قسم الرياضيات

Linear Algebra (1)

المرحلة الثــانية

الفصل الدراسي الاول

اساتذة المادة

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The First Lecture
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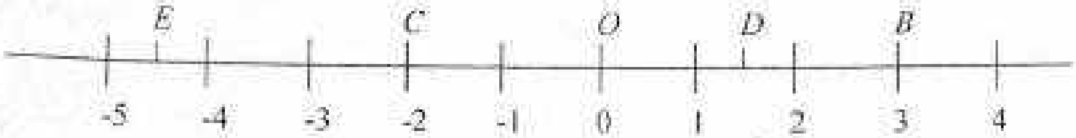
Linear Algebra

Vectors in the plane

Coordinate systems:

We recall that the real numbers system may be visualized as a straight line L, which is usually taken as a horizontal position. A point O called the **origin,** on L; O corresponds to the number 0. A point A is chosen to the right of O and fixing the length of OA as 1 and specifying a positive direction. Thus the positive real numbers lie to right of O; the negative real numbers lie to the left of O(see figure 1).



The absolute value $|x|$ of the real number x is defined by

$$|x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

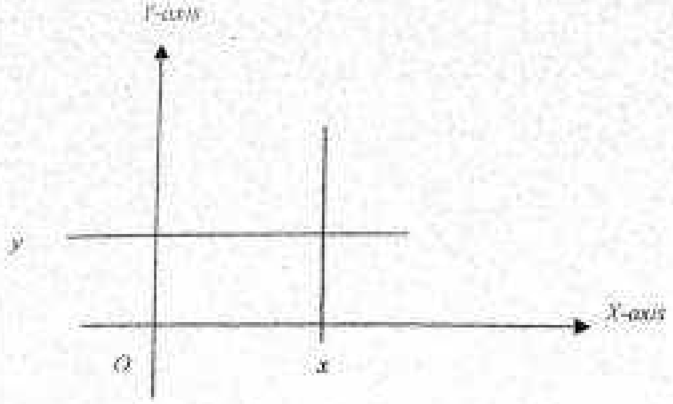
Thus $|3| = 3$, $|-2| = 2$ and $|0| = 0$

If a, b are two points on the line L, then the distance between the point a and b is |b-a|.

Example: If $a=3$, $b=1.5$, the distance between a and b is
$|b-a|=|1.5-(-3)|=4.5$.

In the plane:

We draw a pair of perpendicular lines intersecting at a point O, called the origin. One of the lines, the x-axis, is usually taken in a horizontal position, the other line, the y-axis, is taken in a vertical position.
We now choose a point on the x-axis to right of O and appoint of the y-axis above O to fix the units of length is used for both axis.



Thus with every point in the plane we associate an order pair (x,y) of real numbers its coordinates. The point p with coordinates x and y is denoted by $p(x,y)$.

Conversely, it is easy to see how we can associate a point in the plane with each order pair (x,y) of real number.

The correspondence given above between points in the plane and ordered pairs of real number is called rectangular coordinate system or the Cartesian coordinate system. The set of all points in the plane is denoted by R^2. It is also called *2*-space.

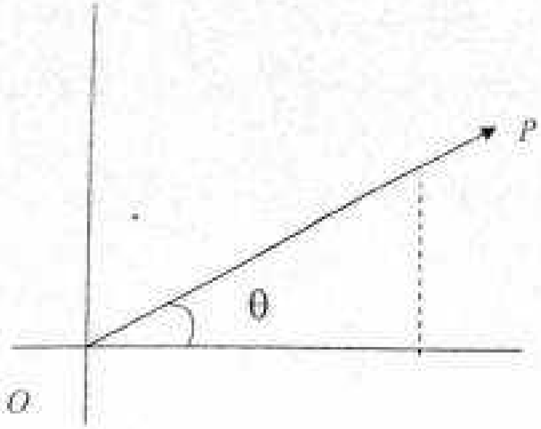
Vectors:

Consider the 2×1 matrix

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

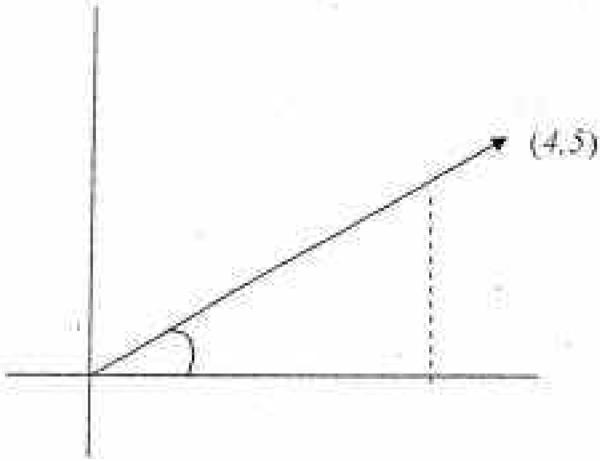
Where x and y are real numbers. With X we associate the directed line segment with initial point at the origin $O\ (0,\ 0)$ and terminal point at $p(x,y)$. It is denoted by $\overline{OP}$; O is called its **tail** and P its **head**.

A directed line segment has a direction, which is the angle made with the positive x-axis, indicated by the arrow at its head



The magnitude of a directed line segment is its length.

Example: Let $X = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ be a vector with the directed line $\overline{OP}$ with head $p(4,5)$.



Definition: A vector in the plane is a 2×1 matrix $X = \begin{bmatrix} x \\ y \end{bmatrix}$ where x and y are real numbers, called the **component** of X.

Note: Since a vector is a matrix, the vectors

$X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $Y = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

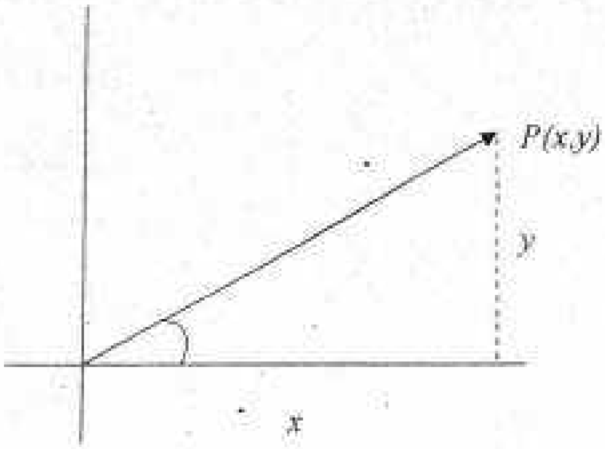
Are said to be **equal** if $x_1 = x_2$ and $y_1 = y_2$. That is two vectors are equal if their respective components are equal.

Example: The vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ are not equal.

Length:

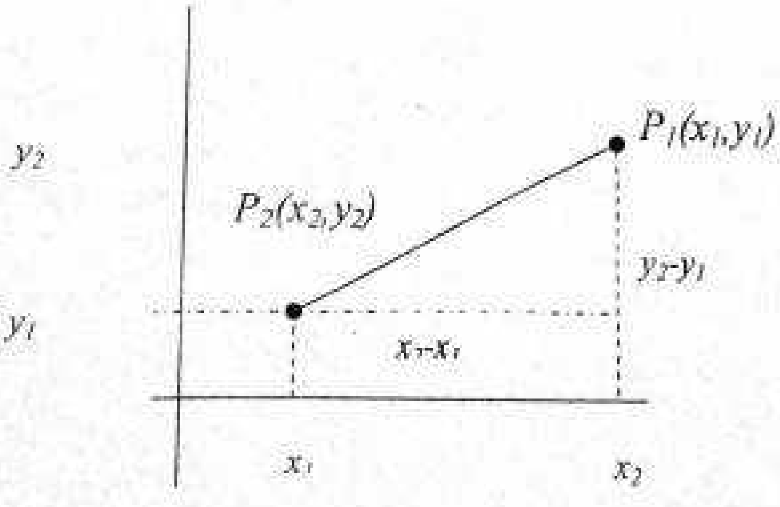
By the Pythagorean theorem the length or magnitude of the vector $X=(x,y)$ is

$\|X\| = \sqrt{x^2 + y^2}$ ... (1)



If $P(x_1,y_1)$ and $Q(x_2,y_2)$ are two points on the plan then by the Pythagorean theorem, the length of the directed segment with initial point $P_1(x_1,y_1)$ and terminal point $P_2(x_2,y_2)$ is

$\|P_1P_2\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ ... (2)



Example: If $X= (2,-5)$, then

$$\|X\| = \sqrt{(2)^2 + (-5)^2} = \sqrt{4+25} = \sqrt{29}$$

Example: The distance between $P\,(3, 2)$ and $Q\,(-1, 5)$ is

$$\|\overline{PQ}\| = \sqrt{(-1-3)^2 + (5-2)^2} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

Note: Two vectors $X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ is said to be **parallel** if $x_1y_2=x_2y_1$ or have the same **slopes**.

Vector operations:

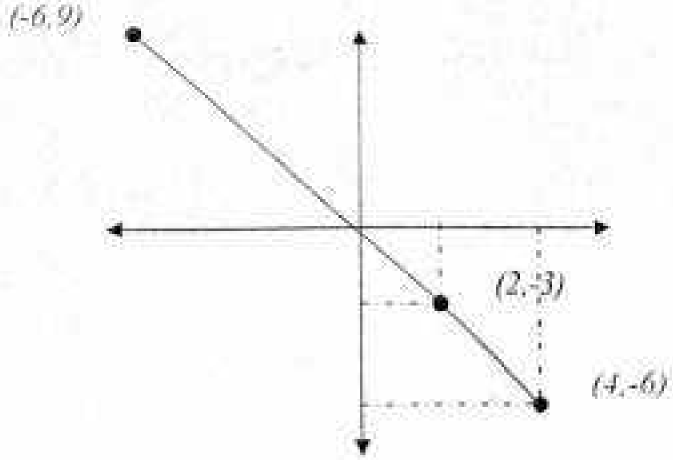
Definition: Let $X=(x_1,y_1)$ and $Y=(x_2,y_2)$ be two vectors in the plane. The **sum** of the vectors X and Y is the vector
(x_1+x_2, y_1+y_2)
And is denoted by $X+Y$.

Example: Let X= (2, 3) and Y= (-5, 6). Then

$Q(x_2,y_2)$ X+Y= (2+ (-5)), 6+3) = (-3, 9)

Definition: If X=(x, y) is a vector and c is a scalar (real number), then the **scalar multiple** cX of X by c is the vector (cx, cy)
If c>0, then cx in the same direction of X
If c<0, then cx in the opposite direction of X

Example: If c=2, d=-3 and X= (2, -3), then
cX=2(2, -3)=(4, -6) and dX= -3(2,-3)=(-6, 9)



Definition: The vector (0,0) is called the **zero vector** and is denoted by O.
If X is any vector, then

$$X+0=X.$$

We can also show that

$$X+(-1)X=O.$$

And we can write (-1) X as –X and call it the **negative** of X.
And we write X+ (-1) Y as and call it the difference between X and Y. The vector X-Y is

The angle between two vectors:

The angle between the nonzero vector $X=(x_1,y_1)$ and $Y=(x_2,y_2)$ is the angle θ ,where $0 \le \theta \le \pi$
Appling the law of cosines to the triangle we have

$$\|X-Y\|^2 = \|X\|^2 + \|Y\|^2 - 2\|X\|\|Y\|\cos\theta \quad \dots (2)$$

From (2)

$$\|X-Y\|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$= x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2(x_1x_2 + y_1y_2)$$
$$= \|X\|^2 + \|Y\|^2 - 2(x_1x_2 + y_1y_2)$$
$$\|X\|^2 + \|Y\|^2 - 2(x_1x_2 + y_1y_2) = \|X\|^2 + \|Y\|^2 - 2\|X\|\|Y\|\cos\theta$$

$\|X\| \ne 0$ and $\|Y\| \ne 0$

Then

$$\cos\theta = \frac{x_1x_2 + y_1y_2}{\|X\|\|Y\|} \quad \dots (3)$$

Definition: The **inner product** or **dot product** of the vectors $X=(x_1,y_1)$ and $Y=(x_2,y_2)$ is defined to be :

$$X \bullet Y = x_1x_2 + y_1y_2 \quad ...(4)$$

Thus we can rewrite (3) as

$$\cos\theta = \frac{X \bullet Y}{\|X\|\|Y\|} \qquad (0 \le \theta \le \pi)...(5)$$

Example: If X=(2,4), Y=(-1,2), then

$X \bullet Y = (2)(-1) + (4)(2) = 6$

Also

$\|X\| = \sqrt{(2)^2 + (4)^2} = \sqrt{20}$ And $\qquad \|Y\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

Hence

$\cos\theta = \frac{6}{\sqrt{20}.\sqrt{5}} = 0.6$ Then $\qquad \theta = 53°8' \qquad$ from table.

Note: If the nonzero vectors X and Y at right angles, then the cosine of the angle θ, $\cos\theta=0$. Hence X.Y=0, conversely if X.Y=0 then $\cos\theta=0$ then $\theta = \frac{\pi}{2} = 90°$

Thus the nonzero vectors X and Y are perpendicular or orthogonal if and only if X.Y=0.

Example:

If X=(2,-4), Y=(4,2)

X.Y=4.2+2(-4)=0 then $X \perp Y$

The properties of the dot product

Theorem: If X,Y and Z are vectors and c is a scalar, then:

(1) $X \bullet X = \|X\|^2 \ge 0$, with equality if and only if X=0,

(2) $X \bullet Y = Y \bullet X$,

(3) $(X+Y) \bullet Z = X \bullet Z + Y \bullet Z$,

(4) $(cX) \bullet Y = X \bullet (cY) = c(X \bullet Y)$.

Proof: H.W.

Unit vectors:

A **unit vector** is a vector whose length is 1 and denoted by U. If X is any nonzero vector, then the vector

$$U = \frac{1}{\|X\|} \bullet X$$

Is a unit vector in the direction of X.

H.W. prove that for any unit vector U, then $\|U\| = 1$.

Example: Let X=(-3,4), then

$\|X\| = \sqrt{(-3)^2 + 4^2} = 5$

Then the vector U is

$U = \frac{1}{5}(-3,4) = (\frac{-3}{5}, \frac{4}{5})$ is the unit vector.

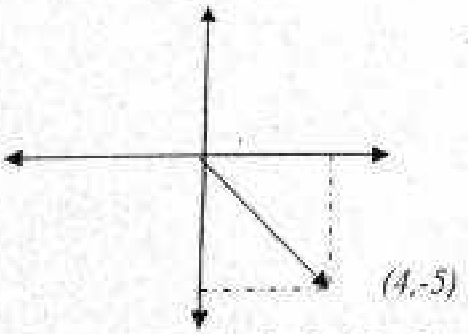
$\|U\| = \sqrt{(\frac{-3}{5})^2 + (\frac{4}{5})^2} = \sqrt{\frac{9+16}{25}} = 1$

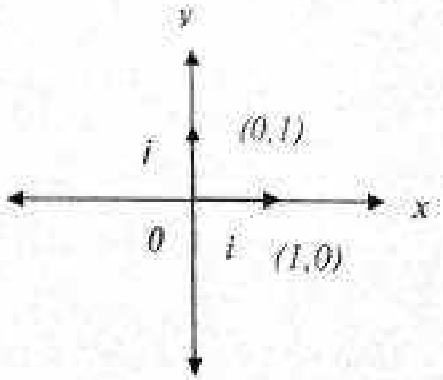
Then U lies in direction of X.

Now, there are two unit vectors in R^2. They are i=(1,0) and j=(0,1).
If X=(x,y) is any vector in R^2, then we write X in the term of I and j as

X=xi+yj

Example: If X=(4,-5) then we can say that X=4i-5j.



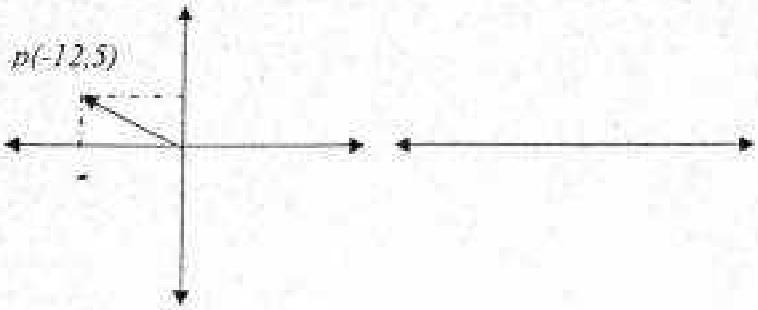


X=4(1,0)+(-5)(0,1)=(4,0)+(0,-5)=(4,-5)

Applications:

Suppose that a force\power of 12 pounds act on a solid in the direction of a negative x-axis and force\power of 5 pounds act on the same solid in the direction of a positive y- axis, find the value and the direction of the magnitude.

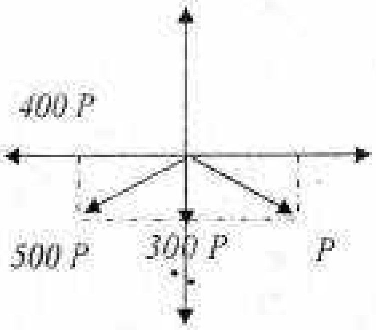
$\|\overrightarrow{OP}\| = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13$



Example: A ship is being pushed by a tugboat with a force of 300 pounds along the negative y-axis while another tugboat is pushed along the negative x-axis with a force of 400 pounds. Find the magnitude and sketch the direction of the resultant force.

Solution:

$\|\overrightarrow{OP}\| = \sqrt{16000 + 90000} = \sqrt{250000} = 500$ pounds.



Exercises :

1- Find X+Y ,X-Y ,2X and 3X-2Y if X =(2,3) ,Y=(-2,5).

2- Let X=(1,2) ,Y=(-3,4),Z=(x,4) and U= (-2,y) find x and y so that

(a) Z=2X (b) $\frac{3}{2}$U=Y (c) Z+U=X

3-Find the length of X=(-4,-5).

4- Find the distance between (0,3),(2,0).

5-Find X • Y, X=(-2,-3), (2,-1), find the cosine of the angle between X,Y.

6- which of the vectors X=(1,2), Y=(0,1), Z=(-2,-4), W=(-2,1), U=(-6,3) are orthogonal, in same direction, in opposite direction

7- Show that if Z orthogonal to X and Y then Z orthogonal to rX+ sY, where r, s are scalars.

n-vectors:

Definition: An n-vectors is an $n \times 1$ matrix

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Where $x_1, x_2, \cdots, x_n$ and real numbers, which are called the **component** of X. Since an n-vector is a matrix, the n-vectors

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Are said to be **equal** if $x_i = y_i$ where $1 \le i \le n$.

Example: The 4-vectors $\begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}$ are not equal.

Note: The set of all n-vectors is denoted by R^n and called **n-space**.

Vector operations:

Definition: Let

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Be two vectors in R^n. The **sum** of the vectors X and Y is the vector

$$\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}.$$

And it is denoted by X+Y.

Example: If

$$X = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}$$

Are vectors in R^3, then

$$X + Y = \begin{bmatrix} 1+2 \\ -2+3 \\ 3+(-3) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Definition: If $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a vector in R^n and c a scalar, then the scalar multiple cX of X and c is the vector $\begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$.

Example: if $X = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ is a vector in R^3 and c=-2, then $cX = (-2)\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 2 \end{bmatrix}$.

Theorem: Let X,Y and Z be any vectors in R^n; let c and d be any scalars. Then:

(a) X+Y is a vector in R^n(that is, R^n is closed under the operation of addition)

(1)X+Y=Y+X,

(2)X+(Y+Z)=(X+Y)+Z,

(3)There is a unique vector **0** in R^n, where $0=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$ such that X+0=0+X=X, 0 is called **zero vector.**

(4)There is a unique vector –X, where $-X=\begin{bmatrix}-x_1\\-x_2\\\vdots\\-x_n\end{bmatrix}$ such that X+(-X)=0.

(b) cX is a vector in R^n

(1)c(X+Y)=cX+cY,

(2)(c+d)X=cX+dX,

(3)c(dX)=(cd)X,

(4)1X=X.

Proof: (a) and (b) are immediately from the definitions for vector sum and scalar multiple.

We prove that (c+d)X=cX+dX

$$(c+d)X=(c+d)\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=\begin{bmatrix}(c+d)x_1\\(c+d)x_2\\\vdots\\(c+d)x_n\end{bmatrix}=\begin{bmatrix}cx_1+dx_1\\cx_2+dx_2\\\vdots\\cx_n+dx_n\end{bmatrix}.$$

(4) $$X=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=\begin{bmatrix}1.x_1\\1.x_2\\\vdots\\1.x_n\end{bmatrix}=\begin{bmatrix}x_1.1\\x_2.1\\\vdots\\x_n.1\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=X.$$

Example: If X and Y are vectors such that $X=\begin{bmatrix}1\\-2\\3\end{bmatrix}$ and $Y=\begin{bmatrix}2\\3\\-3\end{bmatrix}$ then

$$X-Y=\begin{bmatrix}1-2\\-2-3\\3-(-3)\end{bmatrix}=\begin{bmatrix}-1\\-5\\6\end{bmatrix}.$$

Application: The vector in R^n can be used to handle a large amounts of data. Inded a number of computer programming languages.

Example: Suppose that a store handles 100 different items. The inventory on hand at the beginning of the week can be described by the inventory vector A in R^{100}. The number of item sold at the end of the week can be described by the vector S and the vector A-S represents the inventory at the end of the week.

$$A = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{100} \end{bmatrix} \in R^{100}$$

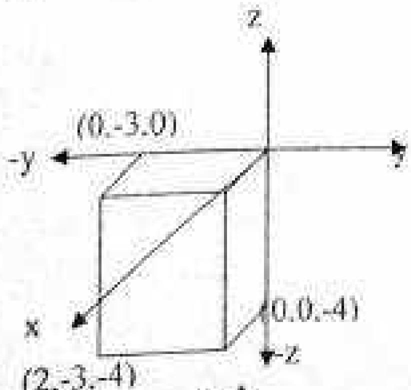
If the store receives a new shipment of goods represented by the vector B. Then its new inventory would be

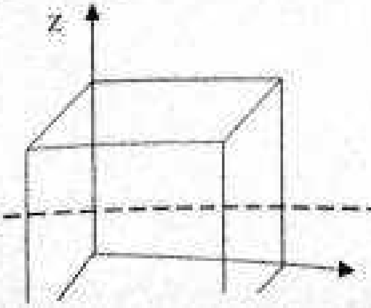
A-S+B

The space R^3:

We draw the three dimension system by fixing the point called **origin** point then we draw the three coordinate axis

Example: Find (2,-3,-4) and (3,5,7) on the coordinate system.

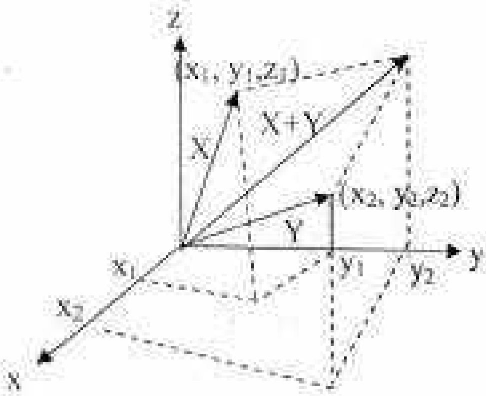




(0,0,7)

(3,5,7)

Note: The sum X+Y of the vectors in R^3 is the diagonal of the parallelogram determined by X and Y.
To illustrate the above note, let $X=(x_1, y_1, z_1)$ and $Y=(x_2, y_2, z_2)$ then:



Definition: The length on norm of the vector $X=(x_1, x_2, \ldots, x_n)$ in R^n is
$$\|X\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$
We also define the distance between the points $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ by
$$\|X - Y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2}$$
Example(*): Let X=(2,3,2,-1) and Y=(4,2,1,3). Then
$$\|X\| = \sqrt{2^2 + 3^2 + 2^2 + (-1)^2} = \sqrt{18}$$
$$\|Y\| = \sqrt{4^2 + 2^2 + 1^2 + 3^2} = \sqrt{30}$$

$$\|X-Y\|=\sqrt{(2-4)^2+(3-2)^2+(2-1)^2+(-1-3)^2}=\sqrt{22}$$

Definition: If $X=(x_1,x_2,\ldots,x_n)$ and $Y=(y_1,y_2,\ldots,y_n)$ are vectors in R^n, then their **inner product** is defined by:

$$X.Y=x_1y_1+x_2y_2+\ldots+x_ny_n$$

Also called **dot product**.

Example: If $X=(2,3,2,-1)$ and $Y=(4,2,1,3)$, then

$$X.Y=(2)(4)+(3)(2)+(-1)(3)$$
$$=13$$

Example:(Revenue Monitoring): Consider the store in the above example with (*), if the vector P denoted the price of each of the 100 items, then the dot product $S.P$ given the total revenue received at the end of the week

$$S.P=\begin{bmatrix}x_1\\x_2\\\vdots\\x_{100}\end{bmatrix}\begin{bmatrix}price\\price\\\vdots\\price\end{bmatrix}$$

Theorem(Properties of the Inner product):

If X, Y and Z are vectors in R^n and c is a scalar then:

1. $X.X=\|X\|^2\geq 0$, with equality if and only if $X=0$,

2. $X.Y=Y.X$

3. $(X+Y).Z=X.Z+Y.Z$

4. $(cX).Y=X.(cY)=c(X.Y)$

Theorem(Cauchy-Schwrz Inequality):

If X and Y are vectors in R^n, then

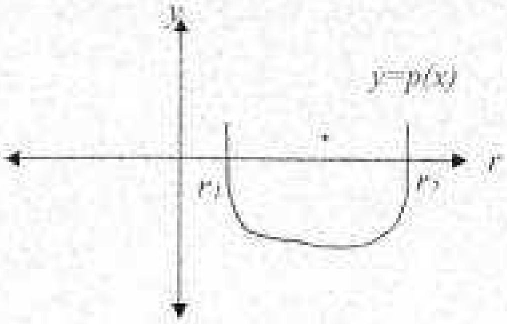
$$|X.Y|\leq\|X\|\,\|Y\|\ldots(3)$$

Proof: If $X=0$, then $\|X\|=0$ and $X.Y=0$, so hold.

Now, suppose that X and Y are nonzero. Let r be a scalar and consider the vector $rX+Y$. Then

$$0\leq(rX+Y).(rX+Y)=r^2X.X+2rX.Y+Y.Y$$
$$=r^2a+2rb+c$$

Where a=X.X, b=X.Y and c=Y.Y



Now , $p(r)=ar^2+2br+c$

P(r) is a quadratic polynomial in r (whose grph is a parabola opening upward, since a>0)

That is nonzero for all values of r? Why?

This means that either this polynomial has no real roots, or it has real roots then both roots are equal(why?)

(The answer) : if p(r) had two distinct root r_1 and r_2, then it would be negative for some value of r.

Recall that the roots of p(r) are given by quadratic formula as $\frac{-2b+\sqrt{4b^2-4ac}}{2a}$ and $\frac{-2b-\sqrt{4b^2-4ac}}{2a}$ (where $a\neq 0$ since $X\neq 0$). Thus we must have

$$4b^2-4ac\leq 0$$

$$4b^2\leq 4ac$$

Which means that :

$$b^2\leq ac$$

Taking square roots of both side and observing that $b\leq\sqrt{a}\sqrt{c}$ Where

$\sqrt{a}=\sqrt{X.X}=\|X\|$ and $\sqrt{c}=\sqrt{Y.Y}=\|Y\|$

Thus

$$|X.Y|\leq\|X\|\|Y\|.$$

Example: If X=(1,2,-1,2) and Y=(3,1,-1,2) then:

$\|X\|=\sqrt{10}$, $\|Y\|=\sqrt{15}$ and $|X.Y|=10\leq\sqrt{10}\sqrt{15}$.

Definition: The angle between two nonzero vectors X and Y is defined as the unique number θ , $0\leq\theta\leq\pi$ such that :

$$\cos\theta = \frac{X.Y}{\|X\|\|Y\|}$$

It follows from the Cauchy-Schwarz inequality that :

$$\left|\frac{X.Y}{\|X\|\|Y\|}\right| \le 1.$$

Example: Let X=(1,0,0,1) and Y=(0,1,0,1) then we have that

$$\|X\| = \sqrt{2},\ \|Y\| = \sqrt{2} \text{ and } X.Y = 1$$

Thus

$$\cos\theta = \frac{1}{2} \text{ and } \theta = 60^{\circ}$$

Definition: Two nonzero vector X and **Y in** R^n are said to be **Orthogonal** if $X.Y = 0$. They are said **Parallel** if $|X.Y| = \|X\|\|Y\|$. They are in the **Same direction** if $X.Y = \|X\|\|Y\|$. That is, they are orthogonal if $\cos\theta = 0$, parallel if $\cos\theta = \pm 1$, and in the same direction if $\cos\theta = 1$.

Example: Consider the vectors X=(1,0,0,1) , Y=(0,1,0,1) and Z=(3,0,0,3) then X.Y=0 and Y.Z=0 (check).
Which implies that X and Y are orthogonal and Y and Z are orthogonal too.
Also X.Z=6, $\|X\| = \sqrt{2}$, $\|Z\| = \sqrt{18}$, and $X.Z = \|X\|\|Z\|$
Hence X and Z are in the same direction.

Theorem: (Triangle Inequality) If X and Y are vectors in R^n, then

$$\|X+Y\| \le \|X\| + \|Y\|$$

Proof: By theorem (*)

$$\|X+Y\|^2 = |X+Y||X+Y|$$
$$= X.X + 2(X.Y) + Y.Y$$
$$= \|X\|^2 + 2(X.Y) + \|Y\|^2$$

By the Cauchy-Schwarz inequality we have:

$$\|X\|^2 + 2(X.Y) + \|Y\|^2 \le \|X\|^2 + 2\|X\|\|Y\| + \|Y\|^2 = (\|X\| + \|Y\|)^2.$$

Example: Let X=(1,0,0,1) and Y=(0,1,0,1) then

$$\|X+Y\| = \sqrt{4} = 2 < \sqrt{2} + \sqrt{2} = \|X\| + \|Y\|$$

Note: If X and Y are vectors in R^n, then

$$\|X+Y\|^2 = \|X\|^2 + \|Y\|^2$$

If and only if X and Y are orthogonal.

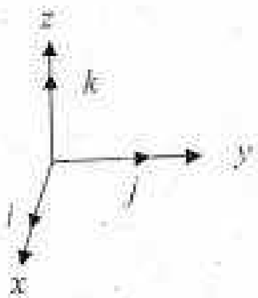
Definition: A **Unit vector** U in R^n is a vector of unit length. Furthermore, if X is a nonzero vector, then the vector

$$U = \left[\frac{1}{\|X\|}\right] X$$

Is a unit vector in the direction of X.

Example: If X= (1,0,0,1), then $\|X\| = \sqrt{2}$ and $U = \frac{1}{\sqrt{2}}(1,0,0,1)$ is a unit vector in the direction of X.

In the case of R^3 the unit vector in the position direction are i=(1,0,0), j=(0,1,0) and k=(0,0,1).
If X=(x, y, z), then X= xi+ yj+ zk.



Example: if X=(2,-1,3), then
X=2i-j+3k

In R^n the unit vector are
$E_1=(1,0,0,\ldots,0)$, $E_2=(0,1,0,\ldots,0)$, …, $E_n=(0,0,\ldots,1)$
And if $X=(x_1, x_2, \ldots, x_n)$ is a vector in R^n, we have
$X= x_1 E_1+x_2 E_2+\ldots+ x_n E_n$.

Exercises :

1- Find X+Y, X-Y, 2X and 3X-2Y if X=(2,3,5), Y=(-2,5,3).
2- Let X=(1,2,2,1), Y=(-3,4,-2,-1), Z=(x,4,0,y) and U= (-2,u,v,4) find x, u, v and y so that
(a) Z=3X (b) Z-Y=Y (c) Z+U=X
3-Find the length of X=(1,6,-4,-5).
4- Find the distance between (0,3,2),(2,0,4).

5-Find X • Y, X=(-2,-3,-4), (2,-1,2), find the cosine of the angle between X,Y.
6- which of the vectors X=(4,2,6,-8), Y =(-2,3,-1,-1), Z =(-2,-1,-3,4), W=(1,0,0,2) are orthogonal, in same direction, parallel.

7- Prove the parallelogram law. $\|X+Y\|^2+\|X-Y\|^2=2\|X\|^2+2\|Y\|^2$

Cross product in R^3

Definition: if $X= x_1i+ x_2j+ x_3k$ and $Y= y_1i+ y_2j+ y_3k$ are two vectors in R^3, then their cross product is the vector $X\times Y$ definedby:

$$X\times Y=(x_2y_3-x_3y_2)i+(x_3y_1-x_1y_3)j+(x_1y_2-x_2y_1)k$$

The cross product $X\times Y$ can also be written as:

$$X\times Y=\begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$X\times Y$ is a vector (from definition)
So

$$X\times Y=\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} i-\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} j+\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} k$$

Example: Let X=2i+j+2k and Y=3i-j-3k. Then

$$X\times Y=\begin{vmatrix} i & j & k \\ 2 & 1 & 2 \\ 3 & -1 & -3 \end{vmatrix}=-i+12j-5k.$$

Properties of cross product:

Theorem: If X , Y and Z are vectors and c is a scalar, then :

1- $X\times Y=-(Y\times X)$
2- $X\times(Y+Z)=X\times Y+X\times Z$
3- $(X+Y)\times Z=X\times Z+Y\times Z$
4- $c(X\times Y)=(cX)\times Y=X\times(cY)$
5- $X\times X=0$
6- $X\times 0=0\times X=0$

7- $(X \times Y) \times Z = (Z.X)Y - (Z.Y)X$

Also $X \times (Y \times Z) = (X.Z)Y - (X.Y)Z$

8- $(X \times Y).Z = X.(Y \times Z)$

Proof:

$X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3)$ and $Z = (z_1, z_2, z_3)$

$$X \times Y = \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \underbrace{(x_2y_3 - x_3y_2)}_{u_1}i + \underbrace{(x_3y_1 - x_1y_3)}_{u_2}j + \underbrace{(x_1y_2 - x_2y_1)}_{u_2}k$$

$$(X \times Y) \times Z = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = (u_2z_3 - u_3z_2)i + (u_3z_1 - u_1z_3)j + (u_1z_2 - u_2z_1)k$$

Substituted instead of u's then we get:

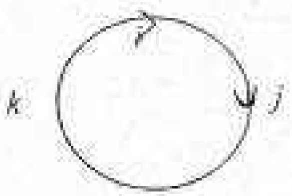
$$= [(x_3y_1 - x_1y_3)z_3 - (x_1y_2 - x_2y_1)z_2]i + [(x_1y_2 - x_2y_1)z_1 - (x_2y_3 - x_3y_2)z_3]j + [(x_2y_3 - x_3y_2)z_2 - (x_3y_1 - x_1y_3)z_1]k$$

Example: From definition we have that:

$i \times i = j \times j = k \times k = 0$ and $i \times j = k$, $j \times k = i$, $k \times i = j$,

Also

$j \times i = -k$, $k \times j = -i$, $i \times k = -j$



Example: Let X=2i+j+2k, Y=3i-j-3k and Z=i+2j+3k

$X \times Y = -i + 12j - 5k$ $(X \times Y).Z = 8$

$Y \times Z = 3i - 12 + 7k$ $X.(Y \times Z) = 8$

Which illustrate equation(8). Also to illustrate (*)

$X \times (Y \times Z) = 31i - 8j - 27k$, $X.Z = 10$, $X.Y = -1$

$(X.Z)Y = 30i - 10j - 30k$, $(X.Y)Z = -i - 2j - 3k$

Hence

$(X.Z)Y - (X.Y)Z = 31i - 8j - 27k$

Also from (8) ,(1) and (5) we get:

$(X \times Y).Y = X.(Y \times Y) = X.0 = 0$

And

$(X \times Y).X = -(Y \times X).X = -Y.(X \times X) = -Y.0 = 0$

Note: If $X \times Y \neq 0$, then $X \times Y$ is orthogonal to both X and Y and to the plane determined by them.

The Fifth Lecture
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To find the angle between X and Y :

$\|X \times Y\|^2 = (X \times Y).(X \times Y)$

$= X.[Y \times (X \times Y)]$ by (8)

$= X.[(Y.Y)X - (Y.X)Y]$ by (7)

$= (X.X)(Y.Y) - (Y.X)(Y.X)$ by(2) and (4)

$= \|X\|^2\|Y\|^2 - (X \times Y)^2$ by (1)

$X \times Y = \|X\|\|Y\|\cos\theta$

$\|X \times Y\|^2 = \|X\|^2\|Y\|^2 - \|X\|^2\|Y\|^2 \cos\theta$

$= \|X\|^2\|Y\|^2(1 - \cos^2\theta)$

$\therefore \|X \times Y\|^2 = \|X\|^2\|Y\|^2 \sin^2\theta$

$\therefore \|X \times Y\| = \|X\|\|Y\|\sin\theta \quad \ldots(9)$

Note: (1) We do not have $|\sin\theta|$, since $\sin\theta$ is nonnegative for $0 \leq \theta \leq \pi$

(2) X and Y are not parallel if and only if $X \times Y = 0$

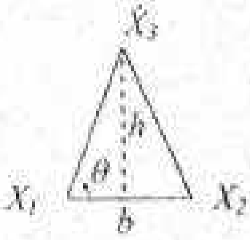
Applications:

Area of triangle:

Area of triangle consider a triangle with vertices X_1,X_2, and X_3. The area of this triangle is

$A_T = \frac{1}{2}bh$

Where b is the base and h is the height.



$b = \|X_2 - X_1\|$ and $h = \|X_3 - X_1\|\sin\theta$

Where

$\sin\theta = \frac{h}{\|X_3 - X_1\|}$ so $A_T = \frac{1}{2}\|X_2 - X_1\|\|X_3 - X_1\|\sin\theta$

From(9)

$A_T = \frac{1}{2}\|(X_2 - X_1)\times(X_3 - X_1)\|$

Example: Find the area of the triangle with vertices $X_1=(2,2,4)$, $X_2=(-1,0,5)$ and $X_3=(3,4,3)$

Solution:

$X_2 - X_1 = -3i - 2j + k$

$X_3 - X_1 = i + 2j - k$

$A_T = \frac{1}{2}\|(-3i - 2j + k)\times(i + 2j - k)\|$

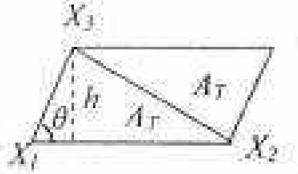
$= \frac{1}{2}\|-2j - 4k\|$

$= \|-j - 2k\| = \sqrt{5}$

Area of a parallelogram:

The area A_P of parallelogram with sides X_2-X_1 and X_3-X_1 is $2A_T$.

$A_P = \|(X_2 - X_1)\times(X_3 - X_1)\|$



Example: If $X_1=(2,2,4)$, $X_2=(-1,0,5)$ and $X_3=(3,4,3)$.

Then

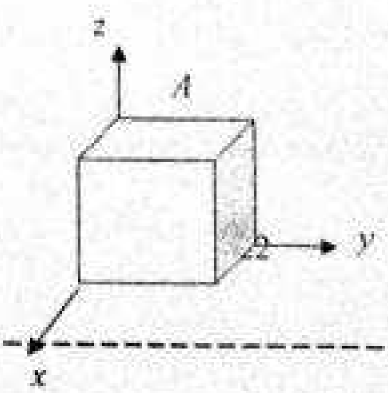
$A_P = 2\sqrt{5}$ (check).

Consider the parallelepiped with vertex at the origin and edges X, Y and Z. The volume of the parallelepiped is the product of the area of the face containing Y, Z and the distance h from the face parallel to it.

$h = \|X\||\cos\theta|$

θ between X and $Y\times Z$, the area of the face determined by Y and Z is $\|Y\times Z\|$

$\therefore V = \|Y\times Z\|\|X\||\cos\theta| = |X.(Y\times Z)|$



Example: X=i-2j+3k, Y=i+3j+k and Z=2i+j+2k (H.W.)

Exercises :

1-Show that X and Y are parallel iff $X \times Y = 0$.

2-Show that $\|X \times Y\|^2 + (X.Y)^2 = \|X\|^2 \|Y\|^2$

3-Prove the Jacobi identity :

$(X \times Y) \times Z + (Y \times Z) \times X + (Z \times X) \times Y = 0$

Vector space

In the following lectures we study the vector space , subspace ,study the linear Independence , basis and the rank of a matrix .

Definition :

Areal vector space is a set V of elements with two operations⊕ and ⊙ defined with the following properties .

(a) If X and Y are any elements in V . then X⊕Y is in V (that is closed under the operation ⊕) .

1- X⊕Y = Y⊕X for all X ,Y in V .

2- X⊕ (Y⊕Z) = (X⊕Y) ⊕Z for all X ,Y,Z in V

3- There is a uniqueelement 0 in V such that X⊕0 = 0⊕X = Xfor every X in V .

4- For each X in V there exists a unique -X in V such that X⊕-X = 0

(b) If X is any element in V and c is any realnumber then c⊙X is in V .

5-c⊙(X⊕Y) = c⊙ X⊕c⊙Y for any X, Y in V , and any real number c.

6-(c+d) ⊙X = c⊙X⊕ d⊙Y for any X in v and any real numbers c and d .

7-c⊙(d⊙X) = (cd)⊙X for any X in v and real numbers c and d .

8-1 ⊙ X = X for any X in v .

(V, ⊕, ⊙) is vector space . The operation ⊕ is called vector addition .

The operation ⊙ is called scalar multiplication .

The vector 0 is called Zero vector .

Example 1:

Let R^n be the set of ordered n- tuples (a_1 , a_2 , ……. , a_n) where we define ⊕ by (a_1 , a_2 , ……. , a_n)⊕(b_1 , b_2 , ……. , b_n)

=(a_1+ b_1 , a_2+ b_2 , ……. , a_n +b_n) and ⊙ by c⊙(a_1 , a_2 , …. , a_n)

=(ca_1 , ca_2 , ……. , ca_n)

R^n is a vector space .

Example 2:

Let V be the set of ordered triples of real number (a_1 , a_2 , 0) where we define ⊕ by (a_1 , a_2 ,0)⊕(b_1 , b_2 , 0)

=(a_1+ b_1 , a_2+ b_2 ,0) and ⊙ by c⊙(a_1 , a_2 ,0)= (ca_1 , ca_2 ,0)

V is a vector space .

Example 3:

Let V be the set of ordered triples of real number (x , y , z) where we define ⊕ by (x , y ,z)⊕(x′ , y′ ,z′)

=(x+x′ , y+ y′ ,z+z′)) and ⊙ by c⊙(x , y ,z)= (cx , y ,z)

V is not vector space the property (c+d) ⊙X = c⊙X⊕ d⊙X fails to hold

thus (c+d) ⊙(x , y ,z)=((c+d)x , y ,z),

On other hand c⊙(x , y ,z)⊕d⊙(x , y ,z)=(cx , y ,z)⊕(dx , y ,z)

=(cx+dx , y+y ,z+z)=((c+d)x ,2y ,2z).

Example 3:

Let V be the set of 2×3 matrices under usual operation of matrix addition and scalar multiplication

V is vector space c.h.

Example 4:

Let V be the set of all real –valued function on R .if f and g are in V we define $f \oplus g$ by $(f \oplus g)(t) = f(t) \oplus g(t)$ and if f and c is a scalar define $c \odot f$ by $c \odot f = c\,f(t)$.

V is vector space c.h.

Example 5:

Let p_n be the set of all real polynomials of degree $\le n$ with zero polynomial .if $p(t) = a_0t^n + a_1t^{n-1} + \dots\dots + a_{n-1}t + a_n$ and $q(t) = b_0t^n + b_1t^{n-1} + \dots\dots + b_{n-1}t + b_n$ are in V we define $p(t) \oplus q(t)$ by $p(t) \oplus q(t) = (a_0+b_0)t^n + (a_1+b_1)t^{n-1} + \dots\dots + (a_{n-1}+b_{n-1})t + (a_n+b_n)$ and if c is a scalar define $c \odot p(t)$ by

$c \odot p(t) = (ca_0)t^n + (ca_1)t^{n-1} + \dots\dots + ca_{n-1}t + ca_n$

the above definition show that the degree of $p(t) \oplus q(t)$ and $c \odot p(t) \le n$

$-p(t) = -a_0t^n - a_1t^{n-1} + \dots\dots - a_{n-1}t - a_n$ is negative of $p(t)$ and since $a_i + b_i = b_i + a_i$ then $p(t) \oplus q(t) = q(t) + p(t)$

And

$(c+d) \odot p(t) = (c+d)a_0t^n + (c+d)a_1t^{n-1} + \dots\dots + (c+d)a_{n-1}t + (c+d)a_n$

$= c(a_0t^n + a_1t^{n-1} + \dots\dots + a_{n-1}t + a_n) + d(a_0t^n + a_1t^{n-1} + \dots\dots + a_{n-1}t + a_n)$

$= c \odot p(t) \oplus d \odot p(t)$

V is vector space c.h.

Theorem :

If V is a vector space then .

1- $0 \odot X = 0$ for any vector X in V

2- $c \odot 0 = 0$ for any scalar c

3- If $c \odot X = 0$ then either $c = 0$ or $X = 0$

4- $(-1) \odot X = -X$ for any X in V.

Proof:

1) $0X = (0+0)X = 0X + 0X$ by (6) of def. adding $-0X$

$0 = 0X + (-0X) = (0X + 0X) + (-0X)$

$= 0X + [0X + (-0X)]$

$=0X+0=0X.$

2) $c.0=c.(0+0)=c.0+c.0$

$c.0-c.0=c.0+c.0-c.0$

$0=c.0$

3) suppose $cX=0$ and $c\neq 0$ then

$0=(\frac{1}{c}).0=(\frac{1}{c})(cx)=[(\frac{1}{c})c]X=1.X$

4) $(-1)X+X=(-1)X+(1)X=(-1+1)X=0X=0$ so that $(-1)X=-X$

Definition :

Let V be a vector space and W a nonempty subset of V if W is a vector space with respect to the same operations as these in V , then W is called **a subspace** of V .

Example : If $(V,\oplus,\odot)$ is vector space then $\{0\}\subseteq V$, $V\subseteq V$ are two subspaces.

Example :

Let W be the set of ordered triples of real number $(a_1, a_2, 0)$ where we define $\oplus$ by $(a_1, a_2, 0)\oplus(b_1, b_2, 0)$

$=(a_1+b_1, a_2+b_2, 0)$ and $\odot$ by $c\odot(a_1, a_2, 0)=(ca_1, ca_2, 0)$

Then $(W,\oplus,\odot)$ is subspace of $(R^3,\oplus,\odot)$.

Theorem:

Let $(V,\oplus,\odot)$ be a vector space and let **W** be a nonempty subset of V. W is a subspace of V if and only if the following condition hold

1- If **X** , Y are any vectors in W then **X**$\oplus$**Y** is in W

2- If **c** is any real number and X is **any** vector in W then $c\odot X$ is in **W** .

Proof : H.W.

Example:

Let W be the set of all 2×3 matrices of form

$W=\left\{\begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix}, a,b,c\in R\right\}$,W is subset of vector space V of all 2×3 matrices under usual operations of matrices addition and scalar multiplication then W is subspace of V.

Solution :

Consider $X=\begin{bmatrix} a_1 & b_1 & 0 \\ 0 & c_1 & d_1 \end{bmatrix}$ and $Y=\begin{bmatrix} a_2 & b_2 & 0 \\ 0 & c_2 & d_2 \end{bmatrix}$ in W then

$X+Y=\begin{bmatrix} a_1+a_2 & b_1+b_2 & 0 \\ 0 & c_1+c_2 & d_1+d_2 \end{bmatrix}$ is in W also let $r\in R$

$rX=\begin{bmatrix} ra_1 & rb_1 & 0 \\ 0 & rc_1 & rd_1 \end{bmatrix}$ is in W ,W is subspace of V.

Example:

Let W be the sub set of(R^3 ,⊕,⊙).

W is ordered triples of real number (a , b , 1),

let X=(a_1 , a_2 , 1),Y=(b_1 , b_2 , 1)

X+Y =(a_1+ b_1 , a_2+ b_2 ,2) Then W is not subspace of (R^3 ,⊕,⊙).

Example 5:

Let W be the set of all real polynomials of degree exactly=2

W is subset of p_2 but not subspace of p_2 since

$2t^2+3t+1$ and $-2t^2+t+2$ is polynomial of degree 1 is not in W.

Exercises:

1- Let $W = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, a = 2c+1 \right\}$ W is subset of vector space V of all 2×3 matrices under usual operations of matrices addition and scalar multiplication is W is subspace of V.

2- Let $W = \{(a,b,c), b = 2a+1\}$ subset of vector space R^3 is W is subspace?

Definition(1 – 7)

Let $X_1, X_2, \ldots\ldots, X_n$ be vectors in a vectors space V. A vector X in V is called linear combination of this vectorsif it can written as $X = c_1X_1+c_2X_2, \ldots\ldots+c_nX_n$ for some real number where $c_1, c_2 \ldots\ldots c_n$ are scalers.

Example: Consider the vector space R^4 . let $X_1=(1,2,1,-1)$, $X_2=(1,0,2,-3)$, $X_3=(1,1,0,-2)$ the vector $X=(2,1,5,-5)$ is linear combination of X_1, X_2, X_3 if we find c_1, c_2, c_3 s.t..

$$X = c_1X_1+c_2X_2+c_3X_3$$

$(2,1,5,-5)=c_1(1,2,1,-1)+c_2(1,0,2,-3)+c_3(1,1,0,-2)$

$(2,1,5,-5)=(c_1,2c_1,c_1,-c_1)+(c_2,0,2c_2,-3c_2)+(c_3,c_3,0,-2c_3)$

$c_1+c_2+c_3=2$

$2c_1+c_3=1$

$c_1+2c_2 \quad =5$

$-c_1-3c_2-2c_3=-5$

solving this linear system by Gauss-**Jordan** we obtain $c_1=1$, $c_2=2$, $c_3=-1$

then X is linear combination of X_1, X_2, X_3 .

Example: Consider the vector space R^3 . let $X_1=(1,2,-1)$, $X_2=(1,0,-1)$, is the vector $X=(1,0,2)$ is linear combination of X_1, X_2

if we find c_1, c_2 s.t..

$$X = c_1X_1+c_2X_2$$

$(1,0,2)=c_1(1,2,-1)+c_2(1,0,-1)$

$c_1+c_2=1$

$2c_1=0$

$-c_1-2c_2=2$

Which has no solution then X is not linear combination of X_1, X_2.

Example: Consider the vector space R^3 . let $X_1=(1,0,1)$, $X_2=(-1,1,0)$, $X_3=(0,0,1)$ is the vector $X=(1,1,1)$ is linear combination of X_1, X_2, X_3 if we find c_1, c_2, c_3 s.t..

$$X = c_1X_1+c_2X_2+c_3X_3$$

$(1,1,1)=c_1(1,0,1)+c_2(-1,1,0)+c_3(0,0,1)$

$c_1-c_2=1$

$\quad c_2=1$

$c_1 \quad + c_3 = 1$
solving this linear system by Gauss-Jordan we obtain $c_1=2, c_2=1, c_3=-1$
then X is linear combination of X_1, X_2, X_3

Definition :
Let $S = \{X_1, X_2, \ldots\ldots, X_n\}$ be the set of vectors in a vectors space V. the set spans V , or V is spanned by S , if every vector in V is a linear combination of vector in S.

Example :
Let V be the vector space R^3 , let$S=\{X_1, X_2, X_3\}$ set of vectors where
$X_1=(1,2,1)$, $X_2=(1,0,2)$, $X_3=(1,1,0)$ is the set S spans V ?
SOL.:
Let $X=(a,b,c)$ be any vector in R^3 , and

$$X = c_1X_1+c_2X_2+c_3X_3$$

$(a,b,c)=c_1(1,2,1)+c_2(1,0,2)+c_3(1,1,0)$

$c_1+c_2+c_3=a$
$2c_1+c_3=b$
$c_1+2c_2 \quad = c$
solving this linear system by Gauss-Jordan we obtain
$c_1=\dfrac{-2a+2b+c}{3}$, $c_2=\dfrac{a-b+c}{3}$, $c_3=\dfrac{4a-b-2c}{3}$
since we obtained a solution for every choice of a ,b and c then
$S=\{X_1, X_2, X_3\}$ spans V.

<u>Example :</u>
Let V be the vector space R^3 . $S=\{i, j, k\}$ spans V
Since for every $X=(a,b,c)$ vector in R^3 .

$(a,b,c)=(a,0,0)+(0,b,0)+(0,0,c) = ai + bj + ck$.

Example :

Let $V=P_2$ be the vector space all polynomials of degree ≤ 2 if and
Let $S=\{P_1(t), P_2(t)\}$ where $P_1(t)=t^2+2t+1$ and $P_2(t)=t^2+2$ is S spans V?

Sol : let $P(t)= at^2+bt+c$ polynomial in P_2 where a,b,c are real number suppose $P(t)=c_1P_1(t)+c_2P_2(t)$ then

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$at^2+bt+c = c_1(t^2+2t+1)+c_2(t^2+2)$

$at^2+bt+c = (c_1+c_2)t^2+2c_1t+(c_1+2c_2)$

$c_1+c_2=a$

$2c_1 \quad = b$

$c_1+2c_2=c$

we obtain

$$\begin{bmatrix} 1 & 1 & : & a \\ 2 & 0 & : & b \\ 1 & 2 & : & c \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & : & 2a-c \\ 0 & 1 & : & c-a \\ 0 & 0 & : & b-4a+2c \end{bmatrix}$$

If $b-4a+2c \neq 0$ then there is no solution to this system hence S does not span P_2.

Linear independence

Definition :Let $S=\{X_1, X_2, \ldots\ldots, X_n\}$ be the set of vectors in a vectors space V. thenS is said to be linearly dependent if there exist constants $c_1, c_2 \ldots\ldots c_n$ not all zero , such that
$c_1X_1+c_2X_2, \ldots\ldots+c_nX_n=0$,other wise . S is called linearly independent
That is S is linearly independent if the equation
$c_1X_1+c_2X_2, \ldots\ldots+c_nX_n=0$ hold only if $c_1=c_2=\ldots\ldots=c_n=0$

Example: Consider the vector space R^4 . let $X_1=(1,0,1,2)$, $X_2=(0,1,1,2)$, $X_3=(1,1,1,3)$is $S=\{X_1,X_2, X_3\}$ is linearly independent

S0l;

Let $c_1X_1+c_2X_2+c_3X_3=0$ where $c_1, c_2, c_3 \in R$

$c_1(1,0,1,2), +c_2(0,1,1,2)+c_3(1,1,1,3)=(0,0,0,0)$

$(c_1,0,c_1,2c_1), +(0,c_2,c_2,2c_2)+(c_3,c_3,c_3,3c_3)=(0,0,0,0)$

$c_1+c_3=0$

$c_2+c_3=0$

$c_1+c_2+c_3=0$

$2c_1+2c_2+3c_3=0$

we obtain $c_1=0, c_2=0, c_3=0$ then S is linearly independent.

Example :

Let V be the vector space R^3. let$S=\{X_1,X_2, X_3,X_4\}$ set of vectors where $X_1=(1,2,-1)$, $X_2=(1,-2,1)$, $X_3=(-3,2,-1)$, $X_4=(2,0,0)$ is the set S linearly independent ?

SOL.:

Let $c_1X_1+c_2X_2+c_3X_3+c_4X_4=0$

$c_1(1,2,-1)+c_2(1,-2,1)+c_3(-3,2,-1)+c_4(2,0,0)=0$

$c_1+c_2-3c_3+2c_4=0$

$2c_1-2c_2+2c_3=0$

$-c_1+c_2-c_3=0$

There are infinitely many solution like $c_1=1, c_2=2, c_3=1, c_4=0$

, then S is linearlydependent.

Example :

Let V be the vector space R^3. $S=\{i,j,k\}$ is linearly independent.

Since

$(0,0,0)=(c_1,0,0)+(0,c_2,0)+(0,0,c_3)$

Then $c_1=0, c_2=0, c_3=0$

In fact $E_1,E_2,\ldots\ldots\ldots,E_n$ are linearly independent in R^n.

Example:

Let $V=P_2$ be the vector space all polynomials of degree ≤ 2 if and Let $S=\{P_1(t), P_2(t), P_3(t)\}$ where $P_1(t)=t^2+t+2$ and $P_2(t)=2t^2+t$, $P_3(t)=3t^2+2t+2$ is S is linearly independent?

Sol : $c_1P_1(t)+c_2P_2(t)+c_3P_3(t)=0$

$c_1(t^2+t+2)+c_2(2t^2+2)+c_3(3t^2+2t+2)=0$

then

$c_1+2c_2+3c_3=0$

$c_1+c_2+2c_3=0$

$2c_1 \quad +2c_3=0$

There are infinitely many solution like $c_1=1, c_2=1, c_3=-1$, then S is linearly dependent.

Remark 1: If $S=\{X_1, X_2, \ldots\ldots, X_n\}$ be the set of vectors and A matrix whose columns are these vectors .S is linearly independent set Iff $|A| \neq 0$.

Remark 2: If $S=\{X_1, X_2, \ldots\ldots, X_n\}$ be the set of vectors and X is zero vector then S is linearly dependent set.Why?

Remark 3: Let S_1, S_2 be finite subsets of vector space V and S_1 subset of S_2 then

1- If S_1 is linearly dependent set so is S_2.

2-If S_2 is linearly independent so is S_1.

Proof 1) : let $S_1=\{X_1, X_2, \ldots\ldots, X_k\}$,

$S_2=\{X_1, X_2, \ldots X_k, X_{k+1} \ldots, X_n\}$ $k \leq n$

c.h?

2) $S_1=\{X_1, X_2, \ldots\ldots, X_k\}$,

$S_2=\{X_1, X_2, \ldots X_k, X_{k+1} \ldots, X_n\}$ $k \leq n$

Let $i \leq k$, $c_1X_1+c_2X_2, \ldots\ldots+c_iX_i=0$

$c_1X_1+c_2X_2, \ldots+c_iX_i+0X_{k+1}\ldots+0X_n=0$

S_n is linearly independent then $c_1 = c_2 = \ldots\ldots = c_n = 0$

Meaning of linearly independence in R^2 and R^3

Suppose that $\{X_1, X_2\}$ are linearly dependentin R^2 then there exist scalars c_1, c_2 not both zero such that $c_1X_1 + c_2X_2 = 0$ and
Either $c_1 \neq 0$ or $c_2 \neq 0$

If $c_1 \neq 0$ then $X_1 = (\frac{-c_2}{c_1})X_2$,if $c_2 \neq 0$,then $X_2 = (\frac{-c_1}{c_2})X_1$

Thus one of the vectors is a multiple of the other
Conversely , suppose that $X_1 = cX_2$ then $1X_1 - cX_2 = 0$ since $1 \neq 0$ then X_1 and X_2 are linearly dependent thus $\{X_1, X_2\}$ are linearly dependentin R^2 iff one of the vectors is a multiple of the other .

<u>**In - R^3**</u>

Suppose that $\{X_1, X_2, X_3\}$ are linearly dependent set of vectors in R^3.thenwecan write $c_1X_1 + c_2X_2 + c_3X_3 = 0$
Where c_1, c_2, c_3 are not all zero say that $c_3 \neq 0$ then

$X_3 = (\frac{-c_1}{c_3})X_1 + (\frac{-c_2}{c_3})X_2$ which is means that X_3 is span
$\{X_1, X_2\}$ then $c_1X_1 + c_2X_2 = X_3$, $-c_1X_1 - c_2X_2 + 1.X_3 = 0$
then $\{X_1, X_2, X_3\}$ are linearly dependent

Theorem: Let $S = \{X_1, X_2, \ldots\ldots, X_n\}$ be the set of non zero vectors in a vectors space V. thenS is linearly dependent iff the vector X_i is a linear combination of the preceding vectors in S.

Proof: suppose X_i is linear combination of the preceding vectors in S.
Then $X_i = c_1X_1 + c_2X_2, \ldots + c_{i-1}X_{i-1}$
Then $c_1X_1 + c_2X_2, \ldots + c_{i-1}X_{i-1} + (-1)X_i + \ldots\ldots + 0X_n = 0$
Since $c_i = -1 \neq 0$ thus S is linearly dependent .
Conversely , suppose that S is linearly dependent then there exist scalars c_1, $c_2, \ldots\ldots, c_n$ not all zeros such that

$$c_1X_1 + c_2X_2, \ldots\ldots + c_nX_n = 0$$

now let i be the largest subscript for which $c_i \neq 0$

if $i > 1$ then $\mathbf{X}_i = \left(\frac{-c_1}{c_i}\right)X_1 + \left(\frac{-c_2}{c_i}\right)X_2, \ldots + \left(\frac{-c_{i-1}}{c_i}\right)X_{i-1}$

if $i=1$ then $c_i X_i=0$ which implies that $X_i=0$

contradiction to the hypothesis that nonzero of the vectors in S is the zero vector .

Exercises:

1-Is the vector (3,6,3,0) is linear combination of $X_1=(1,2,1,0)$,$X_2=(4,1,-2,3)$,$X_3=(1,2,6,-5)$,$X_4=(-2,3,-1,2)$

2-Do the polynomials t^3+2t+1 ,t^2-t+2 ,t^3+2 ,$-t^3+t^2-5t+2$ span P_3?

3-Which of the following set of vectors span R^3?

$X_1=(1,-1,2)$, $X_2=(0,1,1)$.

$X_1=(2,2,3)$, $X_2=(-1,-2,1)$, $X_3=(0,1,0)$

4-Which of the following set of vectors are linearly dependent in R^3?

$X_1=(4,2,1)$, $X_2=(2,6,-5)$,$X_3=(1,-2,3)$

$X_1=(1,2,-1)$, $X_2=(3,2,5)$

5- Suppose that $S=\{X_1,X_2,X_3\}$ is a linearly independent set of vector in vector space V prove that $T=\{Y_1,Y_2,Y_3\}$ is also linearly independent where $Y_1=X_1+X_2+X_3$,$Y_2=X_2+X_3$,$Y_3=X_3$.

Basis and Dimension

Definition:
A set of vectors $S = \{X_1, X_2, \ldots\ldots, X_n\}$ in a vector space V is . called a basis for V if S spans V and S is linearly independent .

Example :
In R^n the unit vector are

$E_1=(1,0,0,\ldots,0)$, $E_2=(0,1,0,\ldots,0)$, …, $E_n=(0,0,\ldots,1)$

Form a basis for R^n

Example; Let V be the vector space R^4 .let $S=\{X_1, X_2, X_3, X_4\}$ set of vectors where
$X_1 =(1,0,1,0)$, $X_2=(0,1,-1,2)$, $X_3 =(0,2,2,1)$, $X_4=(1,0,0,1)$ is the set S basis for V?
SOL.:
Let $c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 = 0$

∴

$$\begin{aligned} c_1 \quad\quad\quad + c_4 &= 0 \\ c_2 + 2c_3 \quad &= 0 \\ c_1 - c_2 + 2c_3 \quad &= 0 \\ 2c_2 + c_3 + c_4 &= 0 \end{aligned}$$

Only solution $c_1 = c_2 = c_3 = c_4 = 0$
, then S is linearly independent.to show S spans R^4 let X=(a,b,c,d,) be any vector in R^4
let $c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 = X$ we can find a solution for c_1, c_2, c_3, c_4 by a ,b, c, d .then S spans R^4
then S is a basis for R^4 .

Example :

The set $S=\{t^n, t^{n-1}, \ldots\ldots, t, 1\}$ spans p_n ,since every polynomials of the form $p(t)=a_0t^n+a_1t^{n-1}+\ldots\ldots+a_{n-1}t+a_n$ which is linear combination of elements in S.

S is linearly independent since

$$c_1t^n+c_2t^{n-1}+\ldots\ldots+c_nt+c_{n+1}=0\ldots\ldots(1)$$

holds for every real number t is root of

$p(t)= c_1t^n+c_2t^{n-1}+\ldots\ldots+c_nt+c_{n+1}$.

but nonzero polynomials have only a finite number of roots that (1) only if $c_1=c_2=\ldots\ldots=c_n=c_{n+1}=0$

then S is a basis for p_n

Example: The set $S=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ is a basis for V of all 2×2 matrices to show S is linearly independent

$$c_1\begin{bmatrix}1&0\\0&0\end{bmatrix}+c_2\begin{bmatrix}0&1\\0&0\end{bmatrix}+c_3\begin{bmatrix}0&0\\1&0\end{bmatrix}+c_4\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

then $\begin{bmatrix}c_1&c_2\\c_3&c_4\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ then $c_1=c_2=c_3=c_4=0$

hence S is linearly independent . to show S spans V

Let $c_1\begin{bmatrix}1&0\\0&0\end{bmatrix}+c_2\begin{bmatrix}0&1\\0&0\end{bmatrix}+c_3\begin{bmatrix}0&0\\1&0\end{bmatrix}+c_4\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}$

Then $c_1=a$,$c_2=b$, $c_3=c$, $c_4=d$

then S is a basis for V

Theorem 1:If $S=\{X_1, X_2, \ldots\ldots, X_n\}$ is a basis for a vector space V then every vector in V can be written in one and only one way as a linear combination of the vector in S .

Proof :-

First every vector X in V can be written as a linear combination of the vectors in S because S spans V .

Now let

$X = a_1 X_1 + a_2 X_2 + \dots\dots\dots + a_n X_n$ and

$X = b_1 X_1 + b_2 X_2 + \dots\dots\dots + b_n X_n$ we must show that $a_i = b_i$

for $i = 1, 2, \dots\dots, n$ we have

$0 = (a_1 - b_1) X_1 + (a_2 - b_2) X_2 + \dots\dots + (a_n - b_n) X_n$

Since S is linearly independent , we conclude that .

$a_i - b_i = 0$ for $i = 1, 2, \dots\dots\dots, n$.

So that $a_i = b_i$

Theorem 2:Let $S = \{X_1, X_2, \dots\dots, X_n\}$ set of non-Zero vectors and let W=spans S then some subset of S is basis for W .

Proof :Ex.(like example)

Example: Let V be the vector space R^4 . let $S=\{X_1, X_2, X_3, X_4\}$ set of vectors where

$X_1 = (1,2,-2,1)$, $X_2=(-3,0,-4,3)$, $X_3 = (2,1,1,-1)$, $X_4=(-3,3,-9,6)$ Find a subset of S that is basis for W .

SOL.:

observe that every vector X in W is of the form

$aX_1 + bX_2 + c X_3 + d X_4$ ....(1)

to find a basis for W we first determine the set $S=\{X_1, X_2, X_3, X_4\}$ is linearly independent or not .if S linearly independent then S is basis for W . but S is not linearly independent (ch .)

$$X_1 - X_2 - 2 X_3 + 0 X_4 = 0 \text{.......................}(2)$$

then

$$X_2 = X_1 - 2 X_3 \qquad \text{......................}(3)$$

Substituting (3) in (1) every vector X in W is of the form

$(a+b) X_1 + (c-2b) X_3 + dX_4$ ....(4)

Thus W spanned by X_1, X_3, X_4 . we check the set $S=\{X_1, X_3, X_4\}$ is linearly independent or not.

We find that X_1, X_2, X_4 is linearly dependent and

$-3X_1 + 3X_3 + X_4 = 0$

Then $X_4 = 3X_1 - 3X_3$......... (5)

Substituting (5) in (4)) every vector X in W is linear combination of X_1, X_3 then W spanned by X_1, X_3

we check the set

$\{X_1, X_3\}$ is linearly independent or not.

$\{X_1, X_3\}$ is linearly independent and is basis for W.

Remark:To find a subset of $S = \{X_1, X_2, \ldots, X_n\}$ that is a basis for W=span S is as follows.

Step 1: first determine the set S is linearly independent or not
If S is linearly independent then S is basis for W.

Step 2:If is linearly dependent then there is constants $c_1, c_2 \ldots c_n$ not all zero, such that

$$c_1X_1 + c_2X_2, \ldots + c_nX_n = 0$$

we can then solve for one vectors in S as linear combination of other vectors ,suppose c_j is nonzero

let $S_1 = \{X_1, X_2, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n\}$ then S_1 spans W

now return Step 1 with S_1

Remark: If $\{X_1, X_2, \ldots, X_n\}$ is a basis for V ,then
$\{cX_1, X_2, \ldots, X_n\}$ is also basis if $c \neq 0$ thus a vector space always has infinitely many basis.

Theorem 3 :If $S = \{X_1, X_2, \ldots, X_n\}$ that is a basis for a vector space V and $T = \{Y_1, Y_2, Y_3, \ldots, Y_r\}$ is linearly independent set of vectors in V then $r \leq n$.

Proof: Let $T_1 = \{Y_1, X_1, X_2, \ldots, X_n\}$ since S span V ,So dose T_1 since Y_1 is linear combination of vectors of vector in S ,we find that T_1 Is linearly dependent then by theorem

[Let $S=\{X_1, X_2, \ldots, X_n\}$ be the set of non zero vectors in a vectors space V. then S is linearly dependent iff the vector X_j is a linear combination of the preceding vectors in S].

Some X_j is a linear combination of the preceding vectors in T_1.

Delete X_j let let $S_1=\{Y_1, X_1, X_2, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n\}$ not that S_1 spans V

Let $T_2=\{Y_2, Y_1, X_1, X_2, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n\}$ then T_2 is linearly dependent and some vector in T_2 is a linear combination of the preceding vectors in T_2 .since T is linearly independent this vector can not be Y_1, so it is X_i, $i \neq j$.

repeat this process over and over

If the X vectors are all eliminated before we ran out Y vector then the resulting set of Y vectors is subset of T is linearly dependent then T is linearly dependent a contradiction thus the number r of Y must be less then n.

Corollary4: If $S=\{X_1, X_2, \ldots, X_n\}$ and $T=\{Y_1, Y_2, Y_3, \ldots Y_r\}$ are bases for a vector space V then $r=n$.

Proof: Since T is linearly independent set of vectors by above theorem implies that $r \leq n$

Similarly $n \leq r$ hence $n=r$.

Definition :The dimension of nonzero vector space V is the number of vectors in a basis for V .

We write dimV for The dimension V

Remarks:

1-Since $\{0\}$ is linearly dependent then $\dim(\{0\})=0$

2- The dimension of R^2 is 2 and $\dim(R^4)=4$.

3- $\dim(P_n)=n+1$

4-V is called finite dimensional vector space if the dimension of V is finite number .

5- $C(-\infty,\infty)$ set of continuous function is subspace of the vector space of all real –valued function on R are not finite dimension .

Theorem5: If S is linearly independent set of vectors in finite dimensional vector space V there is basis T for V which contains S .

Proof :Ex. (like example)

This th. Says linearly independent set of vectors can be extended to basis for V .

Example :Suppose that we wish to find a basis for R^4 that contains vectors $X_1=(1,0,1,0)$, $X_2=(-1,1,-1,0)$
We use theorem 5 as follows
Let { E_1 , E_2,E_3 ,E_4 } be natural basis for R^4 where
$E_1=(1,0,0,0)$, $E_2=(0,1,0,0)$, $E_3=(0,0,1,0)$, $E_4=(0,0,0,1)$
Let S_1 = {X_1 ,X_2 , E_1 , E_2, E_3 ,E_4} since S_1 spans R^4 by theorem 2[Let S = {X_1 ,X_2 , ……. , X_n } set of non-Zero vector space V and let W=space V then some subset of S is basis for W] contains a basis for R^4
The basis is obtained by deleting every vector that is linear combination of the preceding vectors
We check if E_1 is linear combination of X_1 ,X_2
The answer is no
Then we retain E_1
We check if E_2 is linear combination of X_1 ,X_2 , E_1
The answer is yes , Then we delete E_2
We check if E_3 is linear combination of X_1 ,X_2 , E_1
The answer is yes , Then we delete E_3
We check if E_4 is linear combination of X_1 ,X_2 , E_1
The answer is no
Then we retain E_4 the basis is {X_1 ,X_2 , E_1 , E_4 }

Theorem 6:Let V be n- dimensional vector space and let
S = {X_1 ,X_2 , ……. , X_n } set of n vectors in V

a) If S is linearly independent then it is basis for V .
b) If S is spans then it is basis for V .

proof :a)Let $S = \{X_1, X_2, \ldots\ldots, X_n\}$ is linearly independent set of n vectors in V .if X any vector in V the set $\{X, X_1, X_2, \ldots\ldots, X_n\}$ islinearly dependent since the maximal number of linearly independent vectors in V is n .thus there is linearly dependent relation

$cX + c_1X_1 + c_2X_2, \ldots\ldots + c_nX_n = 0$

if c=0 a contradiction to linear independence of $X_1, X_2, \ldots\ldots, X_n$

then $c \neq 0$ and then $X = \left(\frac{-c_1}{c}\right)X_1 + \ldots\ldots\ldots + \left(\frac{-c_{n-1}}{c}\right)X_{n-1}$

then S is basis for V .
b) suppose S is spans for V .
by[theorem 2:Let $S = \{X_1, X_2, \ldots, X_n\}$ set of non-Zero vector space V and let W=space V then some subset of S is basis for W .]
but this subset must contain n vectors
thus S is basis for V .

To find a basis for the solution space of homogeneous system AX=0 we do the following :
1-solve the homogeneous system by gauss-Jordan redaction
2- write the solution as linear combination of vectors ,using arbitrary constants as coefficients ,the set of vectors S thus determined spans the solution space W .
3-find a subset of S a basis for W.

Example: find a basis for the solution space of homogeneous system

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 3 & 0 & 3 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 3 & 5 & 0 & 6 & 2 \\ 2 & 3 & 2 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution : solve the homogeneous system by gauss-Jordan redaction

$$\begin{bmatrix} 1 & 0 & 0 & -3 & -1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Every solution is of the form

$$X=\begin{bmatrix} 3s+t \\ -3s-t \\ -s-\frac{1}{2}t \\ s \\ t \end{bmatrix} \text{.........}(*)$$ where s and t are any real number

Every vector in W is a solution and is then of the form given by (*)
Then

$$X= s\begin{bmatrix} 3 \\ -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} +t\begin{bmatrix} 1 \\ -1 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \text{.......}(**)$$

Let s=1 , t=0And then s=0 ,t=1

$$X_1=\begin{bmatrix} 3 \\ -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, X_2=\begin{bmatrix} 1 \\ -1 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

That is X_1 ,X_2 belong to W from(**) we see $\{X_1, X_2\}$ spans W
Since $S=\{X_1, X_2\}$ linearly independent it is a basis for Wand the dimW=2.

Exercises:

1-Is the polynomials t^2+1 , t^2-t+1 is a basis for P_2?

2-Find a basis for subspace W of R^3 spans by

$X_1=(1,2,2)$, $X_2=(3,2,1)$, $X_3=(11,10,7)$, $X_4 = (7,6,4)$ what dimW?

3-Find a basis for subspace W of R^3 such that every vector is (a,b,c) and b= a+c what dimW?

4-Find a basis for the solution space of homogeneous system

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 & 2 \\ 2 & 4 & 3 & 3 & 3 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

What is the dimension ?

5-Proof if dim V =n then any set of (n-1) vectors can not spans V.

6- proof if W is subspace of n- dimensional vector space V and dimW=dim V then W=V .

The Rank of matrixand applications

In the following lecture we obtain a good method for find a basis for vector space V spanned by given set of vectors

Definition:

Let $A=\begin{bmatrix} a_{11} & a_{21} & \cdot & \cdot & a_{12} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \\ a_{m1} & a_{m2} & \cdot & \cdot & a_{mn} \end{bmatrix}$ be an $m \times n$ matrix

The rows of A $X_1 = (a_{11}, a_{12}, \ldots\ldots, a_{1n})$

$X_2 = (a_{21}, a_{22}, \ldots\ldots, a_{2n})$

....

....

$X_m = (a_{m1}, a_{m2}, \ldots\ldots\ldots, a_{mn})$

considered as vectors in R^n , span a subspace of R^n called the row space also columns of A , considered as vectors in R^m called the column space of A.

Theorem: If A and B are two $m \times n$ row equivalent matrices then the row space of A and B are equivalent.

Proof :Ex.

Example :
Let V be the subspace of R^5 spanned by
S={(1,-2,0,3,- 4),(3,2,8,1,4),(2,3,7,2,3),(-1,2,0,4,-3)} Find a basis for V

Solution :

Let V be the row space of the matrix $A = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix}$

Now A is row equivalent to the following matrix B in reduced row echelon.

From $B = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ the row spaces of A and **B** are identical and a basis for the row space of B consists of nonzero of **B then**
(1, 0,2,0, 1) , (**0**, 1, 1,0, 1),(0,0,0,1,-1) a basis for V **.**

Definition :The dimension of the row space of A is called the row rank of A and the dimension of the column space of A is called the column rank of A

.

Example :In example (1) the row rank of A is 3.

Example :Find the column rank of A In example (1) .

Sol: We must find the dimension of the column space of A , that is We must find the dimension of the subspace of R^4 spanned by column of A. if we write columns of A as row vectors ;we get :

$$A^T = \begin{bmatrix} 1 & 3 & 2 & -1 \\ -2 & 2 & 3 & 2 \\ 0 & 8 & 7 & 0 \\ 3 & 1 & 2 & 4 \\ -4 & 4 & 3 & -3 \end{bmatrix}$$ Transform A^T to reduced row echelon form then

$$C = \begin{bmatrix} 1 & 0 & 0 & \frac{11}{24} \\ 0 & 1 & 0 & \frac{-49}{24} \\ 0 & 0 & 1 & \frac{7}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ thus the vectors $(1,0,0,\frac{11}{24})$,$(0,1,0,\frac{-49}{24})$,$(0,0,1,\frac{7}{3})$

Form a basis for the row space of A^T . Then the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \frac{11}{24} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{-49}{24} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{7}{3} \end{bmatrix}$$ form a basis for column space of A

And the column rank of A is 3.

Example : Let $S = \{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix} \}$ and let V be the subspace of R^3 given by V =spans S for V

Sol: step I

Let A be the matrix whose column the given vectors in S

$$A^{t} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \\ 1 & -3 & 3 \\ 3 & -5 & 1 \\ 1 & -4 & 5 \end{bmatrix}$$

Step 2: Transform A^{t} to reduced row echelon form then

$$B^{t} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Step3: the nonzero vectors of B^{t} written as columns

$\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ form a basis for V .

Observe that the row and the column rank of A are equal.

Theorem : The row rank and the column rank of $m \times n$ matrix A are equal .

Proof: Let $X_1, X_2, \ldots, X_n$ be row vectors of A where

$$A=\begin{bmatrix} a_{11} & a_{21} & . & . & a_{12} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix}$$ where $X_i = [a_{i1}, a_{i2}, \ldots\ldots, a_{in}]$ $1 \le i \le m$

Let row rank $A = k$ and the set of vectors $\{Y_1, Y_2, Y_3, \ldots Y_k\}$ form a basis for the row space of A where

$Y_i = [b_{i1}, b_{i2}, \ldots\ldots\ldots, b_{in}]$

Now each of the row vectors is linear combination of $Y_1, Y_2, Y_3, \ldots Y_k$ (basis)

That means that

$X_1 = r_{11} Y_1 + r_{12} Y_2 + \ldots\ldots\ldots + r_{1k} Y_k$

$X_2 = r_{21} Y_1 + r_{22} Y_2 + \ldots\ldots\ldots + r_{2k} Y_k$

....

....

$X_m = r_{m1} Y_1 + r_{m2} Y_2 \ldots\ldots\ldots + r_{mk} Y_k$

Where the r_{ij} are uniquely determined real numbers, then

$a_{1j} = r_{11} b_{1j} + r_{12} b_{2j} + \ldots\ldots\ldots + r_{1k} b_{kj}$

$a_{2j} = r_{21} b_{1j} + r_{22} b_{2j} + \ldots\ldots\ldots + r_{2k} b_{kj}$

....

....

$a_{mj} = r_{m1} b_{1j} + r_{m2} b_{2j} \ldots\ldots\ldots + r_{mk} b_{kj}$

Or $$\begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ a_{mj} \end{bmatrix} = b_{1j} \begin{bmatrix} r_{11} \\ r_{21} \\ . \\ . \\ r_{m1} \end{bmatrix} + b_{2j} \begin{bmatrix} r_{12} \\ r_{22} \\ . \\ . \\ r_{m2} \end{bmatrix} + \ldots\ldots + b_{kj} \begin{bmatrix} r_{1k} \\ r_{2k} \\ . \\ . \\ r_{mk} \end{bmatrix}$$

For $j = 1, 2, \ldots, n$

Since every column of A is linear combination of k vectors the dimension of the column space of A is at most **k.**

Or column rank $A \le$ row rank A

Similarly , we get row rank $\leq$ column rank A .hence row rank and the column rank of A are equal .

Definition:-we refer to the rank of A by rank A=The number of non zero rows of B[where B is reduced row echelon form A].

Theorem: The $n \times n$ matrix is nonsingular if and only if rank A = n

Proof :Suppose that A is nonsingular then A is row equivalent to I_n.
So rank A = n .
Conversely ,if rankA = n ,then A is row equivalent to I_n.,then A nonsingular.

Corollary :If A is $n \times n$ matrix then rank A = n if and only if $|A| \neq 0$
Proof : (H.W).

Corollary :Let $S = \{X_1, X_2, \dots, X_n\}$ set of non-Zero vectors in R^n and let A be the matrix whose rows (columns) are the vectors in S . linearly independent iff $|A| \neq 0$.
Proof : (H.W).

Corollary :The homogeneous system AX=0 of linear equation has a nontrivial solution if and only if rank A < n.

Proof :The system AX=0 has a nontrivial solution then A is singular.
By corollary 1 $|A| = 0$ then rank A< n
Conversely , since rank A < n then by corollary 1 $|A| = 0$
Thus AX=0 of linear equation has a nontrivial solution.

Example: let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 1 & 3 \end{bmatrix}$ Transform A to reduced row echelon form B

We find that $B = I_3$ thus rank A = 3 and A is nonsingular. And The system AX=0 has a nontrivial solution.

Example: let $A=\begin{bmatrix}1 & 2 & 0\\1 & 1 & -3\\1 & 3 & 3\end{bmatrix}$ then A is row equivalent to $\begin{bmatrix}1 & 0 & -6\\0 & 1 & 3\\0 & 0 & 0\end{bmatrix}$ hence rank A < 3 and A is singular , And The system AX=0 has a nontrivial solution.

Remark: The following statements are equivalent for $n\times n$ matrix A

1- A is nonsingular.
2- AX=0 has only the trivial solution.
3-A is is row equivalent to I_n.
4- $|A|\neq 0$
5- rankA $= n$
6-the rows(columns) of A form linearly independent set of n vectors in R^n.

The rank of linear system AX=B , Where

$$A=\begin{bmatrix}a_{11} & a_{21} & . & . & a_{12}\\a_{21} & a_{22} & . & . & a_{2n}\\. & . & . & . & \\. & . & . & . & \\a_{m1} & a_{m2} & . & . & a_{mn}\end{bmatrix}$$

. Thus AX=B has solution iff B is a linear combination of the columns of A ,then rank A=rank [A:B]

Exercises:

1-Find the rank of A where $A=\begin{bmatrix}1 & 2 & 1 & 3\\2 & 1 & -4 & -5\\1 & 1 & 0 & 0\\0 & 0 & 1 & 1\end{bmatrix}$

2- Find the rank of A where $A=\begin{bmatrix}1 & -2 & -1\\2 & -1 & 3\\7 & -8 & 3\end{bmatrix}$

3- If A is 3×4 matrix what is the maximum value of rank A.

4- If A is 4×6 matrix, proof that columns A form linearly dependent set

5- If A is 5×3 matrix, Proof that columns of A form linearly dependent set.

Linear transformation

Definition :

Let V and W be vector spaces. A **linear** transformation L of V in to W is a function $L: V \longrightarrow W$
assigning a unique vector L(x) in **W** to each x in V such that .

a - $L(x + y) = L(x) + L(y)$. for every x and y in V
b- $L(cx) = cL(x)$, for every x in V and every scalar c

Not:

If V=W the linear transformation
$L: V \longrightarrow W$ is also called a linear operator on V.

Example : Let $L: R^3 \longrightarrow R^2$ be defined by
$L(x, y, z) = (x, y)$.

To verify that L is linear transformation we let
$X = (x_1, y_1, z_1)$ and $y = (x_2, y_2, z_2)$
Than $L(x + y) = L((x_1, y_1, z_1) + (x_2, y_2, z_2))$
$= L(x_1+x_2, y_1+y_2, z_1+z_2) = (x_1+x_2, y_1+y_2)$
$= (x_1, y_1) + (x_2, y_2) = L(x) + L(y)$
Also if c is a real number .
Then
$L(cx) = L(cx_1, cy_1, cz_1) = (cx_1, cy_1) = c(x_1, y_1)$
$= c\,L(x)$

Example : Let $L: R^3 \longrightarrow R^3$ defined by