Bessel's Functions

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MATHEMATICAL PHYSICS II

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1 - The Factorial Functions

Let $\alpha > 0$, then

$$\int_0^\infty e^{-\alpha x}\,dx = -\frac{1}{\alpha} e^{-\alpha x}\Big|_0^\infty = \frac{1}{\alpha}$$

differentiate both Sides w.r.t. α

$$\int_0^\infty -x e^{-\alpha x}\,dx = -\frac{1}{\alpha^2}$$

$$\Rightarrow \int_0^\infty x e^{-\alpha x}\,dx = \frac{1}{\alpha^2}$$

differentiate agin

$$\Rightarrow \int_0^\infty x^2 e^{-\alpha x}\,dx = \frac{2}{\alpha^3}$$

and agin $\Rightarrow \int_0^\infty x^3 e^{-\alpha x}\,dx = \frac{3!}{\alpha^4}$

In general $\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$

Put $\alpha = 1$ weget

$$\boxed{\int_0^\infty x^n e^{-x}\,dx = n!}$$ where $n = 1, 2, \ldots$

we can find $0!$ by Putting $n = 0$

$$\Rightarrow \int_0^\infty e^{-x}\,dx = 0!$$

$$-e^{-x}\Big|_0^\infty = 0!$$

$$-(0-1) = 0!$$

$$\boxed{1 = 0!}$$

2- Gamma Function

we can define the gamma function as

$$\Gamma p = \int_0^\infty x^{p-1} e^{-x}\, dx$$

Notice that

$$\Gamma p+1 = \int_0^\infty x^p e^{-x}\, dx \qquad \text{integrating by parts}$$

$$= -\cancel{e^{-x} x^p}\Big|_0^\infty + p\int_0^\infty e^{-x} x^{p-1}\, dx$$

$$= p\int_0^\infty x^{p-1} e^{-x}\, dx$$

$$= p\Gamma p \implies \boxed{\Gamma p+1 = p\Gamma p}$$

from this we obtain

$$\Gamma 2 = 1\,\Gamma 1 = 1\int_0^\infty e^{-x} dx = 1$$

$$\Gamma 3 = 2\Gamma 2 = 2\cdot 1 = 2 = 2!$$

$$\Gamma 4 = 3\Gamma 3 = 3\cdot 2\cdot 1 = 6 = 3!$$

$$\vdots$$

$$\Gamma n = (n-1)!$$

Moreover,

$$\Gamma 2.5 = (1.5)\Gamma 1.5 = (1.5)(0.5)\Gamma 0.5$$

and to find the value of gamma of number between 0 and 1 we use

$$\Gamma p = \frac{1}{p}\Gamma p+1 \qquad 0<p<1$$

for example $\Gamma 0.5 = \frac{1}{0.5}\Gamma 1.5$

It is possible to extend the domain of Γp to negative values of p

$$\Gamma(p+1) = p\,\Gamma p \quad \text{----}(1)$$

$$\Rightarrow \Gamma p = \frac{\Gamma(p+1)}{p}$$

$$\Rightarrow \Gamma 0 = \frac{\Gamma 1}{0} \rightsquigarrow \infty$$

Similarly $\Gamma(-1) = \frac{\Gamma 0}{-1} = \frac{\Gamma 1}{(-1)(0)} \rightsquigarrow \infty$

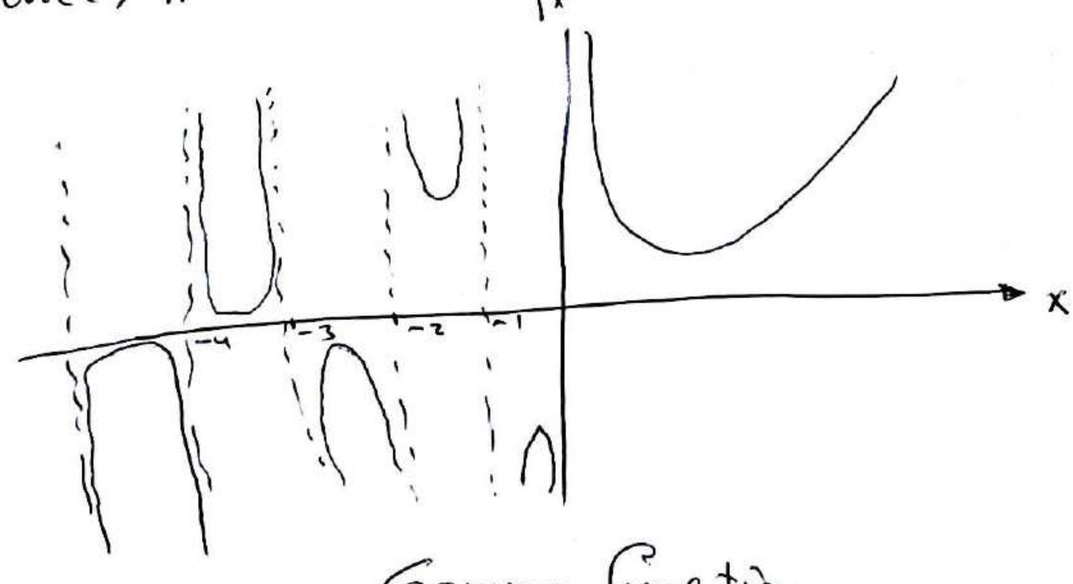
$$\Gamma(-2) = \frac{\Gamma(-1)}{-2} = \frac{\Gamma 1}{(-2)(-1)(0)} \longrightarrow \infty$$

For any other negative value of p, we can compute Γp using (1) until $\Gamma(p+1)$ has positive argument

Examples $\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = \frac{\Gamma\frac{1}{2}}{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)} = \frac{4}{3}\Gamma\frac{1}{2}$

$$\Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}\right)}{-\frac{5}{2}} = \frac{\Gamma\left(-\frac{1}{2}\right)}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)} = \frac{\Gamma\frac{1}{2}}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)} = -\frac{8}{15}\Gamma\frac{1}{2}$$

Hence, Γp is well defined for any $p \in \mathbb{R}$ except $x = 0, -1, -2, \ldots$



Gamma function

The Beta Function

The Beta Function is a two-parameter Composition of gamma functions that has been useful enough in application to gain its own name. Its definition is

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

If $x \geq 1$ and $y \geq 1$, this is a proper integral. If $x > 0$ and $y > 0$ and either or both $x < 1$ or $y < 1$, the integral is improper but Convergent

Properties of Beta function

1) $B(x,y) = B(y,x)$

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

use the tranformation $u = 1-t$

$\Rightarrow du = -dt$ and $t = 1-u$

$$\therefore B(x,y) = -\int_1^0 (1-u)^{x-1} u^{y-1}\,du$$

$$= \int_0^1 u^{y-1}(1-u)^{x-1}\,du = B(y,x) \quad \square$$

2) $B(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta \, \cos^{2y-1}\theta \, d\theta$

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

use the transformation $t = \sin^2\theta$

$\Rightarrow dt = 2\sin\theta\cos\theta\, d\theta$

$$\therefore B(x,y) = \int_0^{\pi/2} (\sin^2\theta)^{x-1}(1-\sin^2\theta)^{y-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} (\sin^2\theta)^{x-1}(\cos^2\theta)^{y-1}\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2x-2}\theta \cos^{2y-2}\theta \sin\theta \cos\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2y-1}\theta \, d\theta \quad \square$$

3) $$B(x,y) = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}} \, dt$$

$$B(x,y) = \int_0^{1} t^{x-1}(1-t)^{y-1} \, dt$$

Put $t = \frac{u}{1+u} \Rightarrow dt = \frac{(1+u)du - udu}{(1+u)^2} = \frac{du}{(1+u)^2}$

$$\therefore B(x,y) = \int_0^{\infty} \left(\frac{u}{1+u}\right)^{x-1} \left(1 - \frac{u}{1+u}\right)^{y-1} \frac{du}{(1+u)^2}$$

$$= \int_0^{\infty} \left(\frac{u}{1+u}\right)^{x-1} \left(\frac{1+u-u}{1+u}\right)^{y-1} \frac{du}{(1+u)^2}$$

$$= \int_0^{\infty} u^{x-1} \cdot \left(\frac{1}{1+u}\right)^{x-1} \left(\frac{1}{1+u}\right)^{y-1} \left(\frac{1}{1+u}\right)^{2} du$$

$$= \int_0^{\infty} u^{x-1} \cdot \left(\frac{1}{1+u}\right)^{x-1+y-1+2} du$$

$$= \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}} \, du$$

replace each u by t we get

$$B(x,y) = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}} \, dt \quad \square$$

4) $B(m,n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma m+n}$

$$\Gamma m = \int_0^\infty t^{m-1} e^{-t} dt$$

let $t = x^2 \Rightarrow dt = 2x\,dx$

$$\Rightarrow \Gamma m = \int_0^\infty (x^2)^{m-1} e^{-x^2} \cdot 2x\,dx$$

$$= 2\int_0^\infty x^{2m-2+1} e^{-x^2} dx$$

$$= 2\int_0^\infty x^{2m-1} e^{-x^2} dx$$

Similarly,

$$\Gamma n = 2\int_0^\infty y^{2n-1} e^{-y^2} dy$$

$$\Rightarrow \Gamma m \, \Gamma n = 4\int_0^\infty e^{-x^2} x^{2m-1} dx \cdot \int_0^\infty e^{-y^2} y^{2n-1} dy$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dx\,dy$$

Put $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow \Gamma m \, \Gamma n = 4 \int_0^\infty \int_0^{\pi/2} e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} r\,d\theta\,dr$$

$$= 4\int_0^\infty e^{-r^2} r^{2m-1+2n-1+1} dr \cdot \int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta$$

$$= 4\int_0^\infty e^{-r^2} r^{2m+2n-1} dr \int_0^{\pi/2} \cos\theta^{2m-1} \sin\theta^{2n-1} d\theta$$

$$= \underline{\underline{(2)}}(2)\int_0^\infty e^{-r^2} r^{2(m+n)-1} dr \int_0^{\pi/2} \cos\theta^{2m-1} \sin\theta^{2n-1} d\theta$$

$$= \Gamma m+n \; B(m,n)$$

Series Solutions of ODEs

- Power Series Method

The power Series method is the standard for solving linear ODEs with variable coefficients. It gives solutions in the form of power series. These series can be used for computing values, graphing curves, proving formulas, and exploring properties of solutions.

From Calculus we remember that a power series (in powers of $x - x_0$) is an infinite series of the form

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots$$

Here, x is a variable. $a_0, a_1, a_2, \ldots$ are constants, called the coefficients of the series. x_0 is a constant, called the center of the series. In particular, if $x_0 = 0$, we obtained a power series in powers of x

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots$$

we shall assume tha all variables and constants are real, and m is positive integer (neither negative nor fractional

Example 1 Familiar power Series are the Maclaurin Series

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \quad (|x|<1,\ \text{geometric Series})$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

The basic idea of the power Series method for Solving differention equations is very Simple as we can See in the next example,

Example 2 Solve $y' - y = 0$

let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \cdots = \sum_{m=0}^{\infty} a_m x^m$

$$\Rightarrow y' = a_1 + 2a_n x + 3a_3 x^2 + \cdots = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

Substitute in the original equation

$$(a_1 + 2a_2 x + 3a_3 x^2 + \cdots) - (a_0 + a_1 x + a_2 x^2 + \cdots) = 0$$

$$\Rightarrow (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \cdots = 0$$

Equating the Coefficient of each power of x to Zero, we have

$$a_1 - a_0 = 0\ ,\ 2a_2 - a_1 = 0\ ,\ 3a_3 - a_2 = 0 \cdots$$

$$\Rightarrow a_1 = a_0 \ , \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2!} \ , \ a_3 = \frac{a_2}{3} = \frac{a_0}{3!} \ , \ldots$$

$$\Rightarrow y = a_0 + a_1 x + a_2 x^2 + \cdots = a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$= a_0 e^x$$

<u>Example 3</u> Solve $y'' + y = 0$

$$\text{let } y = \sum_{m=0}^{\infty} a_m x^m$$

$$\Rightarrow y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$\Rightarrow y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

$$\therefore \quad \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$2(1) a_2 + 3(2) a_3 x + 4(3) a_4 x^2 + \cdots + a_0 + a_1 x + a_2 x^2 + \cdots = 0$$

$$(2a_2 + a_0) + (6a_3 + a_1) x + (12 a_4 + a_2) x^2 + \cdots = 0$$

$$2a_2 + a_0 = 0 \Rightarrow a_2 = -\frac{a_0}{2!}$$

$$6a_3 + a_1 = 0 \Rightarrow a_3 = -\frac{a_1}{6} = -\frac{a_1}{3!}$$

$$12 a_4 + a_2 = 0 \Rightarrow a_4 = -\frac{a_2}{12} = \frac{a_2}{4!} \ , \ldots$$

a_0 and a_1 are arbitrary

$$y = a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 + \cdots$$

$$= a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots\right) + a_1 \left(x - \frac{x^2}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$y = a_0 \cos x + a_1 \sin x$$

Legendre's Equation, Legendre Polynomials $P_n(x)$

Legendre's differential equation

$$(1-x^2)y''-2xy'+n(n+1)y=0 \quad \text{---- (1)}$$
(n Constant)

is one of the most important ODEs in physics. It arises in numerous problems, particularly in boundary value porblems for spheres.

The equation involves a parameter n, whose value depends on the physical or engineering problem. So (1) is actually a whole family of ODEs. For $n=1$ we solved it in last section. Any solution of (1) is called a Legendre function. The study of these and other "higher" functions is called the theory of special functions.

Deviding (1) by $1-x^2$, we obtain the standard form needed in the perivriouse theorem and we see that the coefficients $\frac{-2x}{(1-x^2)}$ and $\frac{n(n+1)}{(1-x^2)}$ of the new equation are analytic at $x=0$, so that we may apply the power series method, substituting

$$y=\sum_{m=0}^{\infty} a_m x^m \quad \text{---- (2)}$$

and its derivatives into (1), and denoting the constant $n(n+1)$ simply by k, we obtain

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} + k\sum_{m=0}^{\infty} a_m x^m = 0$$

By writing the first expression as two separate series we have the equation

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0$$

To obtain the general power x^s in all four series, set $m-2=s$ (thus $m=s+2$) in the first series and simply write s instead of m in the other three series. This gives

$$\sum_{s=0}^{\infty} (s+2)(s+1)a_{s+2} x^s - \sum_{s=2}^{\infty} s(s-1)a_s x^s - \sum_{s=1}^{\infty} 2s a_s x^s + \sum_{s=0}^{\infty} k a_s x^s = 0$$

Now,

$$2\cdot 1 a_2 + 3\cdot 2 a_3 x^1 + \sum_{s=2}^{\infty}(s+2)(s+1)a_{s+2}x^s - \sum_{s=2}^{\infty} s(s-1)a_s x^s$$
$$-2\cdot 1 a_1 x - \sum_{s=2}^{\infty} 2s a_s x^s + k a_0 + k a_1 x + \sum_{s=2}^{\infty} k a_s x^s = 0$$

$$\Rightarrow 2\cdot 1 a_2 + k a_0 + (3\cdot 2 a_3 - 2\cdot 1 a_1 + k a_1)x +$$
$$\sum_{s=2}^{\infty}\left[(s+2)(s+1)a_{s+2} + (-s(s-1) - 2s + k)a_s\right]x^s = 0$$

$$\therefore \quad 2\cdot 1 a_2 + k a_0 = 0$$

$$2\cdot 1 a_2 + n(n+1) = 0$$

$$\Rightarrow a_2 = -\frac{n(n+1)}{2!}a_0 \qquad \cdots\cdots (3a)$$

and

$$3\cdot 2a_3 + \left[-2\cdot 1 + n(n+1)\right] a_1 = 0$$

$$\Rightarrow a_3 = -\frac{-2+n^2+n}{3!}a_1 = -\frac{n^2+n-2}{3!}a_1$$

$$a_3 = -\frac{(n-1)(n+2)}{3!}a_1 \qquad ---- \quad (3b)$$

and

$$(s+2)(s+1)a_{s+2} + \underbrace{\left[-s(s-1)-2s+n(n+1)\right]} a_s = 0 \quad ---- (3c)$$

$$\begin{aligned} & -s^2+s-2s+n^2+n \\ &= -s^2-s+n^2+n \\ &= n-s+n^2-s^2 \\ &= n-s+(n-s)(n+s) \\ &= (n-s)(n+s+1) \end{aligned}$$

$$\Rightarrow \boxed{a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)}a_s \qquad (s=0,1,2,\cdots)} \quad (4)$$

This is called a recurrence relation. It gives each coefficient in terms of the second one preceding it, except for a_0 and a_1, which are left as arbitrary constants. We find successively

$$a_2 = -\frac{n(n+1)}{2!}a_0 \qquad\qquad a_3 = -\frac{(n-1)(n+2)}{3!}a_1$$

$$a_4 = -\frac{(n-2)(n+3)}{4\cdot 3}a_2 \qquad\qquad a_5 = -\frac{(n-3)(n+4)}{5\cdot 4}a_3$$

$$= \frac{(n-2)n(n+1)(n+3)}{4!}a_0 \qquad\qquad = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1$$

and so on

Extended power Series Method: Frobenius Method

Theorem 1: (Frobenius Method)

Let $b(x)$ and $c(x)$ be any functions that are analytic at $x=0$. Then the ODE

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \qquad \text{---(1)}$$

has at least one Solution that Can be represented in the form

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \quad (a_0 \neq 0) \qquad \text{---(2)}$$

where the exponent r may be any (real or complex) number (and r is chosen so that $a_0 \neq 0$).

The ODE (1) also has a second Solution (Such that these two Solutions are linearly independent) that may be Similar to (2) (with a different r and diffrent Coefficients) or may Contain a logarithmic term.

For example, Bessel's equation

$$y'' + \frac{1}{x} y' + \left(\frac{x^2 - \nu^2}{x^2}\right) y = 0 \qquad (\nu \text{ a parameter})$$

is of the form (1) with $b(x) = 1$ and $c(x) = x^2 - \nu^2$ analytic at $x=0$, So that the theorem applies. This ODE Could not be handled in full generality by the power Series method.

Regular and Singular Points

The following terms are practical and Commonly used. A regular point of the ODE

$$y'' + p(x)y' + q(x)y = 0$$

is a point x_0 at which the coefficients p and q are analytic. Similarly, a regular point of the ODE

$$\tilde{h}(x)y'' + \tilde{p}(x)y' + \tilde{q}(x)y = 0$$

is an x_0 at which $\tilde{h}, \tilde{p}, \tilde{q}$ are analytic and $\tilde{h}(x_0) \neq 0$ (so what we can divide by $\tilde{h}$ and get the previous standard form). Then the power series method can be applied. If x_0 is not a regular point, it is called a <u>singular point</u>.

<u>Example</u>

$$2x^2y'' + 3xy' - (x^2+1)y = 0$$

$$y'' + \frac{3x}{2x^2}y' - \frac{x^2+1}{2x^2}y = 0$$

$$y'' + p(x)y' + Q(x)y = 0$$

$$p(x) = \frac{3}{2x} \qquad Q(x) = -\frac{x^2+1}{2x^2}$$

If $p(0)$ or $Q(0)$ is undefined $\rightarrow$ $x=0$ is a singular point.

There are two types of singular point

1. regular singular point (we can use Frob-meth.)
2. Irregular singular point (we cannot use Frob. meth)

Now, how we can test the singular point. Rewrite $p(x)$ and $Q(x)$ as

$$p(x) = \frac{3}{2x} = \frac{\overbrace{(3/2)}^{p(x)}}{x} \quad \text{and} \quad Q(x) = -\frac{x^2+1}{2x^2} = \frac{\overbrace{\left(-\frac{x^2+1}{2}\right)}^{q(x)}}{x^2}$$

if $p(0)$ and $q(0)$ is defined we say $x=0$ is regular Singular point $\Rightarrow$ we can use Frobenius method.

Summary

Suppose $y'' + P(x)y' + Q(x)y = 0$

has $P(0)$ or $Q(0)$ undefined (i.e. singularity)

Then define

$$p(x) = xP(x) \quad , \quad q(x) = x^2 Q(x)$$

If $p_0 = p(0)$, $q_0 = q(0)$ exist , then it is regular Singular point and we have

$$r(r-1) + p_0 r + q_0 = 0 \qquad \text{(indicial equation)}$$

Indicial Equation, Indicating the Form of Solutions

we shall now explain the Frobenius method for solving (1). Multiplication of (1) by x^2 gives the more Convention form

$$x^2 y'' + x b(x) y' + c(x) y = 0 \quad \text{---- } (1')$$

We first expand $b(x)$ and $c(x)$ in power Series

$$b(x) = p_0 + p_1 x + p_2 x^2 + \cdots , \qquad c(x) = q_0 + q_1 x + q_2 x^2 + \cdots$$

or we do nothing if $b(x)$ and $c(x)$ are polynomials. Then we differentiate (2) term by term, finding

$$y'(x) = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} = x^{r-1}\left[r a_0 + (r+1) a_1 x + \cdots \right]$$

$$y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}$$

$$= x^{r-2}\left[r(r-1) a_0 + (r+1) r a_1 x + \cdots \right]$$

The Series Solution of Bessel's Equation

One of the most important of all variable-coefficient differential equations is

$$x^2y''+xy'+(\lambda^2x^2-\nu^2)y=0 \qquad \cdots (1)$$

which is known as Bessel's equation of order ν with a parameter λ.

To simplify equation (1) we will use $\boxed{t=\lambda x}$

$$\Rightarrow \frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\lambda\frac{dy}{dt}$$

and $$\frac{d^2y}{dx^2}=\frac{dy/dx}{dt}\cdot\frac{dt}{dx}=\lambda^2\frac{d^2y}{dt^2}$$

$$\Rightarrow \qquad t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+(t^2-\nu^2)y=0 \qquad \cdots (2)$$

which is known simply as Bessel's equation of order ν.

Now, from the last section (Frobenius Method) we have

$$P(t)=\frac{1}{t} \quad \text{and} \quad Q(t)=\frac{t^2-\nu^2}{t^2}$$

$$\Rightarrow p=tP(t)=1 \qquad \text{and}\ q=t^2Q(t)=t^2-\nu^2$$

$$\Rightarrow p_0=p(0)=1 \qquad \text{and}\ q_0=q(0)=-\nu^2$$

using the indicial equation

$$r(r-1)+p_0r+q_0=0$$

$$\Rightarrow r^2-r+(1)r+(-\nu^2)=0$$

$$\Rightarrow r^2-\nu^2=0$$

$$\Rightarrow r_1=\nu \quad \text{and} \quad r_2=-\nu$$

it is clear that the origin is regular singular point and here we can use the Frobenius method

Let $y_\nu = \sum_{k=0}^{\infty} a_k t^{\nu+k}$

$\Rightarrow y'_\nu = \sum_{k=0}^{\infty} (\nu+k) a_k t^{\nu+k-1}$ & $y''_\nu = \sum_{k=0}^{\infty} (\nu+k)(\nu+k-1) a_k t^{\nu+k-2}$

where $r = \nu$ $(\nu \geq 0)$. Substituting these in the original equation

$$t^2 \sum_{k=0}^{\infty} (\nu+k)(\nu+k-1) a_k t^{\nu+k-2} + t \sum_{k=0}^{\infty} (\nu+k) a_k t^{\nu+k-1} + (t^2-\nu^2) \sum_{k=0}^{\infty} a_k t^{\nu+k} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k \left[(\nu+k)(\nu+k-1) + (\nu+k) - \nu^2\right] t^{\nu+k} + \sum_{k=0}^{\infty} a_k t^{\nu+k+2} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k \, k(2\nu+k) t^{\nu+k} + \sum_{k=0}^{\infty} a_k t^{\nu+k+2} = 0$$

$(\nu+k)(\nu+k-1)+(\nu+k)-\nu^2$
$= (\nu+k)(\nu+k-1+1)-\nu^2$
$= (\nu+k)^2 - \nu^2$
$= \nu^2 + 2\nu k + k^2 - \nu^2$
$= 2\nu k + k^2 = k(2\nu+k)$

here we sub. $k-2$ instead of each k

$$\Rightarrow \sum_{k=0}^{\infty} a_k \, k(2\nu+k) t^{\nu+k} + \sum_{k=2}^{\infty} a_{k-2} t^{\nu+k} = 0$$

$$\Rightarrow a_1(2\nu+1) t^{\nu+1} + \sum_{k=2}^{\infty} a_k \, k(2\nu+k) t^{\nu+k} + \sum_{k=2}^{\infty} a_{k-2} t^{\nu+k} = 0$$

$$\Rightarrow a_1(2\nu+1) t^{\nu+1} + \sum_{k=2}^{\infty} \left[k(2\nu+k) a_k + a_{k-2}\right] t^{\nu+k} = 0$$

Now, $\quad a_1(2\nu+1) = 0 \quad \text{---} \quad (3)$

and $\quad k(2\nu+k) a_k + a_{k-2} = 0 \quad \text{---} \quad (4)$

from (3) and the restriction that $\nu \geq 0$ it is clear that $\boxed{a_1 = 0}$

and from ④ we get

$$a_k = -\frac{a_{k-2}}{k(2v+k)} \qquad k = 2,3,4,\dots \quad ---⑤$$

Since $a_1 = 0 \Rightarrow a_3 = a_5 = a_7 = \dots = a_{2m+1} = 0$

and

$$a_2 = -\frac{a_0}{2(2v+2)} = -\frac{a_0}{2^2 \cdot 1!(v+1)}$$

$$a_4 = -\frac{a_2}{4(2v+4)} = -\frac{a_2}{2^2 \cdot 2(v+2)} = \frac{a_0}{2^4\, 2!\,(v+2)(v+1)}$$

$$a_6 = -\frac{a_4}{6(2v+6)} = -\frac{a_4}{2^2 \cdot 3\,(v+3)} = -\frac{a_0}{2^6 \cdot 3!\,(v+3)(v+2)(v+1)}$$

in general,

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} \cdot m!\,(v+m)(v+m-1)\cdots(v+3)(v+2)(v+1)}, \quad m = 1,2,3\dots$$

Now, a_{2m} is the coefficient of t^{v+2m} in the series of y_v. Hence it would probably be convenient if a_{2m} contained the factor 2^{v+2m}, so we will multiply a_{2m} by $\frac{2^v}{2^v}$

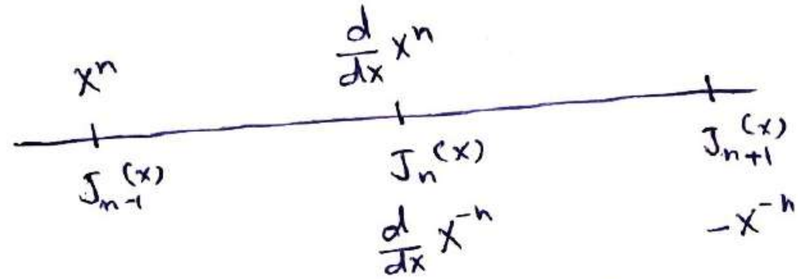
$$\Rightarrow a_{2m} = \frac{(-1)^m}{2^{2m+v} \cdot m!\,(v+m)(v+m-1)\cdots(v+3)(v+2)(v+1)}\,(2^v a_0) \quad ⑥$$

Here we will simplify the denominator using the following note

Recurrence Relations of Bessel's Functions

$\boxed{I}$ $\dfrac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

$\boxed{II}$ $\dfrac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$



If we integrate both sides of I we get

$$\int x^n J_{n-1}(x)\,dx = x^n J_n(x) + c \quad \cdots \quad \boxed{I'}$$

and if we integrate both sides of II we get

$$\int x^{-n} J_{n+1}(x)\,dx = -x^{-n} J_n(x) + c \quad \cdots \quad \boxed{II'}$$

Example Evaluate $\int J_3(x)\,dx$

if we multiply and divide the integrand by x^2, we have

$$\int x^2 \left[x^{-2} J_3(x)\right] dx$$

integrating by parts with

$u = x^2 \qquad dv = x^{-2} J_3(x)\,dx$

$du = 2x\,dx \qquad V = -x^{-2} J_2(x)$ --- Using $\boxed{II'}_{n=2}$

$$\therefore \int J_3(x) = x^2\left(-x^{-2} J_2(x)\right) + \int \left(x^{-2} J_2(x)\right)(2x)\,dx$$

$$= -J_2(x) + 2\int x^{-1} J_2(x)\,dx$$

$$= -J_2(x) - 2x^{-1} J_1(x) + c \quad \text{--- Using } \boxed{II'}_{n=1}$$

<u>Example</u> Evaluate $\int J_{-3}(x)\,dx$

$\because\ J_{-3}(x) = (-1)^3 J_3(x) \Rightarrow J_{-3}(x) = -J_3(x)$

$\Rightarrow J_{-3}(x) = J_2(x) + 2x^{-1} J_1(x) + c$ (By the previous example)

<u>Example</u> Evaluate $\int x^4 J_1(x)\,dx$

$$\int x^4 J_1(x)\,dx = \int x^2 (x^2 J_1)\,dx$$

$u = x^2$ $\qquad dv = x^2 J_1(x)\,dx$

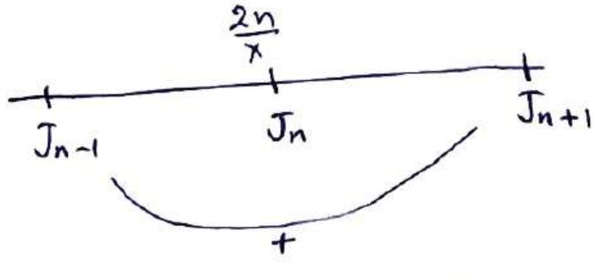
$du = 2x\,dx$ $\qquad v = x^2 J_2(x)$ $\qquad$ using $\boxed{I'}_{n=2}$

$$= x^2 (x^2 J_2(x)) - \int (x^2 J_2(x))(2x)\,dx$$

$$= x^4 J_2(x) - 2\int x^3 J_2(x)\,dx$$

$$= x^4 J_2(x) - 2x^3 J_3(x)\,dx$$ using $\boxed{I'}_{n=3}$

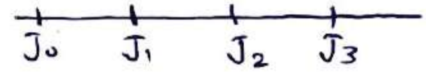
$\boxed{III}$ $\quad \frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$



<u>Example</u>: Express $J_3(x)$ in terms of $J_0(x)$ and $J_1(x)$

using $\boxed{III}$ we get

$J_{n+1} = \frac{2n}{x} J_n - J_{n-1}$



Put $n=2$

$$J_3 = \frac{4}{x} J_2 - J_1 \quad --- \text{①}$$

Similarly put $n=1$

$$J_2 = \frac{2}{x} J_1 - J_0 \quad --- \text{②}$$

Put ② in ①

$$J_3 = \frac{4}{x}\left[\frac{2}{x} J_1 - J_0\right] - J_1$$

$$= \frac{8}{x} J_1 - \frac{4}{x} J_0 - J_1$$

$$= \frac{8-x}{x} J_1 - \frac{4}{x} J_0$$

<u>Example</u> Express $J_{-\frac{5}{2}}$ interms of sin and Cos

$J_{-\frac{5}{2}} \quad J_{-\frac{3}{2}} \quad J_{-\frac{1}{2}} \quad J_{1/2}$

Using $\boxed{III}$ weget

$$J_{n-1} = \frac{2n}{x} J_n - J_{n+1}$$

if $n-1 = -\frac{5}{2} \longrightarrow n = -\frac{3}{2}$

$$\therefore J_{-\frac{5}{2}} = \frac{2\left(\frac{-3}{2}\right)}{x} J_{-\frac{3}{2}} - J_{-\frac{1}{2}}$$

$$J_{-\frac{5}{2}} = -\frac{3}{x} J_{-\frac{3}{2}} - J_{-\frac{1}{2}} \quad ---- \quad \text{①}$$

if $n-1 = -\frac{3}{2} \longrightarrow n = -\frac{1}{2}$

$$\therefore J_{-\frac{3}{2}} = \frac{2\left(-\frac{1}{2}\right)}{x} J_{-\frac{1}{2}} - J_{\frac{1}{2}}$$

$$J_{-\frac{3}{2}} = -\frac{1}{x} J_{-\frac{1}{2}} - J_{\frac{1}{2}} \quad --- \quad \text{②}$$

Bessel's Equation

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6

MATHEMATICAL PHYSICS II

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The Series Solution of Bessel’s Equation

One of the most important of all variable-coefficient differential equations is

$$x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0 \qquad \cdots (1)$$

which is known as Bessel's equation of order ν with a parameter λ.

To simplify equation (1) we will use $\boxed{t = \lambda x}$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \lambda \frac{dy}{dt}$$

and $$\frac{d^2y}{dx^2} = \frac{dy/dx}{dt} \cdot \frac{dt}{dx} = \lambda^2 \frac{d^2y}{dt^2}$$

$$\Rightarrow \qquad t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0 \qquad \cdots (2)$$

which is known simply as Bessel's equation of order ν.

Bessel's Functions

7

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Bessel's Functions

$$J_n = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

if $n=0$

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}$$

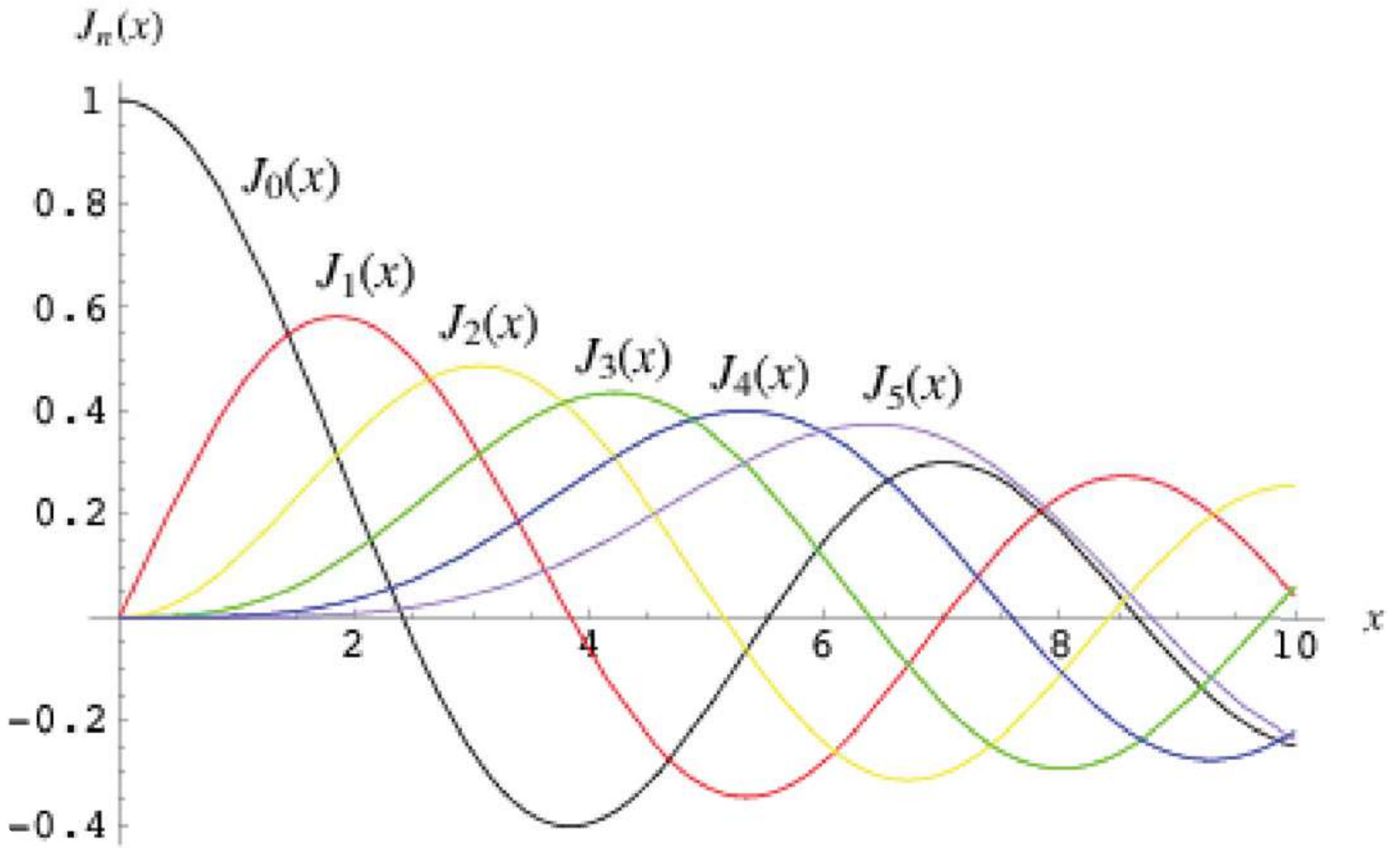
$$\Rightarrow J_0 = 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots$$

$$= 1 - \frac{x^2}{2^2 \cdot (1!)^2} + \frac{1}{2^4 \cdot (2!)^2} x^4 - \frac{x^6}{2^6 (3!)^2} + \cdots$$

if $n=1$

$$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+2)} \left(\frac{x}{2}\right)^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+1)!} \left(\frac{x}{2}\right)^{2k+1}$$

$$\Rightarrow J_1(x) = \frac{x}{2} - \frac{x^3}{2^3\, 1!\, 2!} + \frac{x^5}{2^5\, 2!\, 3!} - \frac{x^7}{2^7\, 3!\, 4!} + \cdots$$


$J_n(x)$
1
0.8
0.6
0.4
0.2
-0.2
-0.4
2
4
6
8
10
x
$J_0(x)$
$J_1(x)$
$J_2(x)$
$J_3(x)$
$J_4(x)$
$J_5(x)$

8

Examples on Bessel's Functions

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Example 1 Show that $J_{-\frac{1}{2}}(x) = J_{\frac{1}{2}}(x) \cot x$

Sol. we know that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad \text{and} \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

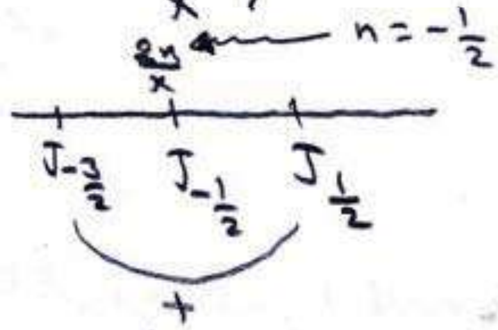
$$\therefore \quad \frac{J_{-\frac{1}{2}}(x)}{J_{\frac{1}{2}}(x)} = \frac{\cancel{\sqrt{\frac{2}{\pi x}}} \cos x}{\cancel{\sqrt{\frac{2}{\pi x}}} \sin x} = \cot x$$

$$\Rightarrow J_{-\frac{1}{2}}(x) = J_{\frac{1}{2}}(x) \cot x \quad \blacksquare$$

Example 2 Show that $J_{-\frac{3}{2}} = -\sqrt{\frac{2}{\pi x}}\left(\sin x + \frac{\cos x}{x}\right)$

Sol.

$$J_{-\frac{3}{2}}(x) = -\frac{1}{x} J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)$$



$$= -\frac{1}{x}\sqrt{\frac{2}{\pi x}}\cos x - \sqrt{\frac{2}{\pi x}}\sin x$$

$$= -\sqrt{\frac{2}{\pi x}}\left(\sin x + \frac{\cos x}{x}\right) \blacksquare$$