Discrete Structures

Lec.1

أ.م عهود سعدي الحسني

Mathematical logical

1. Statement (proposition)

A statement is a declarative sentence which is either true or false but not both. The statement is also known as proposition.

Proposition	Not proposition
Sun rises in the east.	X is a dog.
Two is less than five.	

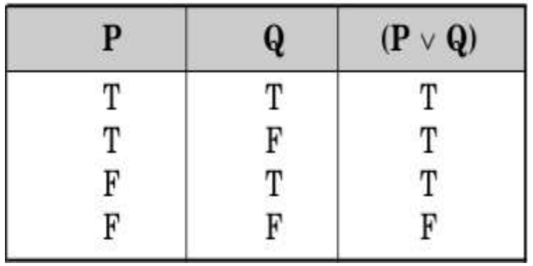
2. Logical connectives

Disjunction

Conjunction

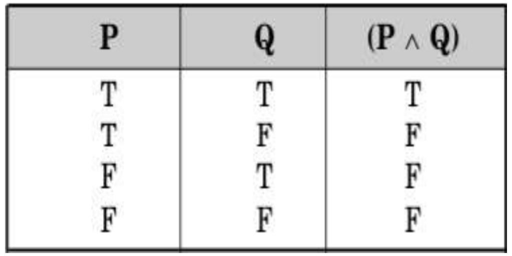
Negation

Truth Table (Disjunction)



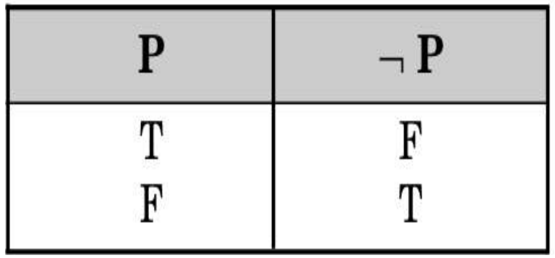
P	Q	(P ∨ Q)
T	T	T
T	F	T
F	T	T
F	F	F

Truth Table (Conjunction)



P	Q	(P ∧ Q)
T	T	T
T	F	F
F	T	F
F	F	F

Truth Table (Negation)



P	¬ P
T	F
F	T

Logical connectives

Negation

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Negation = **not** = ¬p , ~p , p'

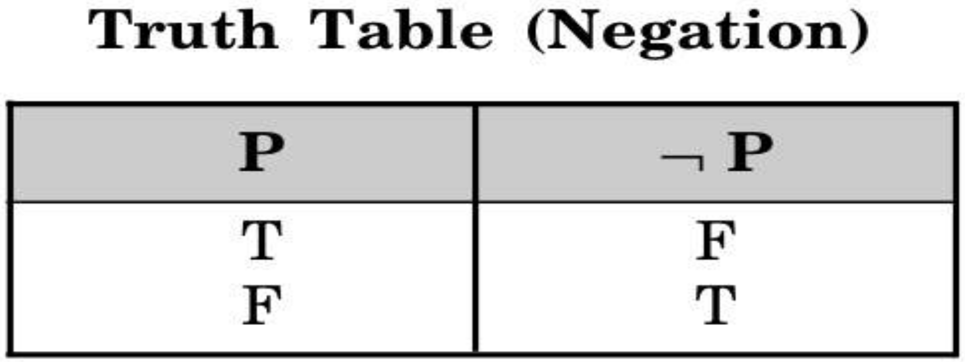
Consider the example of a statement.

P: London is the capital of Iraq.

¬p: London is not the capital of Iraq.

As we all know that Baghdad is the capital of Iraq , the truth value for the statements P is false (F) and ¬p is true (T).

Rule: **If P is true, then ¬p is false and if P is false, then ¬p is true**.



Truth Table (Negation)

P	**¬ P**
T	F
F	T

Logical connectives

Conjunction

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The conjunction of two statements P and Q is also a statement denoted by (P∧Q). We use the connective **And** for conjunction.

Consider the example where P and Q are two statements.

P:2+3=5

Q : 5 is a prime number.

(P ∧Q) : 2 + 3 = 5 and 5 is a prime number.

Rule: (P ∧ Q) is true if both P and Q are true, otherwise false.

Truth Table (Conjunction)

P	Q	(P ∧ Q)
T	T	T
T	F	F
F	T	F
F	F	F

Logical connectives

Disjunction

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The disjunction of two statements P and Q is also a statement denoted by (P ∨ Q). We use the connective **Or** for disjunction.

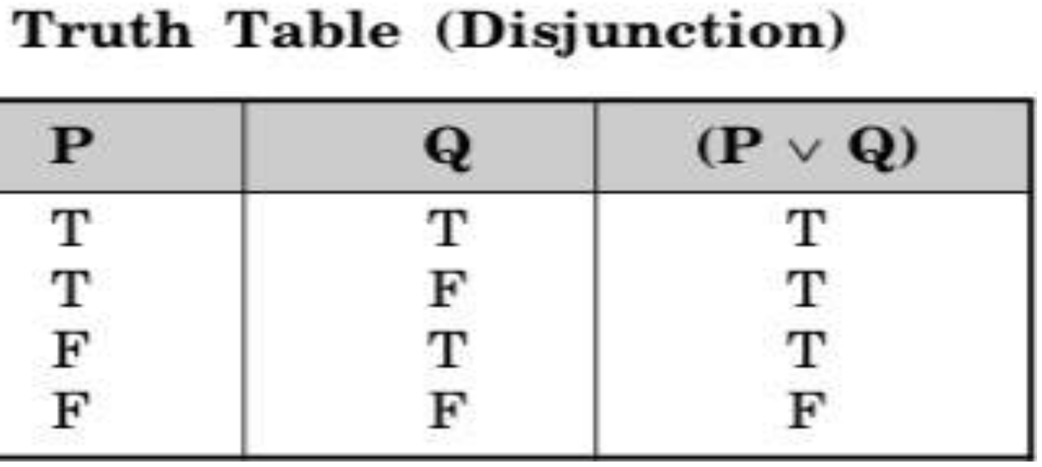
Consider the example where P and Q are two statements

P: 2+3 is not equal to 5

Q : 5 is a prime number

(P ∨ Q) : 2 + 3 is not equal to 5 or 5 is a prime number.

Rule: (P ∨ Q) is true, if either P or Q is true and it is false when both P and Q are false.



Truth Table (Disjunction)

P	Q	$(P \vee Q)$
T	T	T
T	F	T
F	T	T
F	F	F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Discrete mathematics

Lec.2

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3.conditional

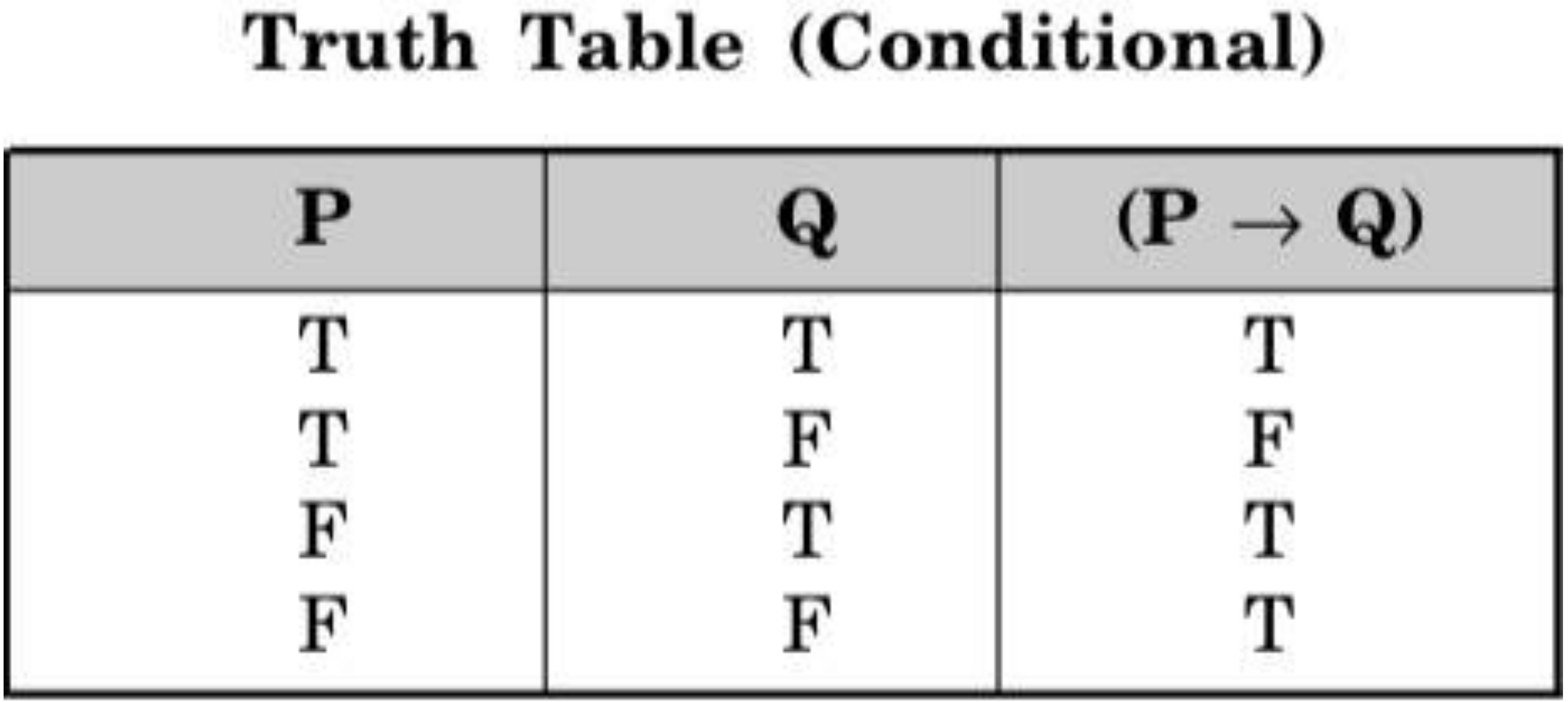
Let P and Q be any two statements. Then the statement P —> Q is called a conditional statement. This can be put in any one of the following forms.

(a) If P , then Q (b) P only if Q

(C) P implies Q (d) Q if P

Ex) "I will take you boating on Sunday, if it is not raining".

Rule: **An implication (conditional) P —> Q is false only when the hypothesis (P) is true and conclusion (Q) is false, otherwise true.**



Truth Table (Conditional)

P	Q	(P → Q)
T	T	T
T	F	F
F	T	T
F	F	T

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4.Bi-conditional

Let P and Q be any two statements. Then the statement P <—>Q is called a bi-conditional statement. This P <—> Q can be put in any one of the following forms.

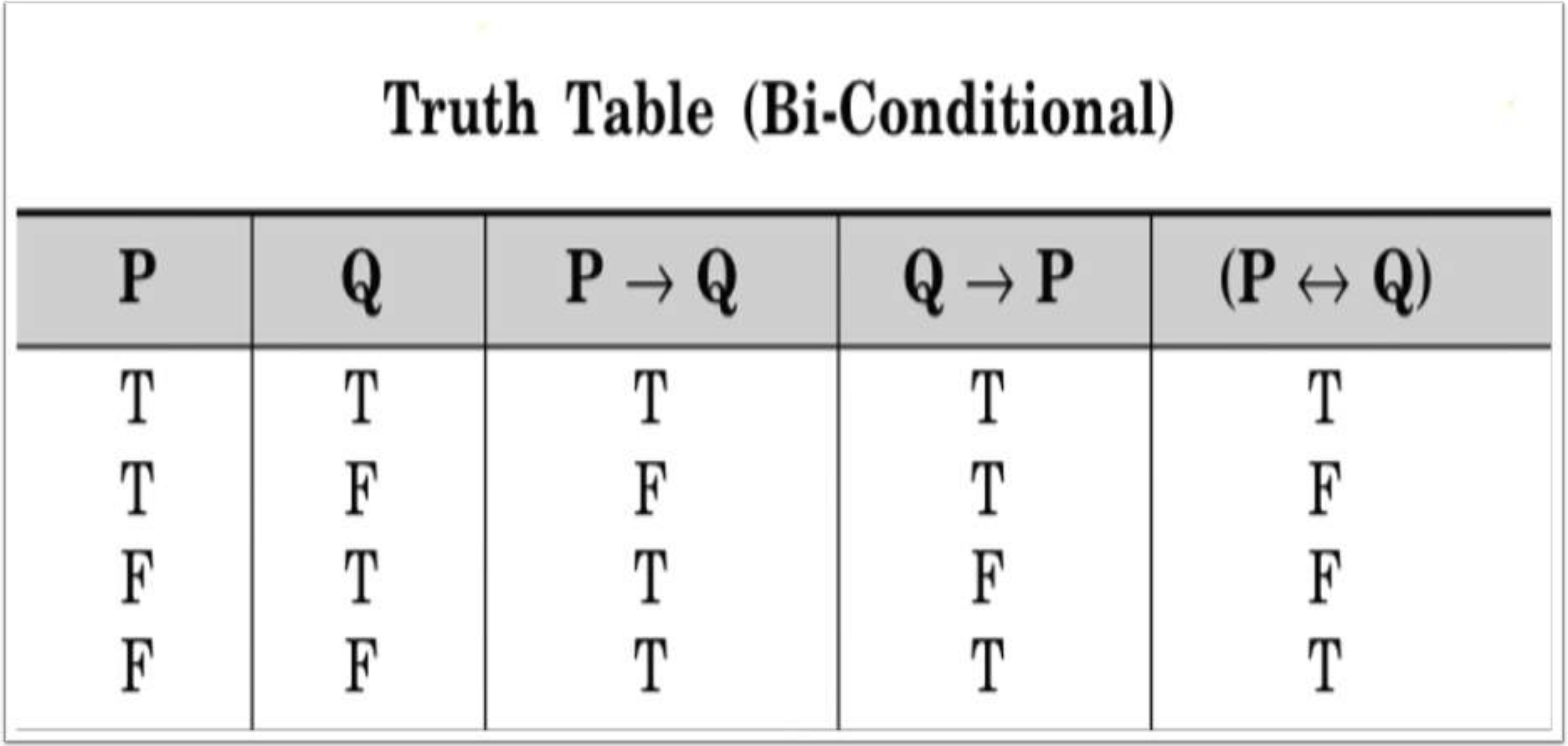
(a) P if and only if Q

(b) P is necessary and sufficient of Q

(c) P is necessary and sufficient for Q

(d) P is implies and implied by Q

Rule: **(P<—>Q) is true only when both P and Q have identical truth values, otherwise false.**



Truth Table (Bi-Conditional)

P	Q	P → Q	Q → P	(P ↔ Q)
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

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5.converse

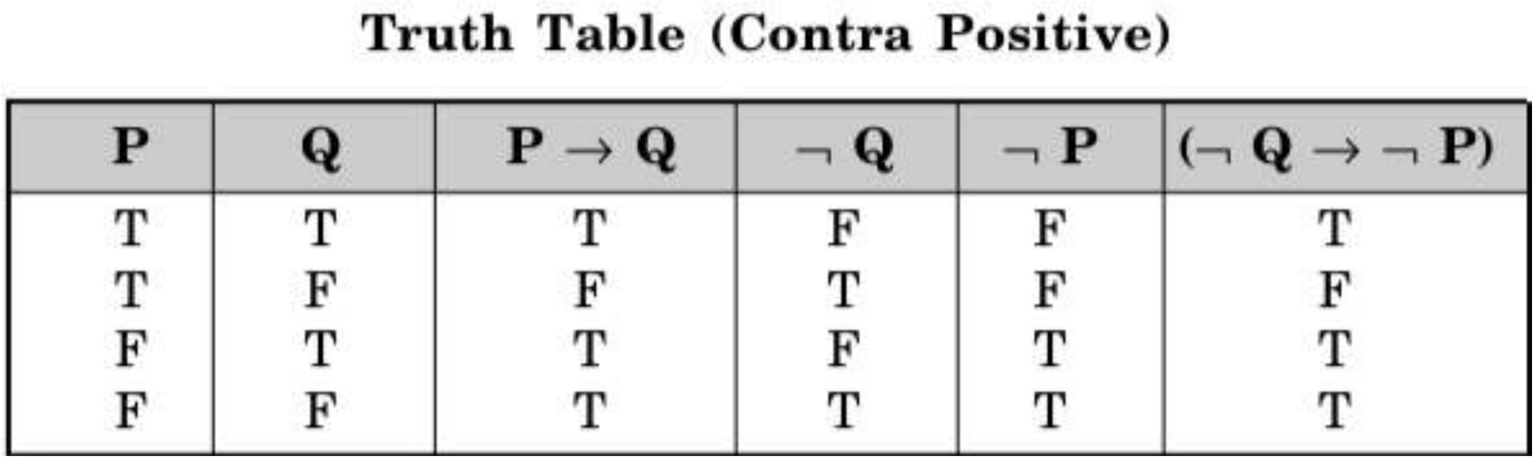
Let P and Q be any two statements. The converse statement of the conditional P —> Q is given as Q —> P.

6.Inverse

Let P and Q be any two statements. The inverse statement of the conditional (P —>Q) is given as (¬p —> ¬Q).

7.Contra positive

Let P and Q be any two statements. The contra positive statement of the conditional (P —>Q) is given As (¬Q —> ¬P).



Truth Table (Contra Positive)

P	Q	$P \rightarrow Q$	$\neg Q$	$\neg P$	$(\neg Q \rightarrow \neg P)$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Rule: From the truth table it is observed that both conditional (P —>Q) and contra positive(¬Q —> ¬P) have same truth values.

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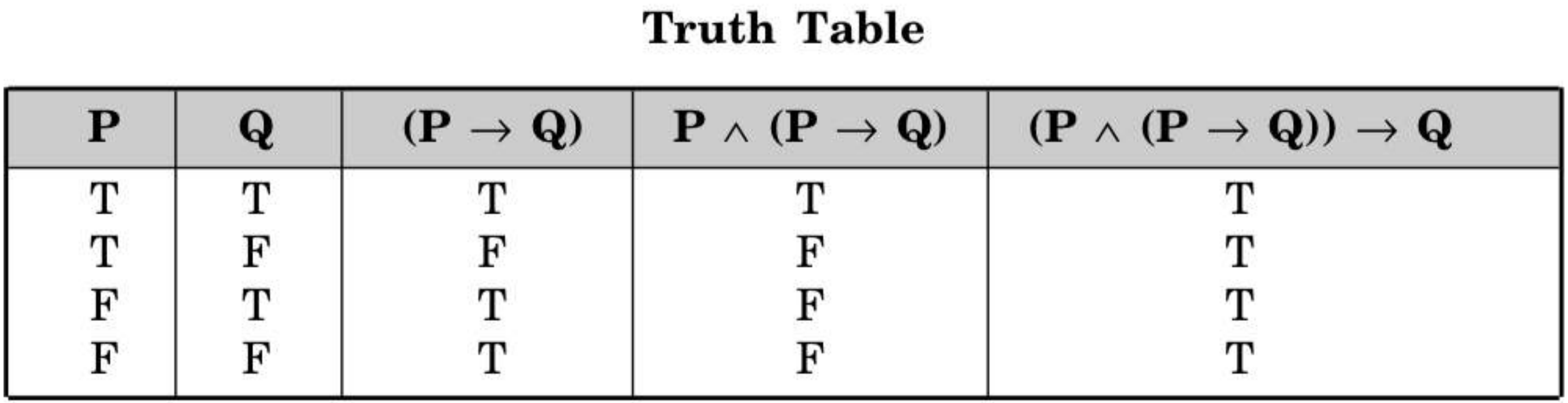
8. Tautology

If the truth values of a composite statement are always true.

So, $(P \wedge (P \rightarrow Q)) \rightarrow Q$ is a tautology.



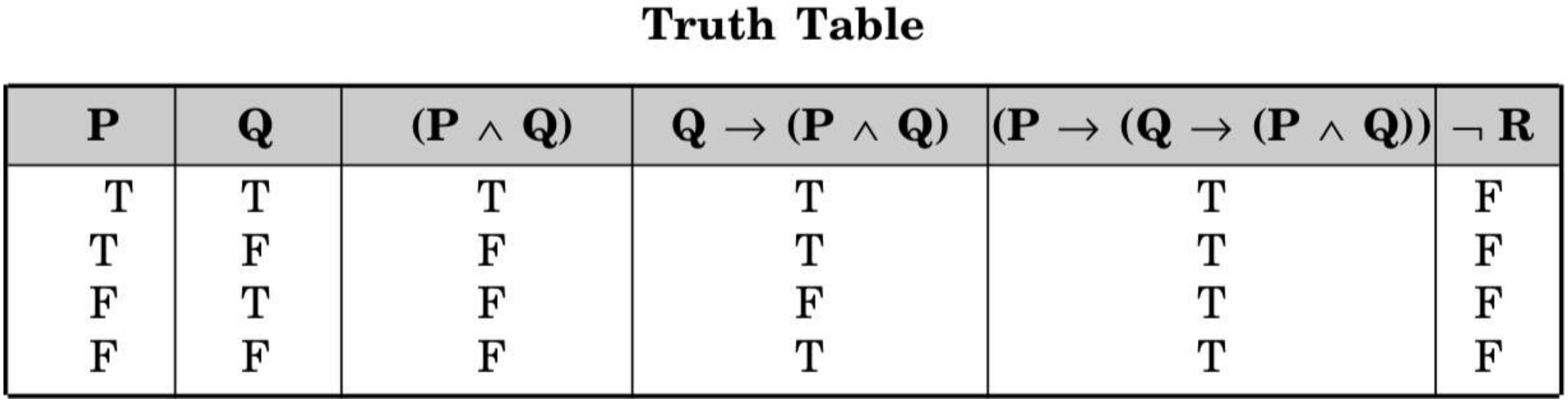
Truth Table

P	Q	$(P \rightarrow Q)$	$P \wedge (P \rightarrow Q)$	$(P \wedge (P \rightarrow Q)) \rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

9. Contradiction

If the truth values of a composite statement are always false.

So, $\neg R \equiv \neg (P \to (Q \to (P \wedge Q)))$ is a contradiction.



Truth Table

P	Q	$(P \wedge Q)$	$Q \to (P \wedge Q)$	$(P \to (Q \to (P \wedge Q))$	$\neg R$
T	T	T	T	T	F
T	F	F	T	T	F
F	T	F	F	T	F
F	F	F	T	T	F

10. Algebra of propositions

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Commutative Laws:

$P \wedge Q \equiv Q \wedge P$ and
$P \vee Q \equiv Q \vee P$

Associative Laws:

$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$ and
$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$

Distributive Laws:

$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ and
$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

Idempotent Laws:

$P \wedge P \equiv P$ and
$P \vee P \equiv P$

Absorption Laws:

$P \vee (P \wedge Q) \equiv P$ and
$P \wedge (P \vee Q) \equiv P$

11. Duality law

Two formulae P and P* are said to be duals of each other if either one can be obtained from the other by interchanging ∧ by ∨ and ∨ by ∧ . The two connectives ∧and∨are called dual to each other.

Consider the formulae

$P \equiv (P \vee Q) \wedge R$ and $P^* \equiv (P \wedge Q) \vee R$

Which are dual to each other.

12.de morgan’s laws

If P and Q be two statements, then

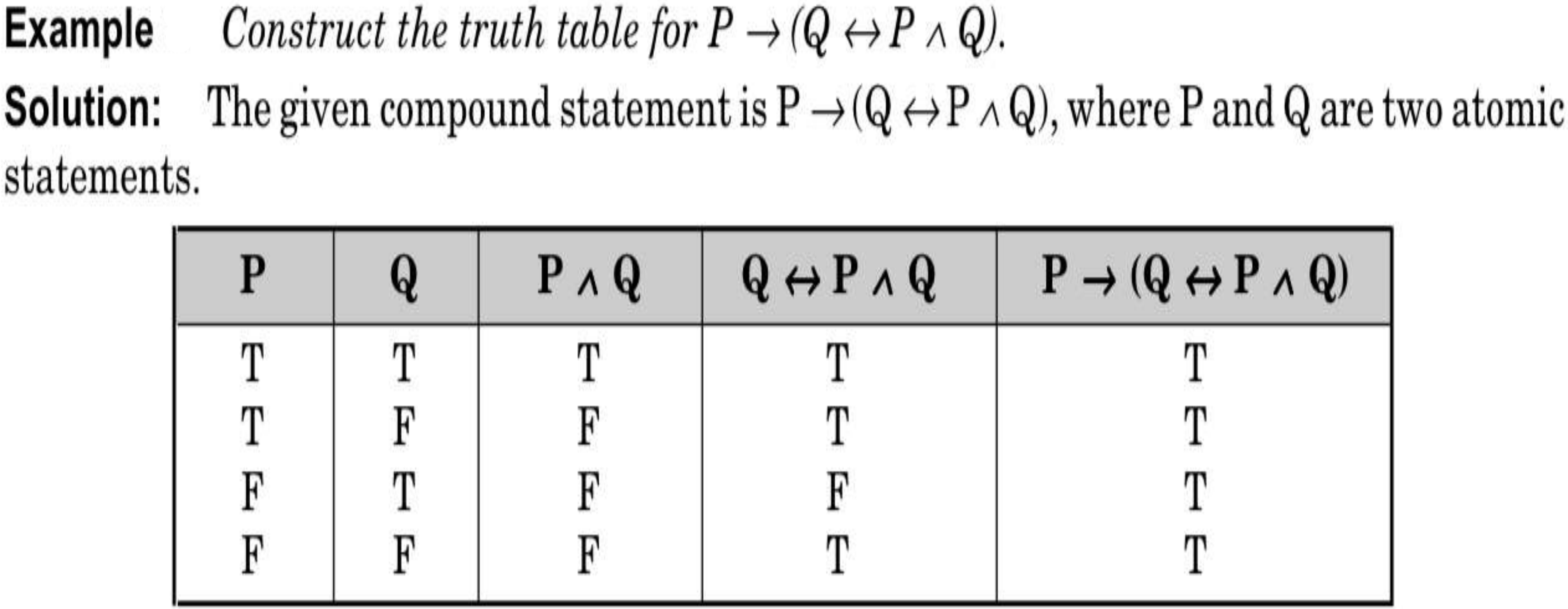
$(i)\ \neg(P \wedge Q) \Leftrightarrow (\neg P) \vee (\neg Q)$ and

$(ii)\ \neg(P \vee Q) \Leftrightarrow (\neg P) \wedge (\neg Q)$

Example *Construct the truth table for* $(P \to Q) \leftrightarrow (\neg P \vee Q)$.

Solution: The given compound statement is $(P \to Q) \leftrightarrow (\neg P \vee Q)$ where P and Q are two atomic statements.

P	Q	$\neg P$	$P \to Q$	$\neg P \vee Q$	$(P \to Q) \leftrightarrow (\neg P \vee Q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T



Example *Construct the truth table for $P \to (Q \leftrightarrow P \wedge Q)$.*

Solution: The given compound statement is $P \to (Q \leftrightarrow P \wedge Q)$, where P and Q are two atomic statements.

P	Q	$P \wedge Q$	$Q \leftrightarrow P \wedge Q$	$P \to (Q \leftrightarrow P \wedge Q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	F	F	T	T

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Discrete mathematics

LEC.3

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SETS

Collection of well defined objects is called a set. Well defined means distinct and distinguishable. The objects are called as elements of the set.

Ex: A = {a, b, c, d} and B = {b, a, d, c} are equal sets.

The symbol ∈ stands for 'belongs to’. X ∈ A means x is an element of the set A.

Sets

Set builder method

Tabular method

1. Tabular Method

Expressing the elements of a set within a parenthesis where the elements are separated by commas is known as tabular method, roster method or method of extension.

Consider the example:

$A = \{1, 3, 5, 7, 9, 11, 13, 15\}$

2. Set Builder Method

$$S = \{x \mid P(x)\}$$

where P(x) is the property that describes the elements of the set.

$A = \{x \mid x = 2n + 1; 0 \leq n \leq 7; n \in I\}$

$= \{1, 3, 5, 7, 9, 11, 13, 15\}$

Types of sets

- Finite set
- Infinite set
- Singleton set
- Pair set
- Empty set
- Set of sets
- Universal set

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1. **Finite set:** A set which contains finite number of elements is known as finite set. Consider the example of finite set as. A = {a, b, c, d, e}.
2. **Infinite set:** A set which contains infinite number of elements is known as infinite set. Consider the example of infinite set as
   N = {1, 2, 3, 4,…}.
   I = {…,- 3, - 2, - 1, 0, 1, 2, 3, ….}
3. **Singleton set:** A set which contains only one element is known as a singleton set. Consider the example. S = {9}
4. **Pair set:** A set which contains only two elements is known as a pair set.

   Consider the examples
   S = {e, f }.
   S = {{a}, {1, 3, 5}}

5. Empty set :A set which contains no element is known as empty set. The empty set is also known as void set or null set. denoted by ∅.

Consider the examples

(i) f={x : x ≠ x}

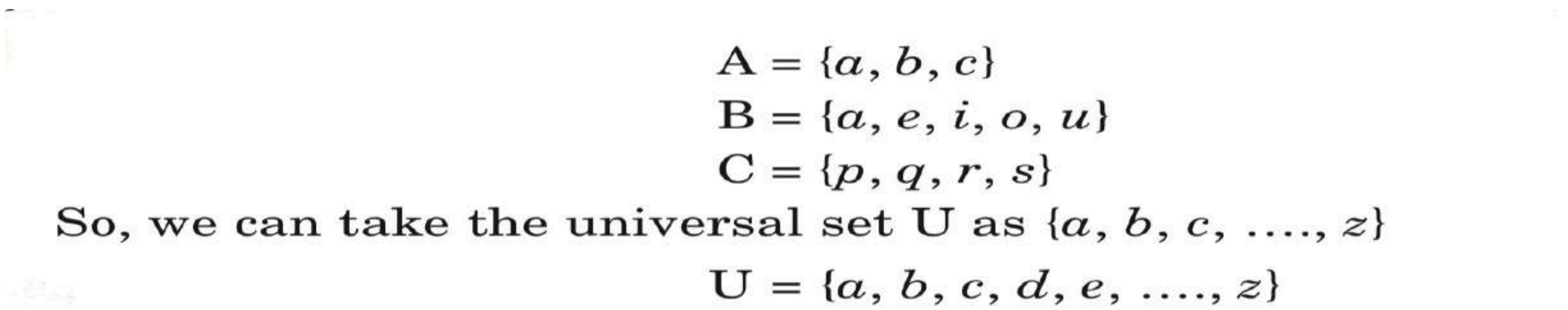
(ii) f = {x : x is a month of the year containing 368 days}6.

6.Set of sets:A set which contains sets is known as set of sets.

Consider the example A = {{a, b}, {1}, {1, 2, 3, 4}, {u, v}, {Book, Pen}}

7.Universal Set:A set which is superset of all the sets under consideration or particular discussion is known as universal set. denoted by U

lets



$$A = \{a, b, c\}$$
$$B = \{a, e, i, o, u\}$$
$$C = \{p, q, r, s\}$$

So, we can take the universal set U as $\{a, b, c, \ldots., z\}$

$$U = \{a, b, c, d, e, \ldots., z\}$$


Thank you!

Discrete Mathematics

LEC.4

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CARDINALITY OF A SET

If S be a set, then the number of elements present in the set S is known as cardinality of S and is denoted by |S|.

$$A = \{2, 4, 8, 16, 32, 64, 128, 256\}$$

$$|A| = 8$$

CARDINALITY OF A SET

Equivalent Sets

Equivalent Sets: Two sets A and B are said to be equivalent if they contains equal number of elements.

In other words A and B are said to be equivalent if they have same cardinality, i.e. $|A| = |B|$.

denoted as $A \approx B$.

SUBSET AND SUPERSET

Set A is said to be a subset of B or set B is said to be the superset of A if each element of A is also an element of the set B.

We write A$\subseteq$ B.

$i.e.$, $\mathrm{A} \subseteq \mathrm{B} \leftrightarrow \{x \in \mathrm{A} \rightarrow x \in \mathrm{B};\ \forall\, x \in \mathrm{A}\}$

Example

(i) Let $\mathrm{A} = \{1, 2, 3, 4, 5, 6\}$

$\mathrm{B} = \{1, 2, 3, 4, 5, 6, 7, 8\}$

So $\mathrm{A} \subseteq \mathrm{B}$.

(ii) Let $A = \{a, b, c\}$

$B = \{b, c, a\}$

so, $A \subseteq B$ and $B \subseteq A$.

(iii) Let $A = \{\ \}$ and $B = \{1, 2, 3\}$

So, $A \subseteq B$.

SUBSET AND SUPERSET

1. Equal Sets
2. Proper Subset

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Equal Sets: Two sets A and B are said to be equal if and only if every element of A is in B and every element of B is in A

i.e. $A \subseteq B$ and $B \subseteq A$. Mathematically

$$A = B \leftrightarrow \{ A \subseteq B \text{ and } B \subseteq A\}$$

i.e.,

$$A = B \leftrightarrow \{ x \in A \leftrightarrow x \in B\}$$

Consider the example: Let $A = \{x, y, z, p, q, r\}$

$$B = \{p, q, r, x, y, z\}$$

So, $B \subseteq A$ and $A \subseteq B$. Thus $A = B$.

SUBSET AND SUPERSET

1.Equal Sets

2. Proper Subset

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Proper Subset: Set A is said to be a proper subset of B if each element of A is also an element of B and set B has at least one element which is not an element of set A.

We write $A \subset B$

$$A \subset B \leftrightarrow \{x \in A \to x \in B \text{ and for at least one } y \in B \to y \notin A\}.$$

Consider an example

Let $A = \{a, b, c, d\}$

$B = \{a, b, c, d, e, f, g\}$

Here for $x \in A$ we have $x \in B$ and $y = e \in B$ such that $y = e \notin A$. Thus $A \subset B$.

Rule: 1. Every set is a subset of itself, *i.e.* $A \subseteq A$.

2. Empty set is a subset of every set, *i.e.* $\phi \subseteq A$.


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LEC.5

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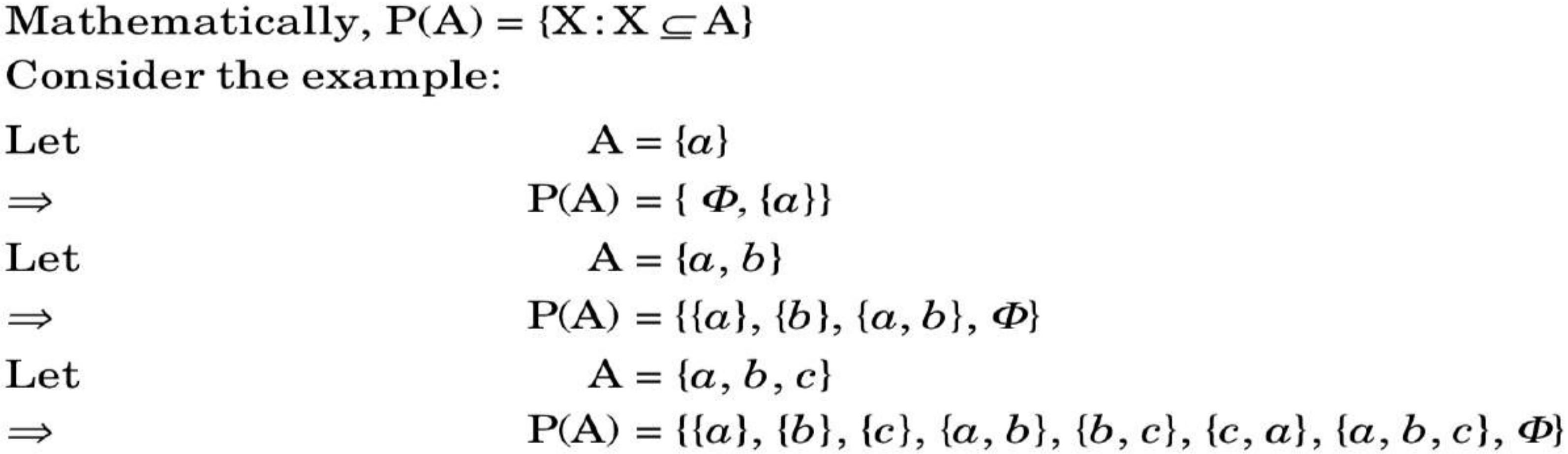
POWER SET

If A be a set, then the set of all subsets of A is known as power set of A and is denoted as P(A)



Mathematically, $P(A) = \{X : X \subseteq A\}$

Consider the example:

Let $A = \{a\}$

$\Rightarrow$ $P(A) = \{\ \Phi, \{a\}\}$

Let $A = \{a, b\}$

$\Rightarrow$ $P(A) = \{\{a\}, \{b\}, \{a, b\}, \Phi\}$

Let $A = \{a, b, c\}$

$\Rightarrow$ $P(A) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}, \Phi\}$

$$|A| = n \Rightarrow |P(A)| = 2^n.$$

Power set

1. Union
2. Intersection
3. Difference
4. Symmetric Difference
5. Complement of a set
6. Theorem

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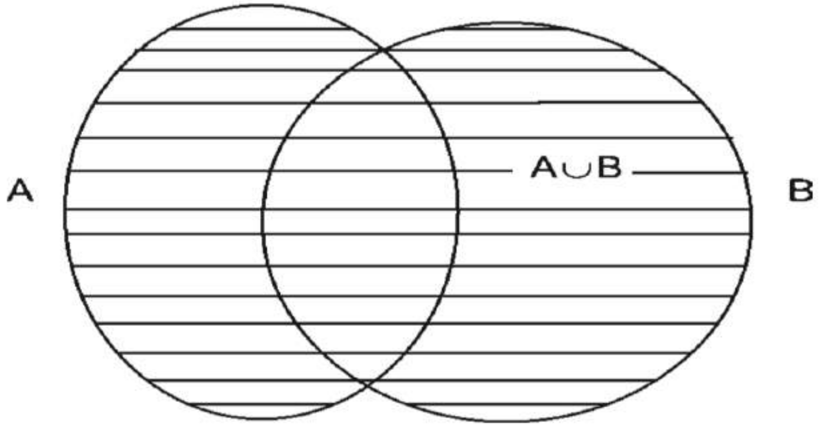
1. Union: If A and B be two sets, then the union ($A \cup B$) is defined as a set of all those elements which are either in A or in B or in both. $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Venn diagram



Consider the example:

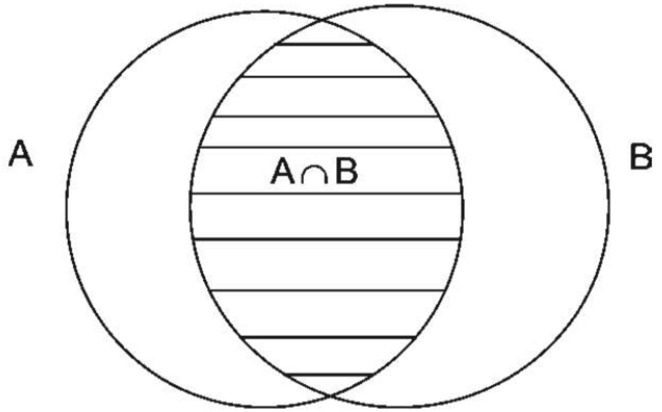
Let $A = \{a, b, c, d, e\}$

$B = \{a, e, i, o, u\}$

Therefore, $(A \cup B) = \{a, b, c, d, e, i, o, u\}$

2.Intersection: If A and B be two sets, then the intersection (A $\cap$ B) is defined as a set of all those elements which are common to both the sets. $(A \cap B) = \{x : x \in A \text{ and } x \in B\}$

Venn diagram



Consider the example:

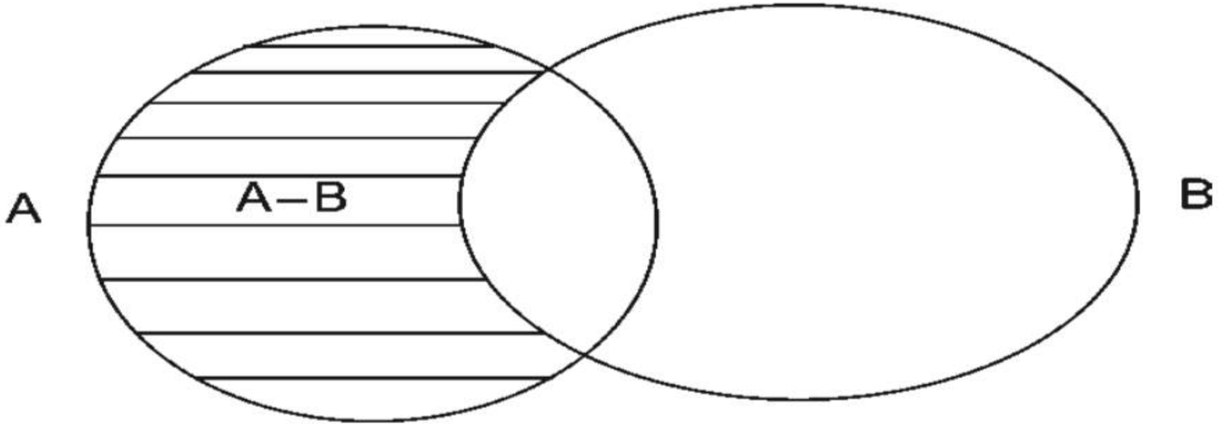
Let $A = \{a, b, c, d, e\}$

$B = \{a, e, i, o, u\}$

Therefore, $(A \cap B) = \{a, e\}$

3.Difference: If A and B be two sets, then the difference (A - B) is defined as a set of all those elements of A which are not in B.

$$(A - B) = \{x \mid x \in A \text{ and } x \notin B\}$$



Consider the example:

Let A = {a, b, c, d, e, f }

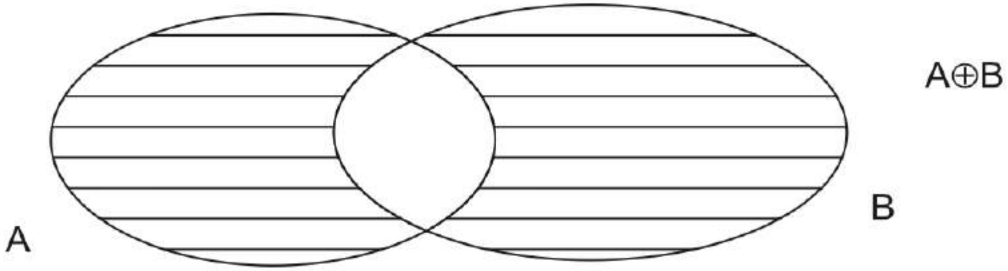
B = {a, c, i, o, u, k}

Therefore (A - B) = {b, d, e, f }

Symmetric Difference: If A and B be two sets, then the symmetric difference $(A \Delta B)$ or $(A \oplus B)$ is defined as a set of all those elements which are either in A or in B but not in both.

$$(A \oplus B) = (A - B) \cup (B - A)$$

Venn diagram



Consider the example:

Let $A = \{a, b, c, k, p, q, r, s\}$

$B = \{b, k, q, m, n, o, t\}$

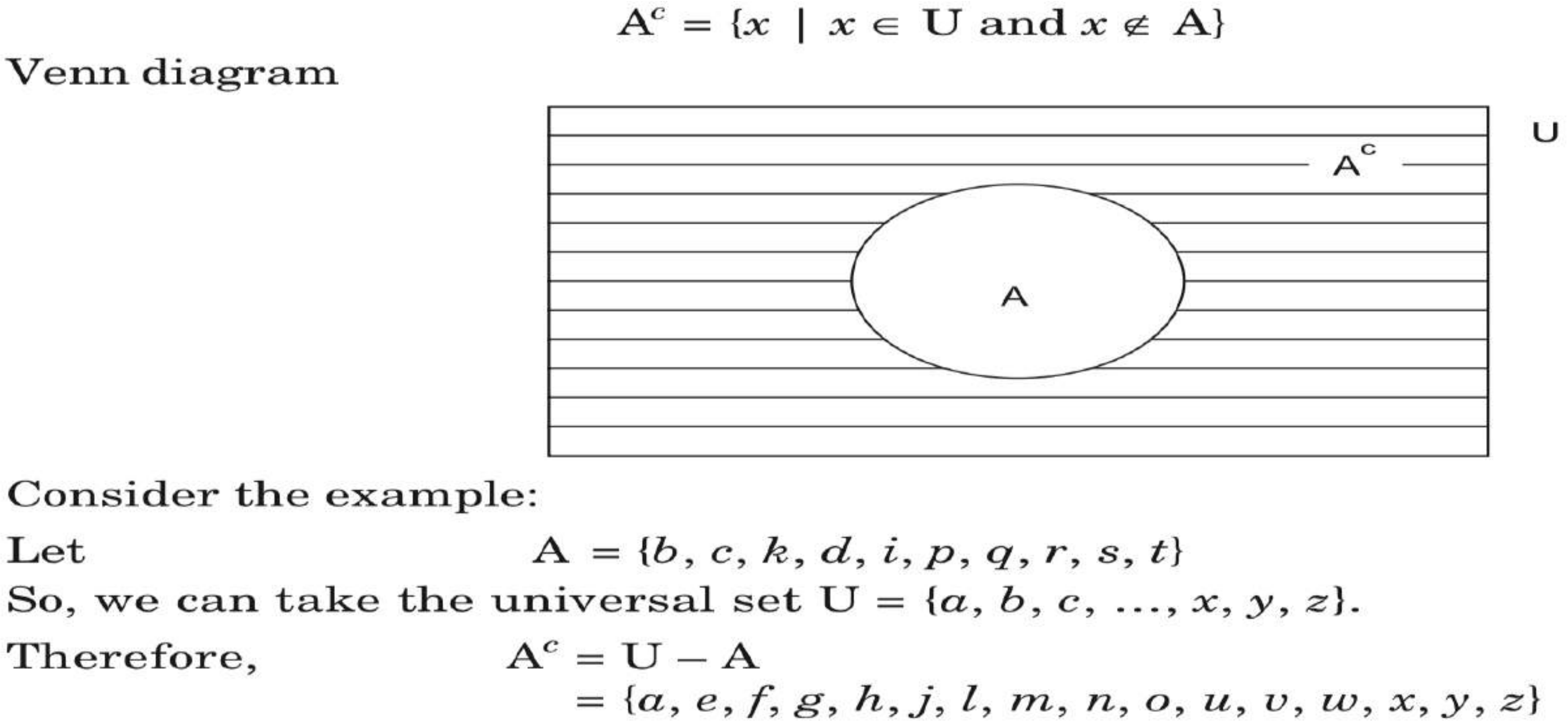
So, $(A - B) = \{a, c, p, r, s\}$

and $(B - A) = \{m, n, o, t\}$

Therefore, $(A \oplus B) = (A - B) \cup (B - A)$

$= \{a, c, p, r, s, m, n, o, t\}$

5.Complement of a Set: If A be a set , then the complement of A is given as A^c, A' or $\overline{A}$ and is defined as a set of all those elements of the universal set U which are not in A.



$$A^c = \{x \mid x \in U \text{ and } x \notin A\}$$

Venn diagram

Consider the example:

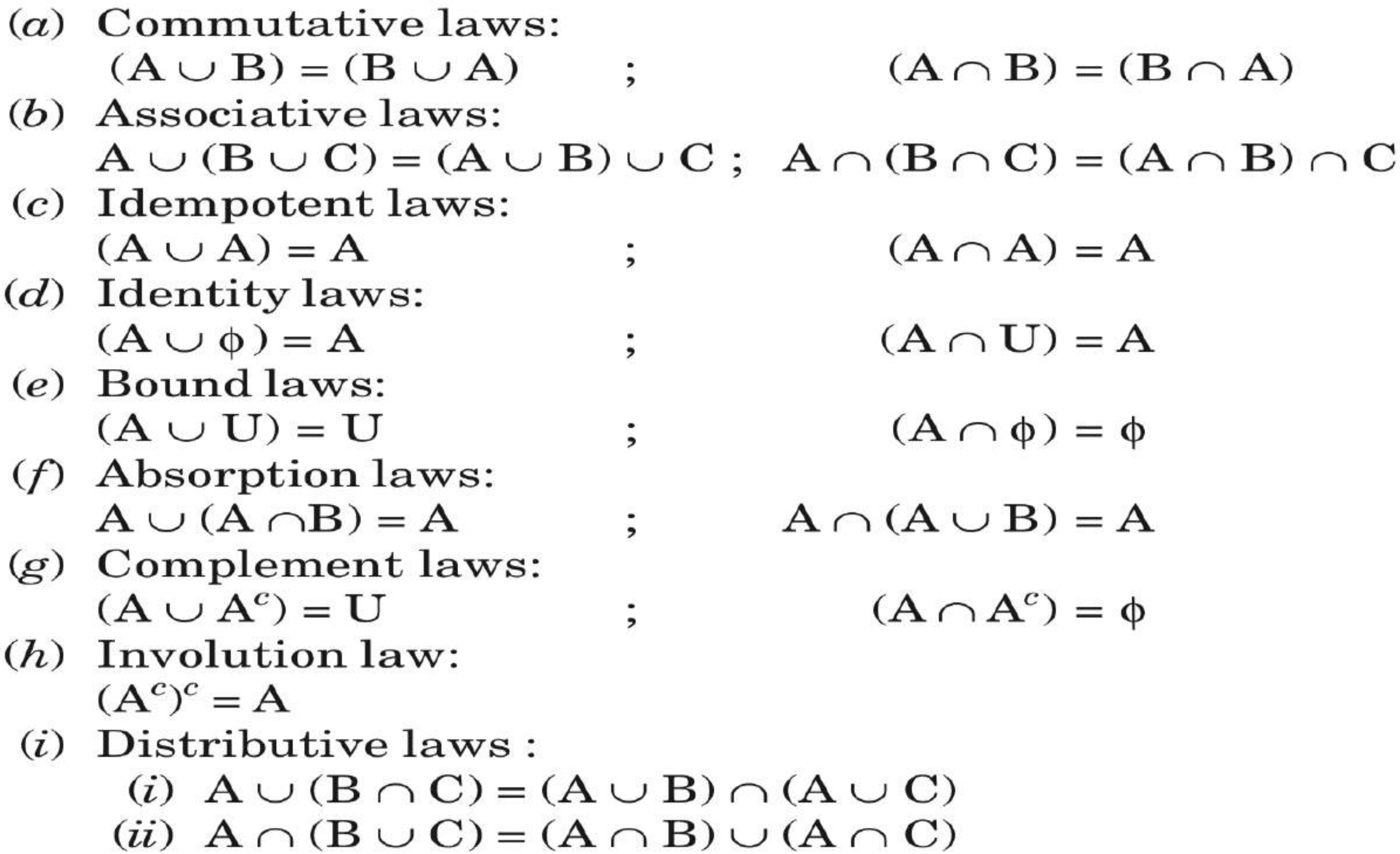
Let $A = \{b, c, k, d, i, p, q, r, s, t\}$

So, we can take the universal set $U = \{a, b, c, \ldots, x, y, z\}$.

Therefore, $A^c = U - A$

$= \{a, e, f, g, h, j, l, m, n, o, u, v, w, x, y, z\}$

6.Theorem: Let A, B and C be subsets of the universal set U. Then the following important laws hold.



(*a*) Commutative laws:

$(A \cup B) = (B \cup A)$; $(A \cap B) = (B \cap A)$

(*b*) Associative laws:

$A \cup (B \cup C) = (A \cup B) \cup C$; $A \cap (B \cap C) = (A \cap B) \cap C$

(*c*) Idempotent laws:

$(A \cup A) = A$; $(A \cap A) = A$

(*d*) Identity laws:

$(A \cup \phi) = A$; $(A \cap U) = A$

(*e*) Bound laws:

$(A \cup U) = U$; $(A \cap \phi) = \phi$

(*f*) Absorption laws:

$A \cup (A \cap B) = A$; $A \cap (A \cup B) = A$

(*g*) Complement laws:

$(A \cup A^c) = U$; $(A \cap A^c) = \phi$

(*h*) Involution law:

$(A^c)^c = A$

(*i*) Distributive laws :

(*i*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(*ii*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$


Thank you!

Discrete mathematics

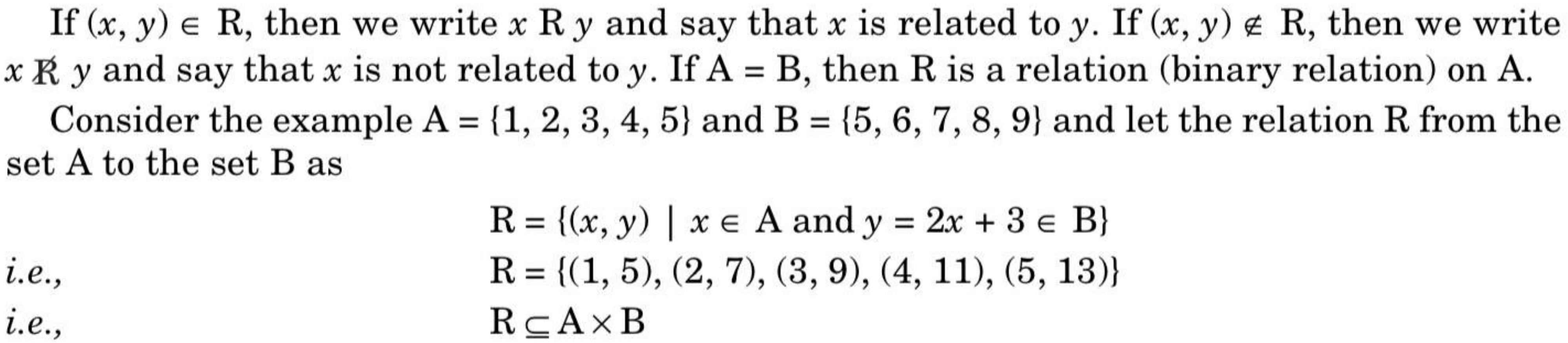
LEC.6

أ.م عهود سعدي الحسني

Binary Relation

Let A and B be two sets. Then any subset R of the Cartesian product (A x B) is a relation (binary relation) from the set A to the set B. Symbolically $R \subseteq (A \times B)$.

$$R = \{(x, y) \mid x \in A \text{ and } y \in B\}$$



If $(x, y) \in R$, then we write $x \, R \, y$ and say that x is related to y. If $(x, y) \notin R$, then we write $x \not R \, y$ and say that x is not related to y. If A = B, then R is a relation (binary relation) on A.

Consider the example A = {1, 2, 3, 4, 5} and B = {5, 6, 7, 8, 9} and let the relation R from the set A to the set B as

$$R = \{(x, y) \mid x \in A \text{ and } y = 2x + 3 \in B\}$$

i.e., $R = \{(1, 5), (2, 7), (3, 9), (4, 11), (5, 13)\}$

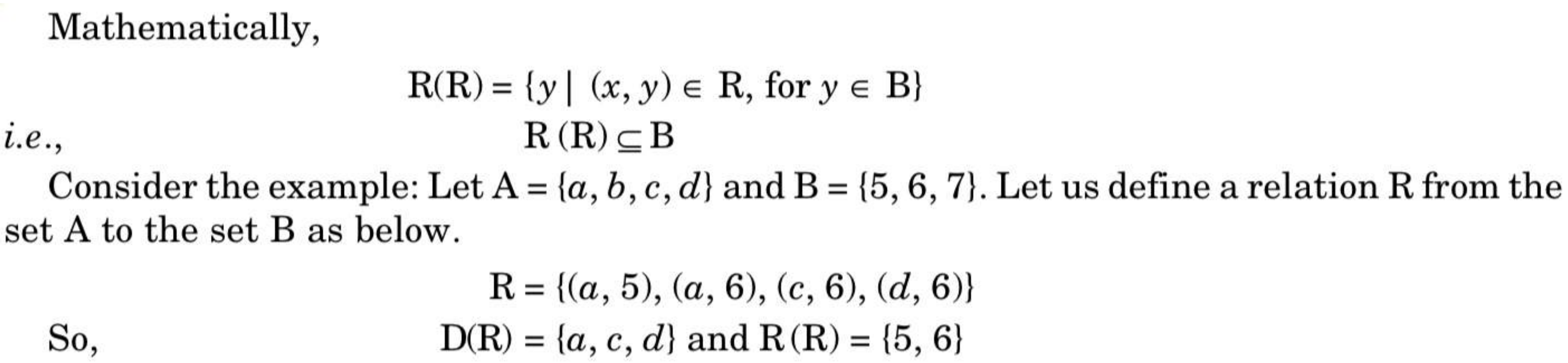
i.e., $R \subseteq A \times B$

1.Domain of a Relation: Let R be a relation from the set A to the set B. Then the set of all first constituents of the ordered pairs present in the relation R is known as domain of R . Denoted by dom. R or D(R). Mathematically,

$$\mathrm{D(R)} = \{x \mid (x, y) \in \mathrm{R}, \text{ for } x \in \mathrm{A}\}$$

$$\mathrm{D(R)} \subseteq \mathrm{A}$$

- **Range of a Relation:** Let R be a relation from the set A to the set B. Then the set of all second constituents of the ordered pairs present in the relation R is known as range of R. Denoted by rng.R or R(R).Mathematically



Mathematically,

$$R(R) = \{y \mid (x, y) \in R, \text{ for } y \in B\}$$

i.e., $$R(R) \subseteq B$$

Consider the example: Let $A = \{a, b, c, d\}$ and $B = \{5, 6, 7\}$. Let us define a relation R from the set A to the set B as below.

$$R = \{(a, 5), (a, 6), (c, 6), (d, 6)\}$$

So, $$D(R) = \{a, c, d\} \text{ and } R(R) = \{5, 6\}$$

2.INVERSE RELATION: Let R be a relation from the set A to the set B. Then the inverse of the relation R is a relation from the set B to the set A. It is denoted by R^{-1} and is defined as

$$R^{-1} = \{(y, x) \mid (x, y) \in R\}$$

Consider the example: Let $A = \{1, 2, 3, 4, 5\}$

and $B = \{4, 9, 16, 17, 25\}$

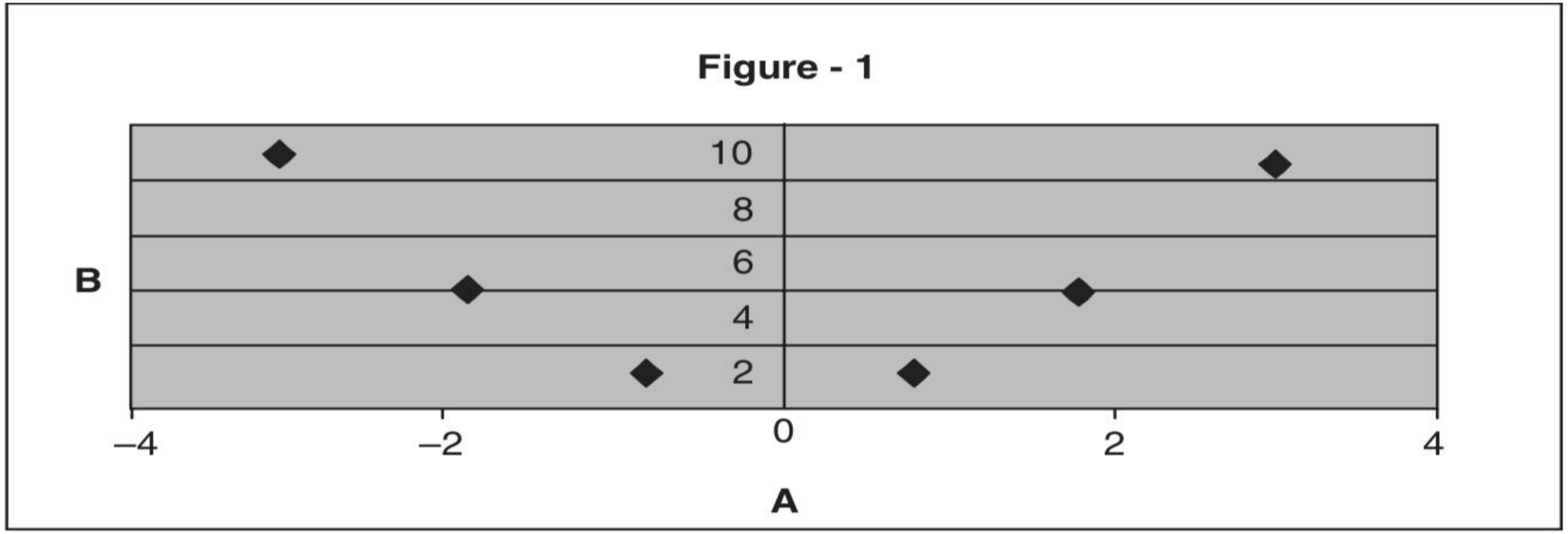
Let us consider the relation R from the set A to the set B as $R = \{(2, 4), (3, 9), (4, 16), (3, 17)\}$

Therefore, $R^{-1} = \{(4, 2), (9, 3), (16, 4), (17, 3)\}$.

3.GRAPH OF RELATION: Let R be a relation from the set A to the set B; that is R is a subset of (A x B). Since (A x B) can be represented by the set of points on the coordinate diagram of (A x B),

Let A = { – 3, – 2, – 1, 1, 2, 3 } and B = {1, 2, 3, 4, 5, 6, 7, 8, 9} and x R y such that $y = x^2$. Thus we have

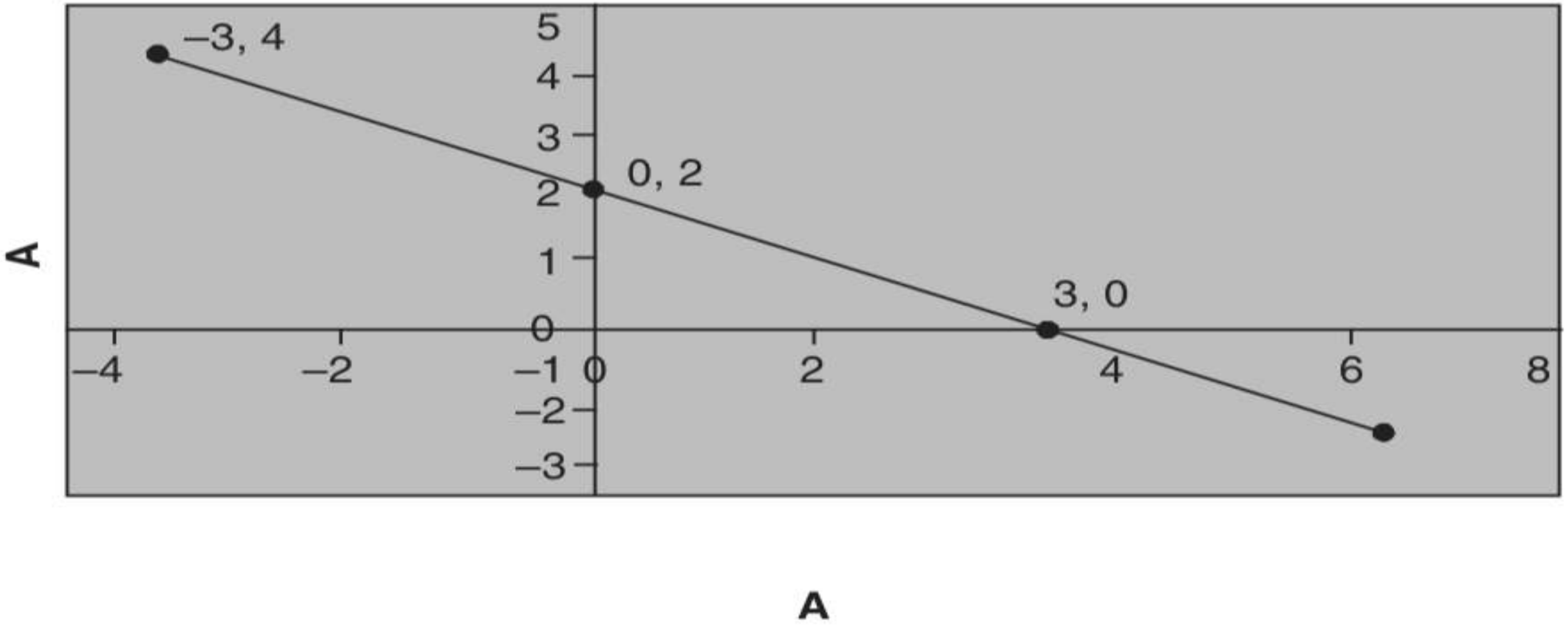
$$R = \{(-1, 1), (1, 1), (-2, 4), (2, 4), (-3, 9), (3,9)\}$$



أ.م عهود سعدي الحسني

Consider another example: Let A = $\{x \mid x$ is a real number$\}$ and x R y such that $2x + 3y \leq 6$. Thus, we have $R = \{(x, y) \mid 2x + 3y \leq 6 \text{ and } x, y \in A\}$.

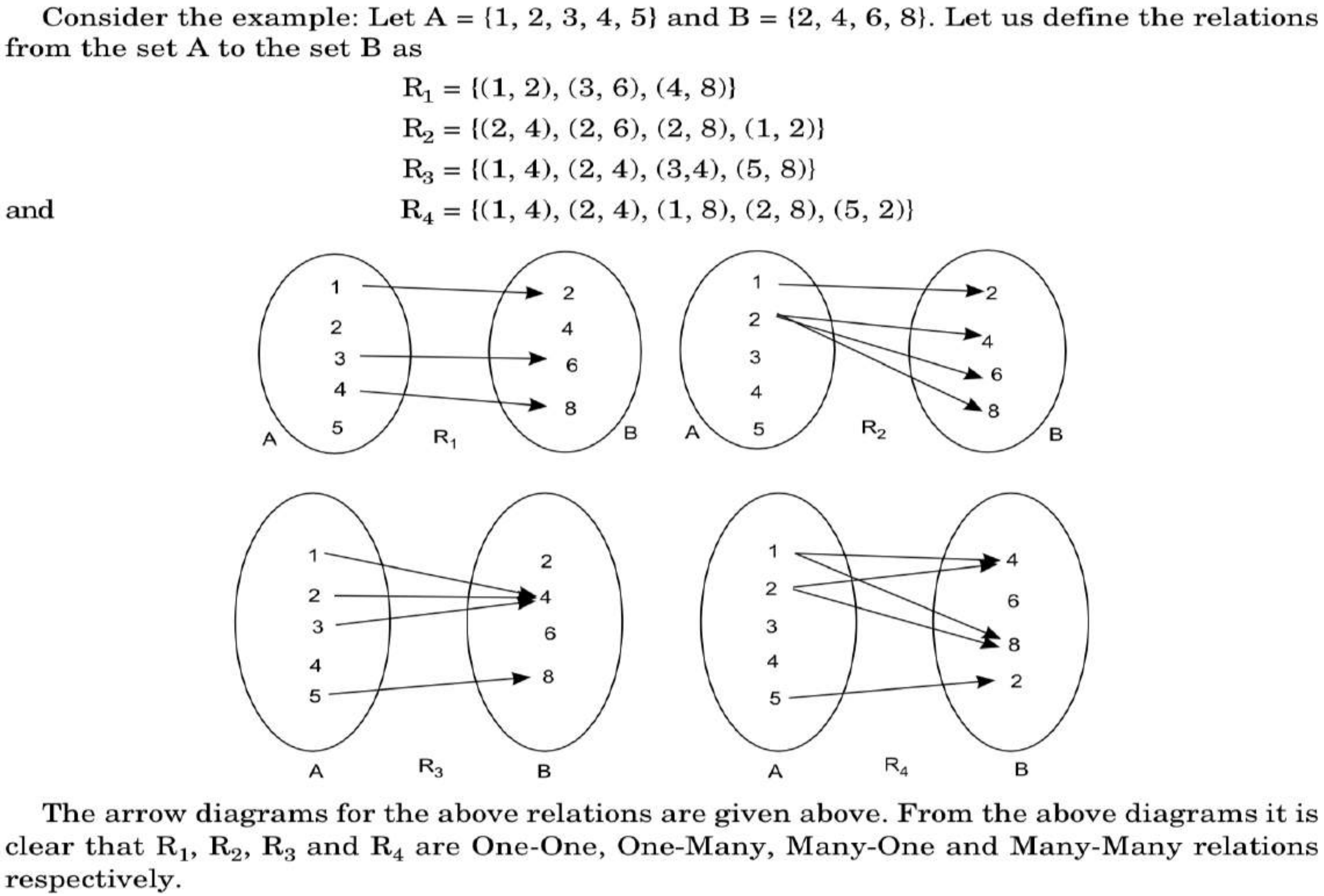
Figure - 2



4.KINDS OF RELATION

- One-One
- One-Many
- Many-One
- Many-Many

5. ARROW DIAGRAM: We use arrow diagrams to represent relations.



Consider the example: Let A = {1, 2, 3, 4, 5} and B = {2, 4, 6, 8}. Let us define the relations from the set A to the set B as

$$R_1 = \{(1, 2), (3, 6), (4, 8)\}$$

$$R_2 = \{(2, 4), (2, 6), (2, 8), (1, 2)\}$$

$$R_3 = \{(1, 4), (2, 4), (3,4), (5, 8)\}$$

and $$R_4 = \{(1, 4), (2, 4), (1, 8), (2, 8), (5, 2)\}$$

The arrow diagrams for the above relations are given above. From the above diagrams it is clear that R_1, R_2, R_3 and R_4 are One-One, One-Many, Many-One and Many-Many relations respectively.


thank you

Discrete mathematics

LEC.7

أ.م عهود سعدي الحسني

6. VOID RELATION

Let R be a relation on a set A; that is R is a subset of (A × A). Then the relation R is said to be an identity relation if $(x, x) \in$ R. Generally denoted by I_A. Mathematically,

$$I_A = \{(x, x) \mid x \in A\}$$

7.IDENTITY RELATION

A relation R from a set A to a set B is said to be a void relation or empty relation if $R = \phi$.

Consider the example: Let $A = \{3, 5, 7\}$; $B = \{2, 4, 8\}$; $R \subseteq A \times B$ and $x \, R \, y \mid x$ divides y; $x \in A$, $y \in B$. Hence, we observe that $R = \phi \subseteq A \times B$ is a void relation from the set A to the set B.

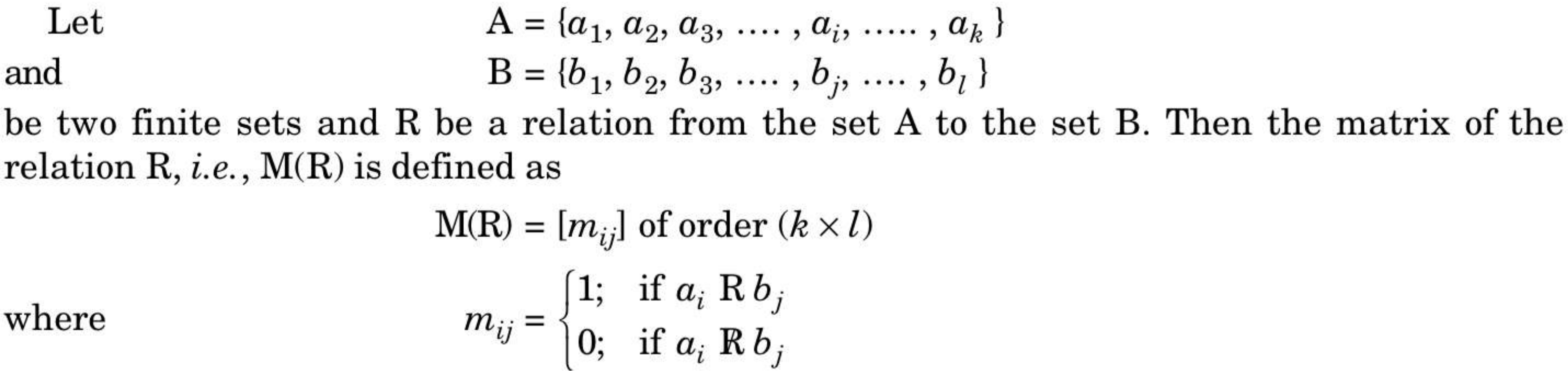
Consider the example: Let $A = \{a, b, c\}$ and I_A be a relation on A such that $I_A = \{(a, a), (b, b), (c, c)\}$. This is an identity relation on A.

8.UNIVERSAL RELATION: relation R from a set A to a set B is said to be an universal relation if R is equal to (A x B). That is R = (A x B).

Let A = {1, 2, 3} and B = {a, b}. Therefore the universal relation R from the set A to the set B is given as

R = {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)}.

9.RELATION MATRIX (MATRIX OF THE RELATION)



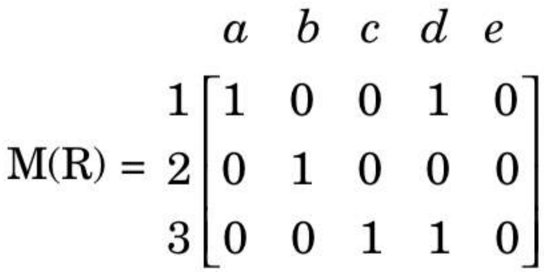
Let $A = \{a_1, a_2, a_3, \ldots, a_i, \ldots, a_k\}$

and $B = \{b_1, b_2, b_3, \ldots, b_j, \ldots, b_l\}$

be two finite sets and R be a relation from the set A to the set B. Then the matrix of the relation R, *i.e.*, M(R) is defined as

$$M(R) = [m_{ij}] \text{ of order } (k \times l)$$

where
$$m_{ij} = \begin{cases} 1; & \text{if } a_i \, R \, b_j \\ 0; & \text{if } a_i \, \not{R} \, b_j \end{cases}$$



$$M(R) = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 1 & 0 \end{array}$$

10 COMPOSITION OF RELATIONS

Let R_1 be a relation from the set A to the set B and R_2 be a relation from the set B to the set C. That is R_1 is a subset of $(A \times B)$ and R_2 is a subset of $(B \times C)$. Then the composition of R_1 and R_2 is given by R_1R_2 and is defined by

$$R_1R_2 = \{(x, z) \in (A \times C) \mid \text{for some } y \in B, (x, y) \in R_1 \text{ and } (y, z) \in R_2\}$$

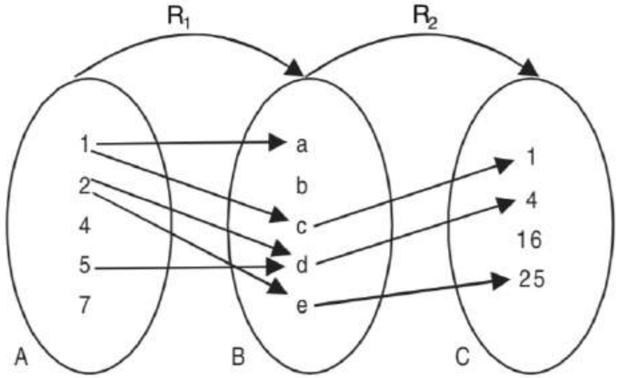
Consider the example: Let $A = \{1, 2, 4, 5, 7\}$;

$$B = \{a, b, c, d, e\}$$

and $$C = \{1, 4, 16, 25\}$$

Consider the relations R_1: A → B and R_2: B → C as

$R_1 = \{(1, a), (1, c), (2, d), (2, e), (5, d)\}$ and $R_2 = \{(c, 1), (d, 4), (e, 25)\}$. The arrow diagram is given as



So, $R_1R_2 = \{(1, 1), (2, 4), (2, 25), (5, 4)\}$

11. TYPES OF RELATIONS

This section discusses a number of different important types of relations on a set A that are important for the study of finite state systems.

أ.م عهود سعدي الحسني

1. Reflexive Relations

A relation R defined on a set A is said to be reflexive if $(x, x) \in$ R for every element $x \in$ A.

i.e., $x \text{ R } x \quad \forall x \in \text{A}$

Consider the following relations on the set A = {1, 3, 5, 7}

$$R_1 = \{(1, 1), (1, 3), (1, 5), (5, 5), (5, 7)\}$$
$$R_2 = \{(1, 3), (1, 5), (5, 7), (3, 7)\}$$
$$R_3 = \{(1, 1), (1, 3), (3, 3), (5, 5), (5, 7), (1, 7), (7, 7)\}$$

From the above relations it is clear that R_3 is a reflexive relation. R_1 is not a reflexive relation as $(3, 3) \notin R_1$ and $(7, 7) \notin R_1$. Similarly, R_2 is also not reflexive.

2. Symmetric Relations

A relation R defined on a set A is said to be symmetric if $(x, y) \in$ R then $(y, x) \in$ R.

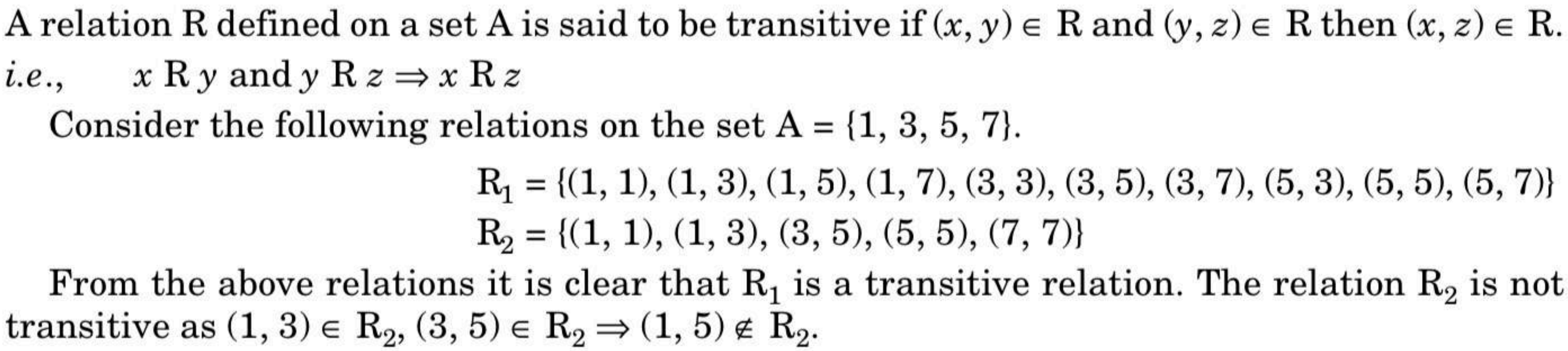
i.e., $x \text{ R } y \Rightarrow y \text{ R } x.$

Consider the following relations on the set A = {1, 3, 5, 7}

$$R_1 = \{(1, 1), (1, 3), (3, 5), (3, 1), (5, 3), (5, 5)\}$$
$$R_2 = \{(1, 1), (1, 3), (3, 1), (3, 5), (5, 3), (5, 7), (7, 7)\}$$

From the above relations it is clear that R_1 is a symmetric relation, but R_2 is not a symmetric relation as $(5, 7) \in R_2 \Rightarrow (7, 5) \notin R_2$.

3. Transitive Relations:



A relation R defined on a set A is said to be transitive if $(x, y) \in$ R and $(y, z) \in$ R then $(x, z) \in$ R.

i.e., $x \text{ R } y$ and $y \text{ R } z \Rightarrow x \text{ R } z$

Consider the following relations on the set A = {1, 3, 5, 7}.

$$R_1 = \{(1, 1), (1, 3), (1, 5), (1, 7), (3, 3), (3, 5), (3, 7), (5, 3), (5, 5), (5, 7)\}$$

$$R_2 = \{(1, 1), (1, 3), (3, 5), (5, 5), (7, 7)\}$$

From the above relations it is clear that R_1 is a transitive relation. The relation R_2 is not transitive as $(1, 3) \in R_2, (3, 5) \in R_2 \Rightarrow (1, 5) \notin R_2$.

12. EQUIVALENCE RELATION: A relation R defined on a set A is said to be an equivalence relation in A if and only if R is reflexive, symmetric and transitive**.**

- **Theorem:** If R be an equivalence relation defined in a set A, then R^{-1} is also an equivalence relation in the set A.

Example *Let R be the relation on the set {1, 2, 3, 4, 5} defined by the rule $(x, y) \in R$ if $x + y \leq 6$. Find the followings.*

(*a*) List the elements of R

(*b*) List the elements of R^{-1}

(*c*) Domain of R

(*d*) Range of R

(*e*) Range of R^{-1}

(*f*) Domain of R^{-1}

Check that domain of R is equal to range of R^{-1} and range of R is equal to domain of R^{-1}.

Solution: Let $A = \{1, 2, 3, 4, 5\}$ and $R = \{(x, y) \in R \mid x + y \leq 6 \, ; x, y \in A\}$

(*a*) $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (5, 1)\}$

(*b*) $R^{-1} = \{(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (1, 2), (2, 2), (3, 2), (4, 2), (1, 3), (2, 3), (3, 3), (1, 4), (2, 4), (1, 5)\}$

(*c*) Domain of R *i.e.,* $D(R) = \{1, 2, 3, 4, 5\}$

(*d*) Range of R *i.e.,* $R(R) = \{1, 2, 3, 4, 5\}$

(*e*) Range of R^{-1} *i.e.,* $R(R^{-1}) = \{1, 2, 3, 4, 5\}$

(*f*) Domain of R^{-1} *i.e.,* $D(R^{-1}) = \{1, 2, 3, 4, 5\}$

From this it is clear that $D(R) = R(R^{-1})$ and $R(R) = D(R^{-1})$.


thank you

Discrete mathematics

LEC.8

أ.م عهود سعدي الحسني

Function

Let A and B be two non-empty sets. A relation f from the set A to the set B is said to be a function if it satisfies the following two conditions.

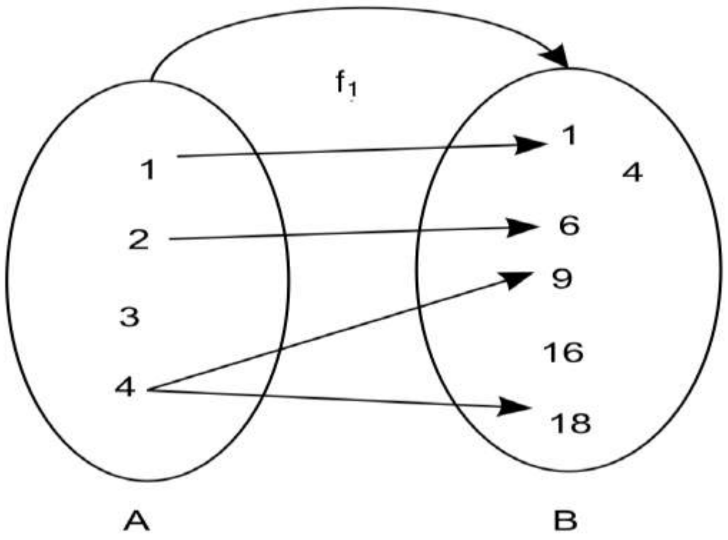
(*i*) $D(f) = A$ and

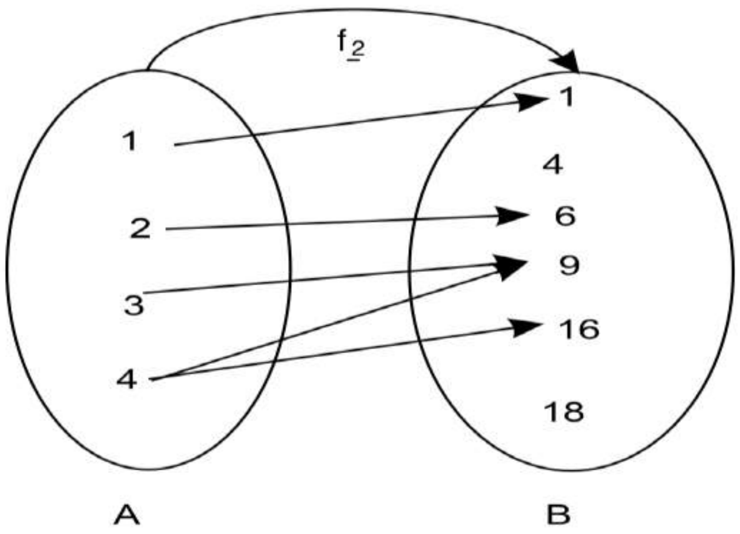
(*ii*) if $(x_1, y_1) \in f$ and $(x_2, y_2) \in f$, then $y_1 = y_2$.

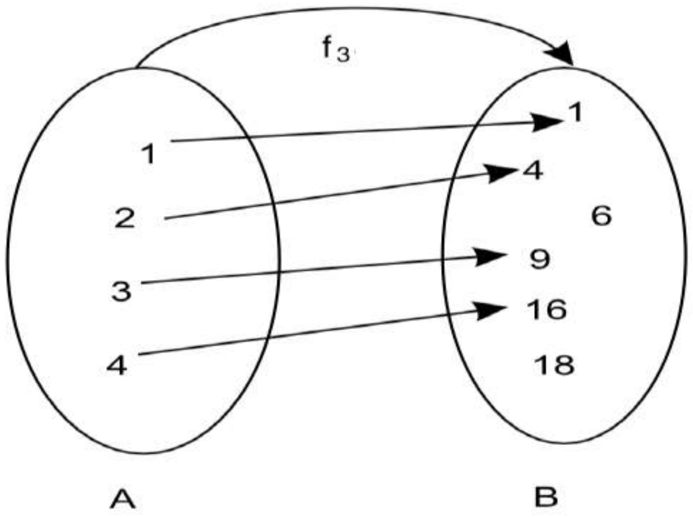
In other words a relation f from the set A to the set B is said to be a function if for each element x in A there exists unique element y in B. A function from A to B is sometimes denoted as $f : A \to B$.

Consider the following relations from the set $A = \{1, 2, 3, 4\}$ to the set $B = \{1, 4, 6, 9, 16, 18\}$.

$$f_1 = \{(1, 1), (2, 6), (4, 9), (4, 18)\}$$
$$f_2 = \{(1, 1), (2, 6), (3, 9), (4, 9), (4, 16)\}$$
$$f_3 = \{(1, 1), (2, 4), (3, 9), (4, 16)\}$$
$$f_4 = \{(1, 1), (2, 4), (3, 9), (4, 9)\}$$


f1
1
2
3
4
1
4
6
9
16
18
A
B


f2
1
2
3
4
1
4
6
9
16
18
A
B


f3
1
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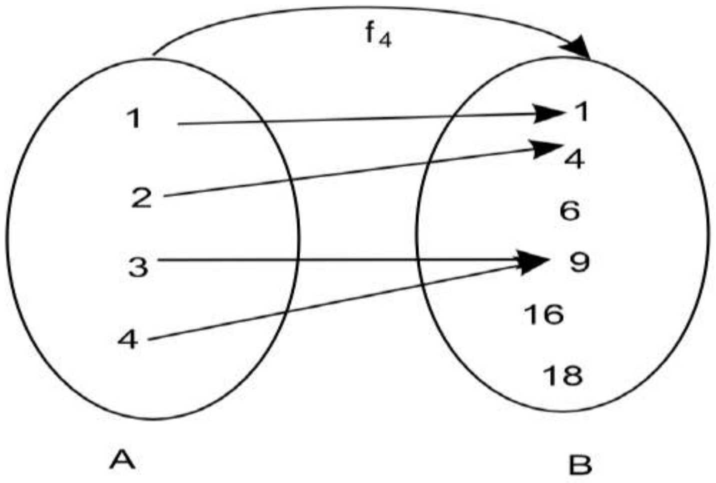
4
1
4
6
9
16
18
A
B


f4
1
2
3
4
1
4
6
9
16
18
A
B

1. a) Domain and Co-domain of a Function

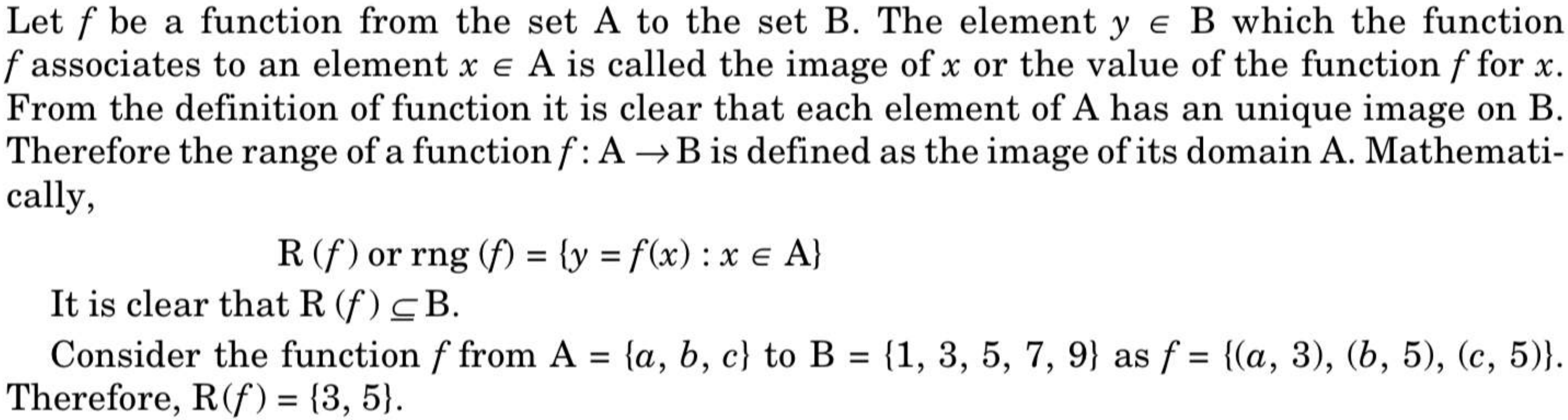
Suppose that f be a function from the set A to the set B. The set A is called the domain of the function f where as the set B is called the co-domain of the function f.

Consider the function f from the set A = $\{a, b, c, d\}$ to the set B = $\{1, 2, 3, 4\}$ as

$$f = \{(a, 1), (b, 2), (c, 2), (d, 4)\}$$

Therefore, domain of $f = \{a, b, c, d\}$ and co-domain of $f = \{1, 2, 3, 4\}$. *i.e.*, D(f) = $\{a, b, c, d\}$ and Co-domain $f = \{1, 2, 3, 4\}$.

b) Range of a Function



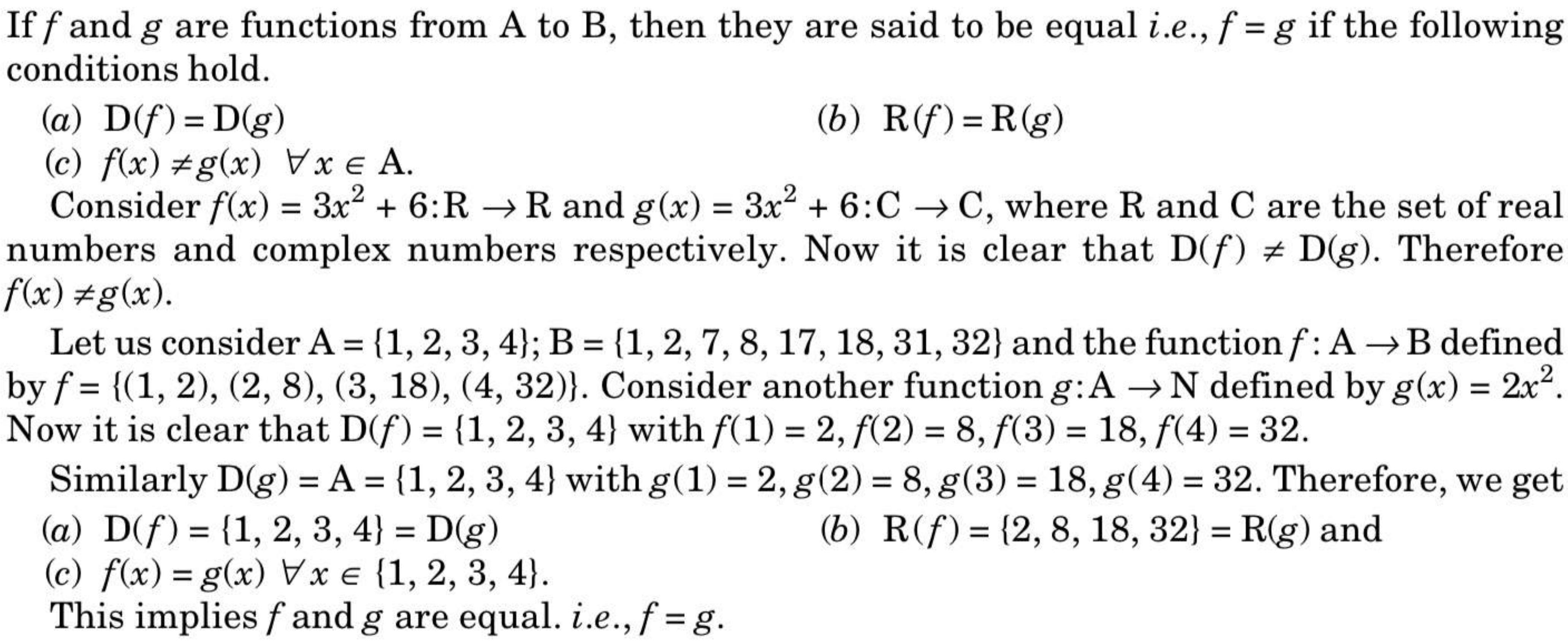
Let f be a function from the set A to the set B. The element $y \in$ B which the function f associates to an element $x \in$ A is called the image of x or the value of the function f for x. From the definition of function it is clear that each element of A has an unique image on B. Therefore the range of a function $f : A \to B$ is defined as the image of its domain A. Mathematically,

$$R(f) \text{ or } rng(f) = \{y = f(x) : x \in A\}$$

It is clear that $R(f) \subseteq B$.

Consider the function f from $A = \{a, b, c\}$ to $B = \{1, 3, 5, 7, 9\}$ as $f = \{(a, 3), (b, 5), (c, 5)\}$. Therefore, $R(f) = \{3, 5\}$.

2.EQUALITY OF FUNCTIONS



If f and g are functions from A to B, then they are said to be equal *i.e.*, $f = g$ if the following conditions hold.

(*a*) $D(f) = D(g)$ (*b*) $R(f) = R(g)$

(*c*) $f(x) \neq g(x) \quad \forall\, x \in A$.

Consider $f(x) = 3x^2 + 6 : R \to R$ and $g(x) = 3x^2 + 6 : C \to C$, where R and C are the set of real numbers and complex numbers respectively. Now it is clear that $D(f) \neq D(g)$. Therefore $f(x) \neq g(x)$.

Let us consider $A = \{1, 2, 3, 4\}$; $B = \{1, 2, 7, 8, 17, 18, 31, 32\}$ and the function $f : A \to B$ defined by $f = \{(1, 2), (2, 8), (3, 18), (4, 32)\}$. Consider another function $g : A \to N$ defined by $g(x) = 2x^2$. Now it is clear that $D(f) = \{1, 2, 3, 4\}$ with $f(1) = 2, f(2) = 8, f(3) = 18, f(4) = 32$.

Similarly $D(g) = A = \{1, 2, 3, 4\}$ with $g(1) = 2, g(2) = 8, g(3) = 18, g(4) = 32$. Therefore, we get

(*a*) $D(f) = \{1, 2, 3, 4\} = D(g)$ (*b*) $R(f) = \{2, 8, 18, 32\} = R(g)$ and

(*c*) $f(x) = g(x) \;\forall\, x \in \{1, 2, 3, 4\}$.

This implies f and g are equal. *i.e.*, $f = g$.

3.TYPES OF FUNCTION

أ.م عهود سعدي الحسني

1. One-One Function:

A function $f : A \to B$ is said to be an one-one function or injective if $f(x_1) = f(x_2)$, then $x_1 = x_2$ for $x_1, x_2 \in A$. *i.e.*, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

Consider a function f: $Q \to Q$ defined by $f(x) = 4x + 3$; $x \in Q$.

Suppose that $f(x_1) = f(x_2)$ for $x_1, x_2 \in Q$.

$$\Rightarrow \quad 4x_1 + 3 = 4x_2 + 3$$
$$\Rightarrow \quad 4x_1 = 4x_2$$
$$\Rightarrow \quad x_1 = x_2$$

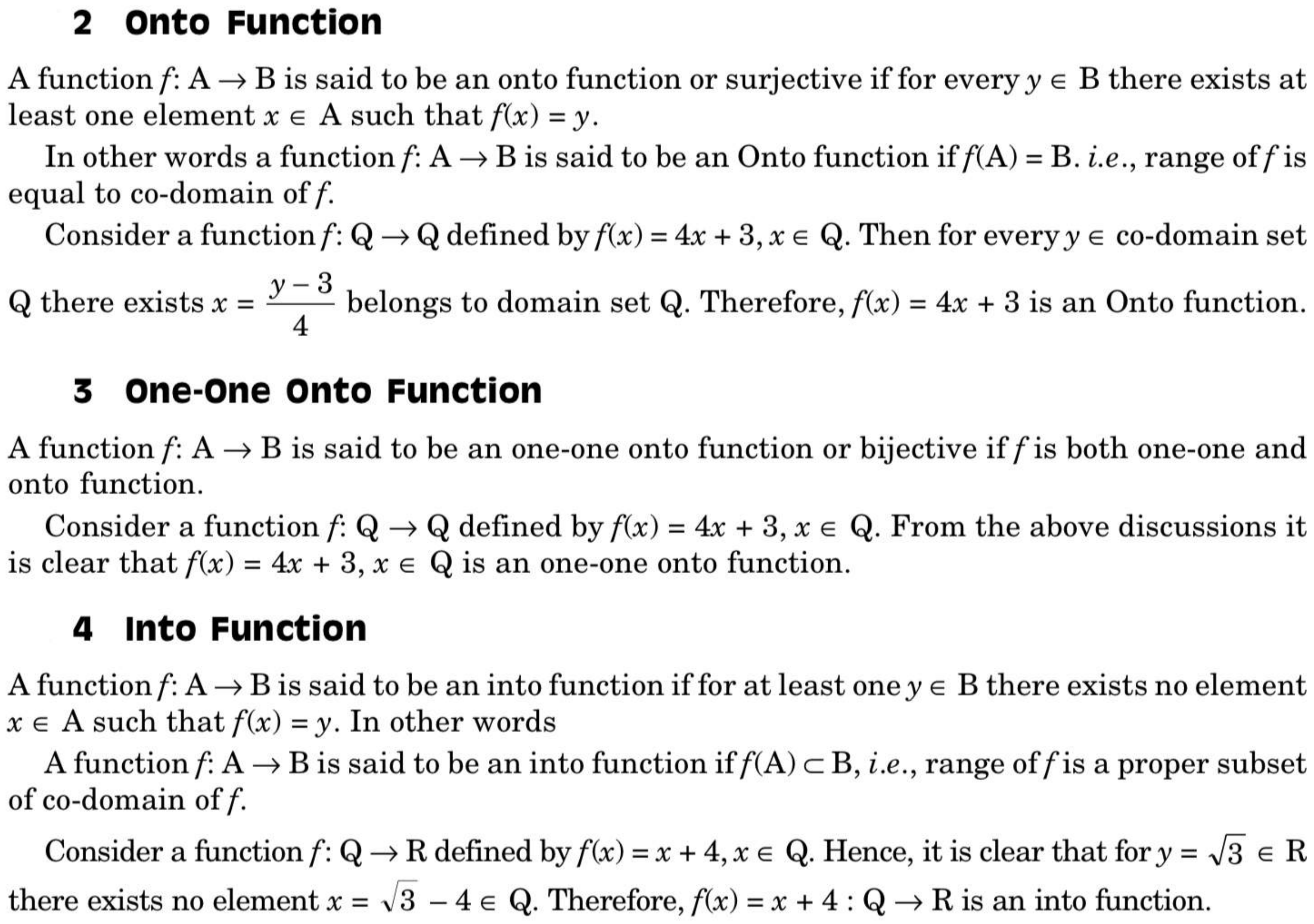
i.e., $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. So, $f(x) = (4x + 3) : Q \to Q$ is One-One.

Consider another function $f : R \to R$ defined by $f(x) = x^2$; $x \in R$. Suppose that $f(x_1) = f(x_2)$

$$\Rightarrow \quad {x_1}^2 = {x_2}^2$$
$$\Rightarrow \quad x_1 = \pm\, x_2$$
$$\Rightarrow \quad x_1 \neq x_2$$

i.e., $f(x_1) = f(x_2) \Rightarrow x_1 \neq x_2$. It is also clear that $f(1) = 1 = f(-1)$; but $1 \neq -1$. Therefore $f(x) = x^2$: $R \to R$; $x \in R$ is not One-One.

3.TYPES OF FUNCTION



2 Onto Function

A function $f: A \to B$ is said to be an onto function or surjective if for every $y \in B$ there exists at least one element $x \in A$ such that $f(x) = y$.

In other words a function $f: A \to B$ is said to be an Onto function if $f(A) = B$. *i.e.*, range of f is equal to co-domain of f.

Consider a function $f: Q \to Q$ defined by $f(x) = 4x + 3$, $x \in Q$. Then for every $y \in$ co-domain set Q there exists $x = \frac{y-3}{4}$ belongs to domain set Q. Therefore, $f(x) = 4x + 3$ is an Onto function.

3 One-One Onto Function

A function $f: A \to B$ is said to be an one-one onto function or bijective if f is both one-one and onto function.

Consider a function $f: Q \to Q$ defined by $f(x) = 4x + 3$, $x \in Q$. From the above discussions it is clear that $f(x) = 4x + 3$, $x \in Q$ is an one-one onto function.

4 Into Function

A function $f: A \to B$ is said to be an into function if for at least one $y \in B$ there exists no element $x \in A$ such that $f(x) = y$. In other words

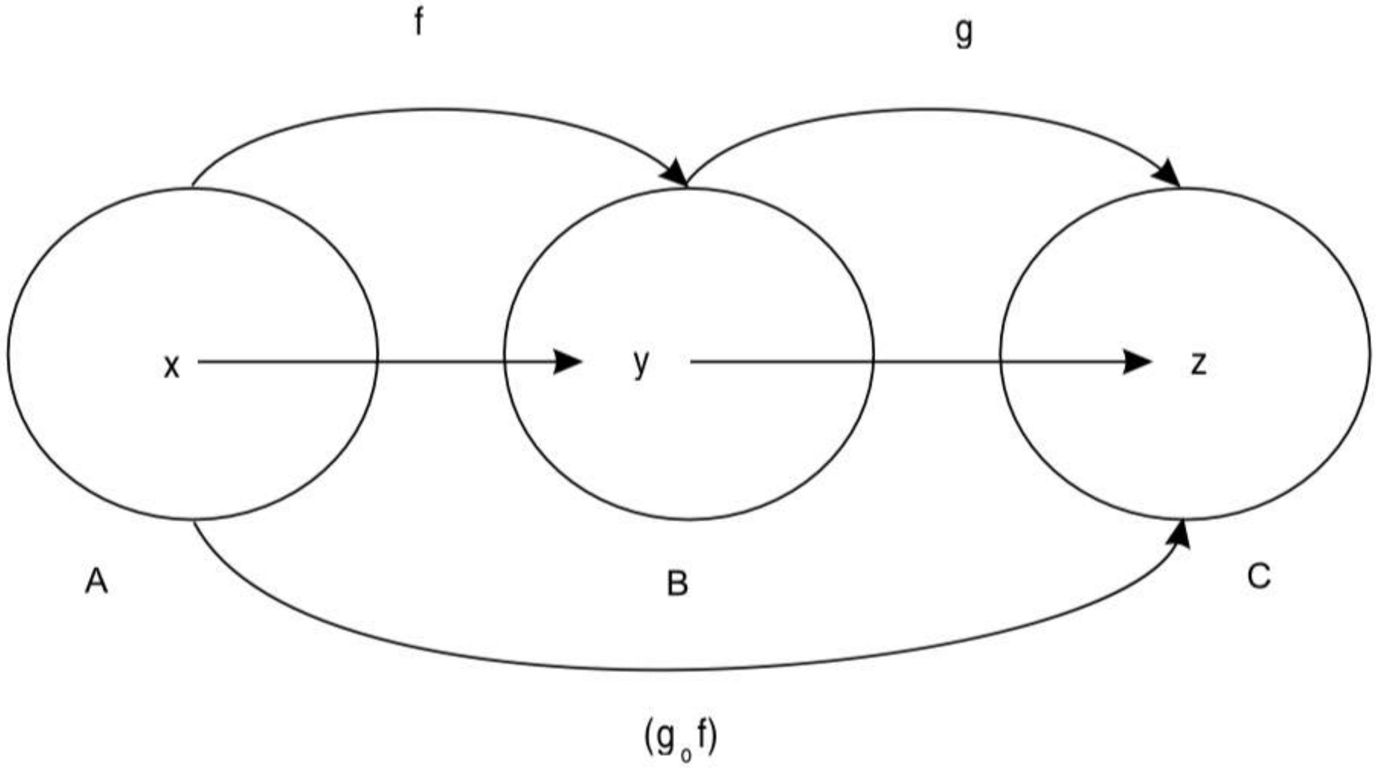
A function $f: A \to B$ is said to be an into function if $f(A) \subset B$, *i.e.*, range of f is a proper subset of co-domain of f.

Consider a function $f: Q \to R$ defined by $f(x) = x + 4$, $x \in Q$. Hence, it is clear that for $y = \sqrt{3} \in R$ there exists no element $x = \sqrt{3} - 4 \in Q$. Therefore, $f(x) = x + 4 : Q \to R$ is an into function.

أ.م عهود سعدي الحسني

4 COMPOSITION OF FUNCTIONS

Let f be a function from the set A to the set B and g be a function from the set B to the set C. Then the composition of the functions f and g is given as $(g \circ f)$ or gf. This is a function from the set A to the set C. It may also be noted that domain of g is equal to co-domain of f.



As f is a function from the set A to the set B, then for every $x \in$ A there exists unique $y \in$ B such that $y = f(x)$. Similarly g is a function from the set B to the set C, then for every $y \in$ B there exists unique $z \in$ C such that $z = g(y)$. Again $(g_o f)$ is a function from the set A to the set C, so we get $(g_o f)(x) = z$ for all $x \in$ A.

i.e., $(g_o f)(x) = g(y)$

i.e., $(g_o f)(x) = g(f(x))$

Consider two functions $f(x) = 2x + 5$ and $g(x) = 3x$.

Therefore $(g_o f)(x) = g(f(x))$

$$= g(2x + 5)$$

$$= 3(2x + 5)$$

i.e., $(g_o f)(x) = 6x + 15$

Similarly, $(f_o g)(x) = f(g(x))$

$$= f(3x)$$

$$= 2(3x) + 5$$

i.e., $(f_o g)(x) = 6x + 5$

4.composition of function

- Theorem

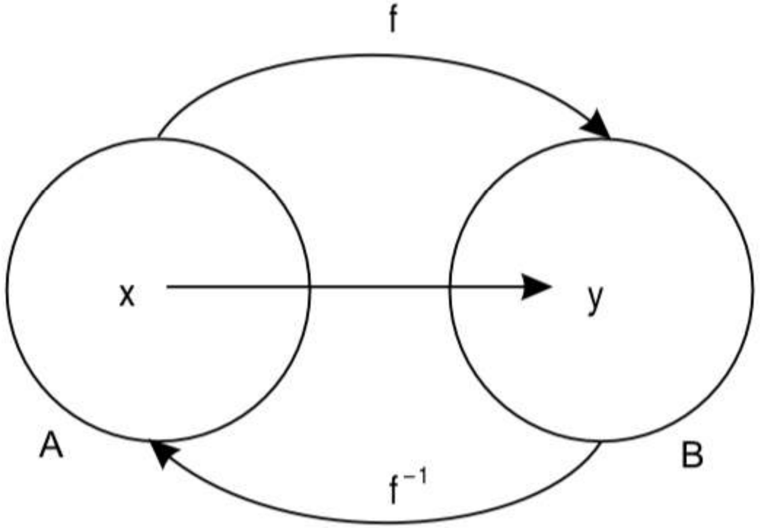
Let $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{C}$ be two functions. Then $(g\ _{o}\, f)$ is one-one if both f and g are one-one and $(g\ _{o}\, f)$ is onto if both f and g are onto.

- Theorem

If f: $\mathrm{A} \to \mathrm{B}$; g: $\mathrm{B} \to \mathrm{C}$ and $h : \mathrm{C} \to \mathrm{D}$, then $h\ _{o}\, (g\ _{o}\, f) = (h\ _{o}\, g)\ _{o}\, f$, *i.e.*, composition of functions holds the associative law.

5. INVERSE FUNCTION

Let $f : \mathrm{A} \to \mathrm{B}$ be a bijective function. Then the inverse of f, *i.e.* f^{-1} be a function from B to A. Since f is a function from A to B, for every $x \in \mathrm{A}$, there exists unique $y \in \mathrm{B}$ such that $f(x) = y$.



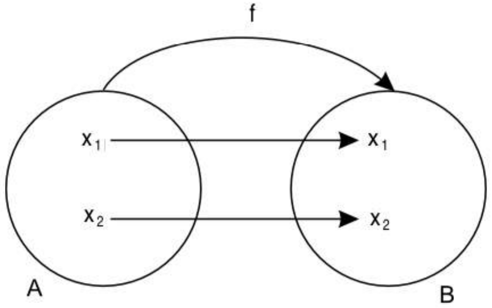
Since $f^{-1}: \mathrm{B} \to \mathrm{A}$ for every $y \in \mathrm{B}$ there exists unique $x \in \mathrm{A}$ such that $f^{-1}(y) = x$, *i.e.*, $f^{-1}(f(x)) = x$.

SOME IMPORTANT FUNCTIONS

أ.م عهود سعدي الحسني

1 Identity Function

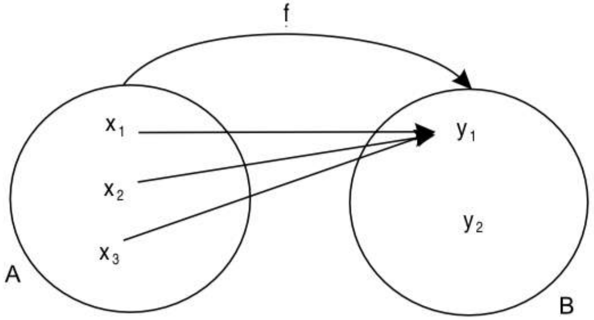
Let A be a set. The function f: A $\rightarrow$ A is said to be an identity function if for every $x \in$ A, $f(x) = x$. Mathematically $f(x) = x \; \forall \; x \in$ A.



2 Constant Function

The function f: A $\rightarrow$ B is said to be a constant function if for every $x \in$ A there exists unique $y \in$ B such that $f(x) = y$. Mathematically,

$$f(x) = y \quad \forall \; x \in \text{A}$$



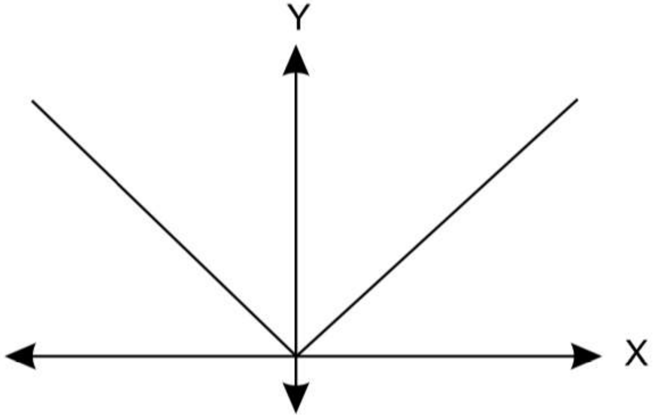
SOME IMPORTANT FUNCTIONS

3 Absolute Function

The absolute function or absolute value function $f(x) = |x|$ is defined as

$$|x| = \begin{cases} x; & \text{if } x \geq 0 \\ -x; & \text{if } x < 0 \end{cases}$$

The graph of $f = \{(x, |x|) : x \in R\}$ is shown in the following figure.



أ.م عهود سعدي الحسني


thank you

جامعة بغداد / كلية علوم الحاسوب

ا.م. عهود سعدي عبد الكريم الحسني

الكورس الثاني

Discrete mathematics

Matrix

Square matrices

A matrix with the same number of rows as columns is called square matrix . A square matrix with n rows and n columns is said to be order n ,and is called an n-square matrix.

Def: Main diagonal

The main diagonal or simply diagonal ,of asquare matrix A=[aij] consists of the numbers: a11,a22,a33,……,ann.

Ex:The matrix $\begin{bmatrix} 1 & -2 & 0 \\ 0 & -4 & -1 \\ 5 & 3 & 2 \end{bmatrix}$

Is asquare matrix of order 3 .The numbers along the main diagonal are 1,-4,2.

Def:(Trace of the matrix)

If A=[a_{ij}]is an nxn matrix , then the trace of A which is denoted by tr(A) is defined as the sum of all elements on the main diagonal of A.

Tr(A)=$\sum_{i=1}^{n} aii$

Ex :The trace of the following matrix $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ **is 4**

Trace of the matrix has the following properties:-

1)Tr(A+B)=Tr(A)+Tr(B)

2)Tr(c A)=c Tr(A)

3)Tr (AB)=Tr(BA)

4)Tr(A^T)=Tr(A)

Def:Identity matrix

The n-sqare matrix with 1,s along the main diagonal and 0,s else where , is called the unit matrix and will be denoted and defined by

$$I_n=\begin{bmatrix} 1 & 0 \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

Note : The unit matrix (I_n)plays the same role in matrix multiplication as the number 1 does in the usual multiplication of number ,specifically

$$A\, I_n=I_n\, A=A$$

For any square matrix A .

Def:Diagonal matrix

Asquare matrix A=[a_{ij}] for which every term off the main diagonal is zero ,that is ,$a_{ij}=0$ for $i\neq j$,is called diagonal matrix.

Ex: $G=\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$,$H=\begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Are diagonal matrices.

Def:Scalar matrix

A diagonal matrix A=[a_{ij}] for which all terms on the main diagonal are equal ,that is ,$a_{ij}=c$ for i=j and $a_{ij}=0$ for $i\neq j$, is called scalar matrix.

Ex: The following are scalar matrices

$$J=\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix},\ A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Def:Upper triangular matrix

A matrix A=[a_{ij}] is called upper triangular matrix if a_{ij}=0 for i $>$ j, and defined by

$$A = \begin{bmatrix} a11 & a12 \ldots \ldots. & a1n \\ 0 & a22 \ldots \ldots. & a2n \\ 0 & 0 & ann \end{bmatrix}$$

Ex:Suppose that $A = \begin{bmatrix} 7 & 5 & 12 \\ 0 & 1 & 18 \\ 0 & 0 & 6 \end{bmatrix}$

Is an upper triangular matrix of order 3.

Def: Lower triangular matrix :

An n- square matrix A=[a_{ij}] is called lower triangular matrix if a_{ij} =0 for i$<$ j ,and defined by $A = \begin{bmatrix} a11 & 0 \ldots \ldots. & 0 \\ a21 & a22 \ldots \ldots. & 0 \\ an1 & an2 \ldots \ldots. & ann \end{bmatrix}$

Ex:Suppose that A= $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 5 & 6 & 0 & 0 \\ 7 & 1 & 8 & 0 \\ 3 & 2 & 4 & 5 \end{bmatrix}$

Is a lower triangular matrix of order 4.

Def:(Periodic matrix)

If A is an n-square matrix .Such that A^{K+1}=A where k is positive integer , then A is called periodic matrix .

Ex:Suppose that A= $A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$,is a periodic matrix.

Def:(Idempotent matrix)

If A is an n-square matrix such that $A^2=A$,then Ais called idempotent matrix.

Ex:Suppose that $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$**,is idempotent matrix.**

Def:(Nilpotent matrix)

If A is an n-square matrix such that $A^P=0$, where P is apositive integer ,then A is called nilpotent matrix.

Ex:Suppose that $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ **is nilpotent matrix of order 3..**

Transpose of matrix :

If $A=[a_{IJ}]$ is an m x n matrix ,then the n x m matrix $A^T=[a_{ij}{}^T]$ where $aij^T=a_{ji}$,

$(1 \leq i \leq m , 1 \leq j \leq n)$is called the transpose of A ,thus the transpose of A is obtained by interchanging the rows and columns of A.

$$\begin{bmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ an & bn & cn \end{bmatrix} = \begin{bmatrix} a1 & a2 & an \\ b1 & b2 & c2 \\ c1 & c2 & cn \end{bmatrix}$$

The transpose operation on matrices satisfies the following properties :-
Theorem:1)$(A+B)^T=A^T+B^T$

2)$(KA)^T=K A^T$,for K ascalar.

3)$(AB)^T=B^T.A^T$

4)$(A^T)^T=A$

Def:(Symmetric matrix)

A matrix A=[a_{ij}] is called symmetric if A^T=A.

That is ,A is symmetric if it is asquare matrix for which aij=aji

If A is symmetric ,then the elements of A are symmetric with respect to the main diagonal of A.

Ex: The matrices A=$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ I_3=$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are symmetric.

Def:Skew symmetric matrix:

Amatrix A=[a_{ij}] is called skew symmetric matrix if A^T=-A that is ,A is skew symmetric matrix if and only if a_{ij}=-a_{ji} , for all i,j

Ex:A=$\begin{bmatrix} 0 & 1 & 4 \\ -1 & 0 & -5 \\ -4 & 5 & 0 \end{bmatrix}$,A is skew symmetric matrix .

Def:(Orthogonal matrix)

If A is an n- square matrix such that A .A^T= A^T .A =In, then A is called orthogonal matrix

Ex:A =$\begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}$is an orthogonal matrix.

Notes:1)we can form powers of asquare matrix A by defining

A^2=A . A ,A^3=A^2 A,.. ...., and A^0 =I

In general ,If p and q be non negative integers and A be asquare matrix ,then

1)A^p .A^q=A^{p+q}

2)$(A^p)^q$ =$A^{p q}$

3)If AB =BA,then $(AB)^p$= A^p B^P

4)(c A)^p =c^p A^P where c is ascalar .

2)We can form polynomials in A . That is ,for any polynomials

$f(x)=a_0+a_1 x + a_2 x^2 +\ldots\ldots\ldots+a_n x^n$

we define f(A) to be the matrix

$f(A)= a_0 I+a_1 A + a_2 A^2 +\ldots\ldots\ldots+a_n A^n$

In the case that f(A) is the zero matrix , then A is said azero or root of the polynomial f(x).

Ex: Let A =$\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$, then A^2=$\begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix}$

If f(x) =x^2 +3 x -10 ,then

f(x)= $\begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix}$+3$\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ +(-10) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$=$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Thus A is azero of the polynomial f(x).

Invertible matrices :

Asquare matrix A is said to be invertible if there exist amatrix B with the property that AB =BA=I(The identity matrix),such amatrix B is unique ,it is called the inverse of A is denoted by A^{-1}.Observe that B is the inverse of A if and only if A is the inverse of B .For example

Suppose that A=$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and B=$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

Then AB=$\begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix}$= $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and BA= $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Thus A and B are inverses.

Note:If is known that AB =I$_n$ if and only if BA =I$_n$, hence it is necessary to test only one product to determine whether two given matrices are inverse as in the following example .

Ex : A=$\begin{bmatrix}1 & 0 & 2\\ 2 & -1 & 3\\ 4 & 1 & 8\end{bmatrix}$ and B=$\begin{bmatrix}-11 & 2 & 2\\ -4 & 0 & 1\\ 6 & -1 & -1\end{bmatrix}$ A.B=$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

Thus the two matrices are invertible and are inverses of each other .

Note :-If A and B are two square matrices with the same size then

$$(A\,B)^{-1} = B^{-1}.A^{-1}$$

Determinants

To each n- square matrix A =[a_{ij}] we assign aspecific real number called the determinant of A ,denoted by det(A) or | A | where

$$|\ A\ | = \begin{vmatrix} a11 & a12 \ldots.. & a1n \\ a21 & a22 \ldots.. & a2n \\ an1 & an2 \ldots.. & ann \end{vmatrix}$$

The determinants of order one ,two and three are defined as follows:

1) | $a11$ |=a11

2) $\overset{+}{}\begin{vmatrix} a11 & a12 \\ a21 & a22 \end{vmatrix}\overset{-}{}$ =a11 a22 _a12 a21

3) $\begin{vmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix}$ = $\left|\begin{matrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{matrix}\right|\begin{matrix} a11 & a12 \\ a21 & a22 \\ a31 & a32 \end{matrix}$

=a11 a22 a33 +a12 a23 a31+a13 a21 a32 – a13 a22 a31 –a11 a23 a32 –a12 a21 a33

Ex:-1) | 6 |=6

2)$\begin{vmatrix} 2 & 1 \\ -4 & 6 \end{vmatrix}$ =12 +4=16 3) $\begin{vmatrix} 2 & 1 & 3 \\ 4 & 6 & -1 \\ 5 & 1 & 0 \end{vmatrix}$ =-81

Properties of determinants:-

1)The determinants of matrix and its transpose are equal

That is $|\ A\ |=|\ A^T\ |$

Ex: Suppose that A=$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$,then$|\ A\ |=|\ A^T\ |$=6

2)If matrix B result from matrix A by interchanging two rows (columns) of A ,then $|\ B\ |=-|\ A\ |$

Ex: A=$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, $|\ A\ |=\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}$=4+3=7

Suppose that B=$\begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}$,then$|\ B\ |=-|\ \begin{matrix} 3 & 2 \\ 2 & -1 \end{matrix}|$=-7

3)If two rows (columns)of A are equal , then $|\ A\ |$=0

Ex:A=$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 7 \\ 1 & 2 & 3 \end{bmatrix}$,then$|\ A\ |$=0

4)If arow(column)of A consists entirely of zero ,then $|\ A\ |=0$

Ex:A =$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$=$|\ A\ |=0$

5)If B is obtained from A by multiplying arow (column) of A by areal number

c ,then $|\ B\ |$ =c$|\ A\ |$

Ex:Suppose that B=$\begin{bmatrix} 2 & -1 \\ 3 & 9 \end{bmatrix}$ and A=$\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

3.$|\ A\ |$=3 .7 =21

$|B|$=21,Observe that $|B|$=3. $|A|$=21.

6)If amatrix A=[a_{ij}] is upper (lower) triangular matrix then

$$|A| = a_{11}\, a_{22} \,\ldots\ldots a_{nn}$$

That is , the determinant of atriangular matrix is the product of the elements on the main diagonal.

Ex:A=$\begin{bmatrix} 7 & 5 & 12 \\ 0 & 1 & 18 \\ 0 & 0 & 6 \end{bmatrix}$,then$|A|$ =7.1 .6 =42

7)The determinant of aproduct of two matrices is the product of their determinants ,that is, $|AB| = |A| . |B|$

Ex:let A =$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$,then $|A|$ =-2 , $|B|$=5

$|A||B|$ =-2 .5 =-10,also

A .B =$\begin{bmatrix} 4 & 3 \\ 10 & 5 \end{bmatrix}$, $|AB| =-\begin{vmatrix} 4 & 3 \\ 10 & 5 \end{vmatrix}$ =20 -30 =-10

Observe that , $|AB| = |A| . |B|$ =-10

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ا.م. عهود سعدي عبد الكريم الحسني

الكورس الثاني

Discrete mathematics

Matrix

Minors and cofactors of the elements of matrix

Let A=[a_{ij}] be an n x n matrix .let M_{ij} be the (n-1)x(n-1) sub matrix of A obtained by deleting the i-th row and j-th column of A .The determinants $|M_{ij}|$ is called minor of a_{ij}.

Ex: let A = $\begin{bmatrix} 5 & 2 & 1 \\ 0 & 4 & 3 \\ -1 & 7 & 8 \end{bmatrix}$

We are disply the minors of a_{ij} of amatrix A

The minor of a_{11} is $|M_{11}| = \begin{bmatrix} 4 & 3 \\ 7 & 8 \end{bmatrix}$

The minor of a_{12} is $|M_{12}| = \begin{bmatrix} 0 & 3 \\ -1 & 8 \end{bmatrix}$

The minor of a_{13} is $|M_{13}| = \begin{bmatrix} 0 & 4 \\ -1 & 7 \end{bmatrix}$

The minor of a_{21} is $|M_{21}| = \begin{bmatrix} 2 & 1 \\ 7 & 8 \end{bmatrix}$

The minor of a_{22} is $|M_{22}| = \begin{bmatrix} 5 & 1 \\ -1 & 8 \end{bmatrix}$

The minor of a_{23} is $|M_{23}| = \begin{bmatrix} 5 & 2 \\ -1 & 7 \end{bmatrix}$

The minor of a_{31} is $|M_{31}| = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$

The minor of a_{32} is $|M_{32}| = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}$

The minor of a_{33} is $|M_{33}| = \begin{bmatrix} 5 & 2 \\ 0 & 4 \end{bmatrix}$

Cofactors and the matrix of cofactors

The cofactor $\boldsymbol{\alpha}_{ij}$ of a_{ij} is defined as $\boldsymbol{\alpha}_{ij}= (- 1)^{i+j} \, |M_{ij}|$

and the matrix of cofactors of a_{ij} is defined as

$$C(A)=[\,\boldsymbol{\alpha}_{ij}]=\begin{bmatrix}\alpha 11 & \alpha 12\ \ldots\ldots & \alpha 1n\\ \alpha 21 & \alpha 22\ \ldots\ldots & \alpha 2n\\ \alpha n1 & \alpha n2\ \ldots\ldots & \alpha nn\end{bmatrix}$$

Ex: let $A=\begin{bmatrix}3 & -2 & 1\\ 5 & 6 & 2\\ 1 & 0 & -3\end{bmatrix}$,compute C(A) ?

Sol.:- The cofactors of A are :

$\alpha 11=(-1)^{1+1}|M_{11}|=\begin{vmatrix}6 & 2\\ 0 & -3\end{vmatrix}=-18-0=-18$

$\alpha 12=(-1)^{1+2}|M_{12}|=(-1)^{3}\begin{vmatrix}5 & 2\\ 1 & -3\end{vmatrix}=-[-15-2]=17$

$\alpha 13=(-1)^{1+3}|M_{13}|=(-1)^{4}\begin{vmatrix}5 & 6\\ 1 & 0\end{vmatrix}=0-6=-6$

$\alpha 21=(-1)^{2+1}|\ M_{21}|=(-1)^{3}\begin{vmatrix}-2 & 1\\ 0 & -3\end{vmatrix}=-[6-0]=-6$

$\alpha 22=(-1)^{2+2}|\ M_{22}\ |=(-1)^{4}\begin{vmatrix}3 & 1\\ 1 & -3\end{vmatrix}=-9-1=-10$

$\alpha 23=(-1)^{2+3}\,|M_{23}|=(-1)^{5}\begin{vmatrix}3 & -2\\ 1 & -0\end{vmatrix}=-[0+2]=-2$

$\alpha 31=(-1)^{3+1}\ |M_{31}|=(-1)^{4}\begin{vmatrix}-2 & 1\\ 6 & 2\end{vmatrix}=-4-6=-10$

$\alpha 32=(-1)^{3+2}\,|M_{32}|=(-1)^{5}\begin{vmatrix}3 & 1\\ 5 & 2\end{vmatrix}=-[6-5]=-1$

$\alpha 33=(-1)^{3+3}\,|M_{33}|=(-1)^{6}\begin{vmatrix}3 & -2\\ 5 & 6\end{vmatrix}=18+10=28$

So $C(A)=\begin{bmatrix}-18 & 17 & -6\\ -6 & -10 & -2\\ -10 & -1 & 28\end{bmatrix}$

Def: adjoint matrix:

Let $A=[a_{ij}]$ be an nx n matrix .The n x n matrix ,adj (A), called adjoint of A ,is the matrix whose i , j th element is the cofactor $\boldsymbol{\alpha}_{ji}$ of a_{ji} ,that is $adj(A)=[\,C\,(A)]^{T}$

Thus adj (A)= $\begin{bmatrix} \alpha 11 & \alpha 21 \ldots\ldots & \alpha n1 \\ \alpha 12 & \alpha 22 \ldots\ldots & \alpha n2 \\ \alpha 1n & \alpha 2n \ldots\ldots & \alpha nn \end{bmatrix}$

Ex:from the previous example ,the adj(A) of amatrix A

A = $\begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}$, adj (A) =[C (A)]^T= $\begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}$

Notes:1)If A and B are n x n matrix ,then adj (A B) =adj (A). adj (B)

2)The magnitude of any determinant for agiven matrix is define by multiplying the elements of any row (column)by its cofactors ,that is ,

$|\ A\ | = \sum_{k=1}^{n} aik\ \alpha$ i k =a i1 α i1 + a i 2 α i 2 +…….+a i n α i n [expansion of $|\ A\ |$ about the i-th row]

$|\ \mathbf{A}\ | = \sum_{k=1}^{n} \mathbf{akj}\ \alpha$ k j =a 1j α 1j + a 2j α2j +…….+a nj αnj [expansion of $|\ \mathbf{A}\ |$ about the j-th column]

Ex:find the value determinant of A = $\begin{bmatrix} 2 & -1 & 3 \\ 5 & 1 & -2 \\ 1 & 2 & -1 \end{bmatrix}$ by using

$|\ A\ | = \sum_{k=1}^{n} aik\ \alpha$ i k

Sol. : $|\ \mathbf{A}\ |$ =a$_{11}$ α_{11} +a$_{12}$ α_{12} +a$_{13}$ α_{13}

=a$_{11}$.(-1)$^{1+1}$$|\ \mathbf{M11}\ |$ +a$_{12}$.(-1)$^{1+2}$ $|\ \mathbf{M12}\ |$ +a$_{13}$.(-1)$^{1+3}$. $|\ \mathbf{M13}\ |$

=2.(-1)^2 . $\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix}$ +(-1).(-1)^3. $\begin{vmatrix} 5 & -2 \\ 1 & -1 \end{vmatrix}$+3 .(-1)^4. $\begin{vmatrix} 5 & 1 \\ 1 & 2 \end{vmatrix}$

=2[-1+4]+[-5 +2]+3[10 -1]

=6-3+27=30

Note:If A=[a$_{ij}$]is an n x n matrix,then ai1 αk1 + ai2 αk2 +.....+ ain αkn = | A | =0 for i$\neq$ k

or

a1j α1k + a2j α2k+.....+ anj αnk = | A | =0 for j$\neq$ k

Ex:find the value of determinant of A = $\begin{bmatrix} 2 & -1 & 3 \\ 5 & 1 & -2 \\ 1 & 2 & -1 \end{bmatrix}$ by using

| A | =$\sum_{k=1}^{n} aik\ \alpha$ i k

Sol. : | A | =$a_{11}\ \alpha_{11}$ +$a_{12}\ \alpha_{12}$ +$a_{13}\ \alpha_{13}$

=a_{11} .$(-1)^{1+1}$| M11 | +a_{12} .$(-1)^{1+2}$ | M12 | +a_{13} .$(-1)^{1+3}$. | M13 |

=2.$(-1)^2$. $\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix}$ +(-1).$(-1)^3$. $\begin{vmatrix} 5 & -2 \\ 1 & -1 \end{vmatrix}$+3 .$(-1)^4$. $\begin{vmatrix} 5 & 1 \\ 1 & 2 \end{vmatrix}$

=2[-1+4]+[-5 +2]+3[10 -1]

=6-3+27=30

Note:If A=[a_{ij}]is an n x n matrix,then

ai1 αk1 + ai2 αk2 +.....+ ain αkn = | A | =0 for i$\neq$ k

or

a1j α1k + a2j α2k+.....+ anj αnk = | A | =0 for j$\neq$ k

Apractical method for finding the inverse of n x n non- singular matrix

Def:(Singular matrix)المصفوفة الشاذة-غير المعتلة

An n x n matrix A is non –singular if and only if $|\ A\ | \neq 0$.

Otherwise ,amatrix A is singular if and only if $|\ A\ |=0$

In other words ,An n x n matrix A is called non –singular (or invertible)if there exist an n x n matrix B such that A B= B A=I_n.

The matrix B is called an inverse of A ,if there exists no such matrix B , then A is called singular(or non- invertible)

Properties of the inverse:

1)If A is a non singular matrix ,then $A^{-1}$is non –singular and $(A^{-1})^{-1}=A$

2)If A and B are non –singular matrices ,then AB is non –singular and

$$(AB)^{-1}=B^{-1}A^{-1}$$

3)If A is a non singular matrix ,then $(A^T)^{-1}=(A^{-1})^T$

4)If A is anon – singular matrix ,and $|\ A\ | \neq 0$ then

$$|A^{-1}| = \frac{1}{|\ A\ |}$$

Notes :1)If A and B be n x n matrices and if AB is singular ,then A or B must be singular.

2)If A is a non –singular matrix ($|\ A\ | \neq 0$) and A B =AC then B=C.

We will illustrate the following method for finding $A^{-1}$of non –singular matrix:-

1)The inverse of anon –singular diagonal matrix:(معكوس مصفوفة قطرية غير شاذة)

Suppose that A is an n x n diagonal matrix

$$A = \begin{bmatrix} k1 & 0 & 0 \\ 0 & k2 & 0 \\ 0 & 0 & kn \end{bmatrix}$$

Then $A^{-1}=\begin{bmatrix} \frac{1}{k1} & 0 & 0 \\ 0 & \frac{1}{K2} & 0 \\ 0 & 0 & \frac{1}{kn} \end{bmatrix}$

2)The inverse of an n x n matrix by using determinants and the cofactors matrix

(أيجاد المعكوس الى مصفوفة بأستخدام المحددات ومصفوفة العوامل المرافقة)

If A is an n x n matrix ,then the inverse of A is defined as

$$A^{-1}=\frac{1}{|A|}.adj(A)$$

<u>Note</u>:If A =[aij] is an n x n matrix ,then A (adj(A))=(adj(A)) A= $|A|. I_n$

Linear Systems

Consider the linear n equations in n- unknowns:

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+\dots\dots\dots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+\dots\dots\dots+a_{2n}x_n=b_2 \\ : \\ : \\ a_{n1}x_1+a_{n2}x_2+\dots\dots\dots+a_{nn}x_n=b_n \end{array}\right\}$$

now, define the following matrices

$$A=\begin{bmatrix} a11 & a12\dots\dots. & a1n \\ a21 & a22\dots\dots. & a2n \\ an1 & an2\dots\dots. & ann \end{bmatrix},\ X=\begin{bmatrix} x1 \\ x2 \\ xn \end{bmatrix}\ ,B=\begin{bmatrix} b1 \\ b2 \\ bn \end{bmatrix}$$

The linear system can be written in matrix form as AX=B

The matrix A is called the coefficient matrix of the linear system.

Alinear system of the form

$$\left.\begin{array}{l}
a_{11} x_1+a_{12} x_2+\ldots\ldots+a_{1n}x_n=0 \\
a_{21} x_1+a_{22} x_2+\ldots\ldots+a_{2n}x_n=0 \\
: \\
: \\
a_{n1} x_1+a_{n2} x_2+\ldots\ldots+a_{nn}x_n=0
\end{array}\right\}$$

is called ahomogeneous system .We can also write in amatrix form as $AX=0$

We have two methods to solve the linear system:

1)By using the inverse of coefficient matrix

$A X =B$

$A^{-1}(A X)=A^{-1}B$

$I^nX=X=A^{-1} B$(If $| \; A | \neq 0$)

We have an uniqe solution.

Ex:Solve the following linear system by using the inverse of coefficients matrix

$x1+x2+x3=8$

$\quad 2x2+3x3=24$

$5x1+5x2+x3=8$

The linear system can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix} \begin{matrix} x1 \\ x2 \\ xn \end{matrix} \begin{matrix} 8 \\ = 24 \\ 8 \end{matrix}$$

$X =A^{-1}B$, Then

$A^{-1}=\frac{1}{|A|}.adj(A)$

Therefore x1=0,x2=0,x3=8

Second(Cramer's –rule): If A is an n x n matrix ,then we can solve the linear system (1) AX=B

If $|\ A\ | \neq 0$,then the system has the unique solution

$x_1=\frac{|A1|}{|\ A\ |}$, $x_2 =\frac{|A2|}{|\ A\ |}$, ……….. $x_n=\frac{|A1|}{|\ A\ |}$

where Ai is the matrix obtained from A by replacing the i th column of A by B

……..

EX: consider the linear system

-2x1 +3x2-x3=1

x1 + 2x2 - x3=4

-2x1 - x2 + x3=-3 , find the solution of the linear system by using Cramer 's rule?

SOL. $|\ A\ | = -2$

x1=2 ,x2 =3 ,x3=4