f(w) \in f(E) \to f(p) \in f(E)$ contradiction

$\therefore f(w) \in \left(N_{f(p)} - \{f(p)\}\right) \cap f(E) \neq \emptyset$ and we have done.

Theorem2.18:- let $f: (X, T_X) \to (Y, T_Y)$ bea homeomorphism and let $E \subseteq X$, then $f\big(b(E)\big) = b\big(f(E)\big)$.

<u>Proof:-</u> first we show that $f\big(b(E)\big) \subseteq b\big(f(E)\big)$ let $w \in f\big(b(E)\big)$? p is a boundary point of E.

Since $w \in f\big(b(E)\big) \Rightarrow \exists p\ s.t\ p \in b(E)$

$w = f(P) \in f\big(b(E)\big)$

We want to show $f(b) \in b\big(f(E)\big)$?

Let $N_{f(p)}$ be a neigh.of $f(p)$ we have to show that

$N_{f(p)} \cap f(E) \neq \phi$ and $N_{f(p)} \cap \big(Y - f(E)\big) \neq \emptyset$

$\because N_{f(p)}$ open in Y and f continuous

$\therefore f^{-1}\left(N_{f(p)}\right)$ open in X

$\because f(p) \in N_{f(p)} \Longrightarrow p \in f^{-1}\left(N_{f(p)}\right)$

And p is a boundry point of E

$\therefore \left(f^{-1}\left(N_{f(p)}\right) \cap E\right) \neq \emptyset$ and $\left(f^{-1}\left(N_{f(p)}\right) \cap (X - E)\right) \neq \emptyset$

Let $a \in f^{-1}\left(N_{f(p)}\right)$ and $a \in E$

Let $b \in f^{-1}\left(N_{f(p)}\right)$ and $b \in X - E$

$\Rightarrow f(a) \in N_{f(p)}$ and $f(a) \in f(E)$

$\Longrightarrow N_{f(p)} \cap f(E) \neq \emptyset$ ----1

And $f(b) \in N_{f(p)}$ and $f(b) \in f(X - E)[f(A - B) \supseteq f(A) - f(B)]$

But f is $1 - 1$

$\Longrightarrow f(X - E) = f(X) - f(E)$

And f is onto $\Longrightarrow f(X) = Y$

$\Longrightarrow f(X - E) = Y - f(E)$

$\Longrightarrow f(b) \in N_{f(p)} \cap \left(Y - f(E)\right)$

$\therefore N_{f(P)} \cap (Y - f(E)) \neq \emptyset$ ---2

By 1 and 2 the proof is complete

By the same way we proof

$b(f(E)) \subseteq f\left(b(E)\right)$ H.W

The subspace topology

<u>Definition3-1:</u>let (X,T) be a topological space and $Y \subseteq X$ let $T_Y = \{Y \cap U; U \in T\}, (Y, T_Y)$

a topological space called sub space topology (relative topology) (induced topology)

<u>Remark3.2</u>: (Y, T_Y) is topological space

Proof: $\emptyset \in T_Y, \emptyset = Y \cap \emptyset\,, \emptyset \in T, \emptyset$ is open in T

1- $Y \in T_Y$

$Y = Y \cap X\,, X \in T$

$= Y$

$\therefore Y \in T_Y$

2- Let $A_\alpha \in T_Y, \forall \alpha \in 1$

We want to show that $\cup_{\alpha \in 1} A_\alpha \in T_Y$

Since $A_\alpha \in T_Y \Longrightarrow \exists u_\alpha \in T$ s.t

$A_\alpha = Y \cap u_\alpha$

$\bigcup_{\alpha\in\Lambda} A_\alpha = \bigcup_{\alpha\in\Lambda}(Y \cap u_\alpha) = Y \cap (\bigcup_{\alpha\in\Lambda} u_\alpha \in T) \in T_Y$

Let $A_1, \dots \dots, A_n \in T_Y, \exists u_1$ s.t. $A_1 = Y \cap u_1, \dots \dots,$

$A_n = Y \cap u_n \Longrightarrow A_1 \cap A_2 \cap \dots \cap A_n = (Y \cap u_1) \cap \dots \cap (Y \cap u_n)$

$= Y \cap (u_1 \cap u_2 \cap \dots \cap u_n) \in T_Y$

<u>Definition3-3:</u>- let (X,T) be a top space and $E \subseteq X$ and $P \in X$, p is called a boundary point of E if every open neighbourhood , $N_p\ of\ p$ has a nonempty intersection with both of E and $X - E$.

i.e. $\forall N_p , N_p \cap E \neq \phi \; and \; N_p \cap (X - E) \neq \phi$

the set of boundary points of E is called the boundary of E and denoted by $b(E)$

$b(E) = [\, P \in X; P \; is \; boundary \; point \; of \; E]$

Ex let $X = \{a, b, c, d\}$

$T = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$

$E = \{a, c\}$

Find $i(E), d(E), \widetilde{E}, b(E)$

Sol

1- $i(E) = \{p \in E; p \; is \; interior \; point \; of \; E\}, \exists \; open \; set \; U \; s.t. \; p \in U \subseteq E$
   $a \in E \quad N_a = \{a\} \to open \; a \in N_a \subseteq E\} \Rightarrow a \in i(E)$
   $c \in E$, $there \; is \; no \; open \; set \; U \; s.t. \;\; c \in U \subseteq E, \therefore c \notin i(E) \Rightarrow i(E) = \{a\}$
2- $d(E) = \{p \in X; p \; is \; a \; limit \; point \; of \; E\} \; \forall \; negh. N_p \; (N_p - \{p\}) \cap E \neq \emptyset$
   $a \in X , N_a = X \; open \; set$
   $(N_a - \{a\}) \cap E =$
   $(X - \{a\}) \cap E =$
   $\{b, c, d\} \cap \{a, c\} = \{c\} \neq \emptyset$
   $N_a = \{a\} \; open \; set$
   $(N_a - \{a\}) \cap E =$
   $\{a\} - \{a\} \cap E =$
   $\emptyset \cap E = \emptyset$
   $thus \; a \; is \; not \; limit \; point \; of \; E.$

Now $\quad b \in X , N_b = X \; open \; set$