Bessel's Functions

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MATHEMATICAL PHYSICS I

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1. Method of forming Partial Differential Equation (PDE)

In a differential equation if there are two or more independant variables and the derivatives are partial derivatives then it is called a partial differential equation (PDE).

① $\left(\frac{\partial^2 z}{\partial x^2}\right)^2 + \left(\frac{\partial^2 z}{\partial y^2}\right)^3 = xy$ $\Big\{$ z - dependent variable; x and y - independent variables

② $x z_x + y z_y + t z_t = xyt$ $\Big\{$ z - dependet variable; x, y and t - independent variables

- The order of a PDE is the order of the highest partial derivative occuring in the equation.
- The degree of a PDE is the greatest exponent of the highest order.

So, ① above is a 2^{nd} order and 3^{rd} degree, while ② is 1^{st} order 1^{st} degree.

1.1. Forming of PDE by Elimination of Arbitrary Constants

Let $f(x, y, z, a, b) = 0$, where a, b are arbitrary constants in addition to this equation we can get two more equations by differentiating it with respect to x and y. Eliminating a and b from these three equations gives

$$\phi(x, y, z, z_x, z_y) = 0$$

which is a PDE

Note that:

- If the number of arbitrary Constants to be eliminated is equal to the number of independent Variables, we obtained a 1^{st} order PDE.
- If the number of arbitrary Constants to be eliminated is more than the number of independent Variables, we get PDE of 2^{nd} or higher order.

Example 1-1: Form a PDE by eliminating the arbitrary constants from $z = ax + by + a^2 + b^2$.

Solution:

$$z = ax + by + a^2 + b^2$$

$$z_x = a \quad , \quad z_y = b$$

$$\Rightarrow \boxed{z = x z_x + y z_y + z_x^2 + z_y^2}$$ 1^{st} order, 2^{nd} degree

Example 1-2: Form a PDE from $z = a^2x + b^2y + ab$

Solution:

$$z = a^2x + b^2y + ab$$

$$\frac{\partial z}{\partial x} = a^2 \quad , \quad \frac{\partial z}{\partial y} = b^2$$

$$\Rightarrow \boxed{z = x z_x + y z_y + \sqrt{z_x z_y}}$$ 1^{st} order, 2^{nd} degree

Example 1.3: Form a PDE from $z = (x+a)(y+b)$

Solution: $z_x = y + b \qquad z_y = x + a$

$$\Rightarrow \boxed{z = z_x z_y}$$ 1^{st} order, 1^{st} degree

Example 1.4: Find the differential equation of all sphere whose Centres lie on the z-axis.

Solution: Any point on the z-axis takes the form $(0,0,c)$

$$(x-0)^2+(y-0)^2+(z-c)^2=a^2 \quad \text{--- (1)}$$

Differentiating with respect to x, gives

$$2x+2(z-c)z_x=0 \Rightarrow x+(z-c)z_x=0 \quad \text{--- (2)}$$

Differentiating with respect to y, gives

$$2y+2(z-c)z_y=0 \Rightarrow y+(z-c)z_y=0 \quad \text{--- (3)}$$

from (1) we get $z-c=-\dfrac{x}{z_x}$

and from (2) we get $z-c=-\dfrac{y}{z_y}$

$$\Rightarrow -\frac{x}{z_x}=-\frac{y}{z_y} \Rightarrow \boxed{xz_y-yz_x=0}$$ 1^{st} order 1^{st} degree

Example 1.5: Form a partial Differential Equation from

$$\ln(az-1)=x+ay+b$$

Solution:

Diff. w.r.t. $x \Rightarrow \dfrac{1}{az-1}\cdot a\dfrac{\partial z}{\partial x}=1 \Rightarrow \dfrac{az_x}{az-1}=1$ --- (1)

Diff. w.r.t. $y \Rightarrow \dfrac{1}{az-1}\cdot a\dfrac{\partial z}{\partial y}=a \Rightarrow \dfrac{z_y}{az-1}=1$ --- (2)

(1) $\Rightarrow az-1=az_x$ and (2) $\Rightarrow az-1=z_y$

(1) & (2) $\Rightarrow \boxed{az_x=z_y}$ 1^{st} order, 1^{st} degree

Example 1.6: Form a PDE by eliminating a and b from

$$z=axe^y+\frac{1}{2}a^2e^{2y}+b$$

Solution: $z_x=ae^y$ ... (1) , $z_y=axe^y+\frac{1}{2}a^2e^{2y}(2)$

$$=axe^y+a^2e^{2y} \quad \text{--- (2)}$$

1.3 Formation of PDE by eliminating of arbitrary function ϕ from $\phi(u,v)=0$, where u and v are functions of x and y

Let $\phi(u,v)=0$, differentiating partially w.r.t. x and y, we get

$$\frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial x}=0 \quad ---(1)$$

$$\frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial y}=0 \quad ---(2)$$

To eliminate ϕ, it is enough to eliminate $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$

From (1) and (2), eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$

From (1) and (2), we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}=0$$

Example 1.14: Form the PDE by eliminating the arbitrary function ϕ from the relation $\phi(x^2+y^2+z^2, xyz)=0$

Solution:

let $u=x^2+y^2+z^2$, $v=xyz$

$\Rightarrow \phi(u,v)=0$

Now $\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}=0$

$u_x = 2x+2zz_x$ $\qquad$ $u_y=2y+2zz_y$

$v_x = xyz_x+yz$ $\qquad$ $v_y = xyz_y+xz$

$$\Rightarrow \begin{vmatrix} 2x+2zz_x & xyz_x+yz \\ 2y+2zz_y & xyz_y+xz \end{vmatrix}=0$$

$$\boxed{xz^2z_x-y^2-y(z^2-y^2)z_y = z(y^2-x^2)}$$

Example 1.15: Form a PDE from $f(x+y+z,\ xy+z^2)=0$

Solution:

let $u = x+y+z$, $v = xy+z^2$

$\Rightarrow \phi(u,v)=0$

$u_x = 1+z_x$ $\qquad u_y = 1+z_y$

$v_x = y+2zz_x$ $\qquad v_y = x+2zz_y$

$$\begin{vmatrix} 1+z_x & y+2zz_x \\ 1+z_y & x+2zz_y \end{vmatrix} = 0$$

$$\Rightarrow \boxed{(x-2z)z_x - (y-2z)z_y = y-x}$$

Example 1.16: Form a PDE from $g\left(\frac{y}{x},\ x^2+y^2+z^2\right)=0$

Solution: let $u = \frac{y}{x}$, $v = x^2+y^2+z^2$

$\Rightarrow g(u,v)=0$

$u_x = -\frac{y}{x^2}$, $u_y = \frac{1}{x}$

$v_x = 2x+2zz_x$, $v_y = 2y+2zz_y$

$$\begin{vmatrix} -\frac{y}{x^2} & 2x+2zz_x \\ \frac{1}{x} & 2y+2zz_y \end{vmatrix} = 0$$

$$-\frac{y}{x^2}(2y+2zz_y) - \frac{1}{x}(2x+2zz_x) = 0$$

$$y(2y+2zz_y) + x(2x+2zz_x) = 0$$

$$2y^2+2yzz_y+2x^2+2xzz_x = 0 \quad /2$$

$$\boxed{(xz_x+yz_y)z + x^2+y^2 = 0}$$

Exercises

Find the PDE by eliminating the arbitrary functions

1. $xyz = \phi(x+y+z)$
2. $z = x f\left(\frac{x}{y}\right)$
3. $z = f\left(\frac{xy}{z}\right)$
4. $z = x^2 + 2f\left(\frac{1}{y} + \ln x\right)$
5. $z = f(x+2y) + g(2x-y)$
6. $z = f(3x+2y) + g(3x+2y)$
7. $z = x f\left(\frac{y}{x}\right) + y\phi(x)$
8. $z = f(x+y)\cdot g(x-y)$
9. $z = f(x^2-y^2)$
10. $z = f(x+iy) + g(x-iy)$
11. $f(xy+z^2, x+y+z) = 0$
12. $z = y^2 + 2f(x + \log y)$
13. $z = x^2 f(y) + y^2 g(x)$
14. $z = f(ax+by) + g(\alpha x + \beta y)$
15. $\phi\left(z^2 - xy, \frac{x}{z}\right) = 0$

3. Solution of PDE by Direct Integration

A partial differential equation can be solved by successive integration in all cases where the dependent variable occurs only in the partial derivatives.

Example 3.1: Solve $\frac{\partial z}{\partial x} = \cos x$

Solution:

integrate with respect to x

$$z = \sin x + g(y)$$

Example 3.2: Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x$

Integrating w.r.t. "x": $\frac{\partial z}{\partial y} = -\cos x + f(y)$

Integrating w.r.t. "y": $z = -y\cos x + F(y) + g(x)$

where $F(y) = \int f(y)\,dy$

Example 3.3: Solve $\frac{\partial^2 z}{\partial x \partial y} = 0$

Integrating w.r.t. 'x': $\frac{\partial z}{\partial y} = f(y)$

Integrating w.r.t. 'y': $z = F(y) + g(x)$

Example 3.4: Solve $\frac{\partial^2 u}{\partial y \partial x} = 4x \sin(3xy)$

Solution:

Integrating w.r.t. 'y': $\frac{\partial u}{\partial x} = -\frac{4x\cos(3xy)}{3x} + f(x)$

$$= -\frac{4}{3}\cos(3xy) + f(x)$$

Integrating w.r.t. 'x': $u = -\frac{4}{3}\frac{\sin(3xy)}{3y} + F(x) + g(y)$

$$u = \frac{-4}{9y}\sin(3xy) + F(x) + g(y)$$

Example 3.5: Solve $\frac{\partial z}{\partial x} = 2x+3y$; $\frac{\partial z}{\partial y} = 3x+\cos y$

$\frac{\partial z}{\partial x} = 2x+3y$, Integrating w.r.t. 'x': $z = x^2+3xy+f(y)$

diffrentiating with respectto y we get:

$$\frac{\partial z}{\partial y} = 3x + f'(y)$$

but we know that

$$\frac{\partial z}{\partial y} = 3x+\cos y$$

$$\therefore f'(y) = \cos y \Rightarrow f(y) = \sin y + c$$

$$\therefore \boxed{z = x^2+3xy+\sin y+c}$$

Example 3-6: Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ for which $\frac{\partial z}{\partial y} = -2\sin y$ when $x=0$ and $z=0$ when y is an odd multiple of $\frac{\pi}{2}$.

Solution:
$$\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$$

Integrating w.r.t. "x": $\frac{\partial z}{\partial y} = -\cos x \sin y + f(y)$

given $\frac{\partial z}{\partial y} = -2\sin y$ when $x=0$

$$\Rightarrow -2\sin y = -\sin y + f(y) \Rightarrow f(y) = -\sin y$$

$$\therefore \frac{\partial z}{\partial y} = -\cos x \sin y - \sin y$$

Integrating w.r.t. 'y': $z = \cos x \cos y + \cos y + g(x)$

given $z=0$ when $y = (2n+1)\frac{\pi}{2}$

$$\Rightarrow g(x) = 0 \quad \left[\text{since } \cos(2n+1)\tfrac{\pi}{2} = 0\right]$$

$$\therefore z = (\cos x + 1)\cos y.$$

Example 3-7: Solve $\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$, given that $u=0$ when $t=0$ and $\frac{\partial u}{\partial t}=0$ when $x=0$. Show also that at $t \to \infty$, $u \to \sin x$.

Solution: $\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$

Integrating w.r.t. 'x' : $\frac{\partial u}{\partial t} = e^{-t}\sin x + f(t)$

$$\text{when } x=0,\ \frac{\partial u}{\partial t}=0 \Rightarrow 0 = f(t)$$

$$\therefore \frac{\partial u}{\partial t} = e^{-t}\sin x$$

Integrating w.r.t. 't': $u = -e^{-t}\sin x + g(x)$

$$\text{when } t=0,\ u=0 \Rightarrow 0 = -\sin x + g(x) \Rightarrow g(x) = \sin x$$

$$\therefore u = -e^{-t}\sin x + \sin x$$

$$\Rightarrow u = (1-e^{-t})\sin x$$

$$\text{as } t \to \infty \text{ we have } e^{-t} \to 0$$

$$\Rightarrow u \to \sin x$$

Exercises: Solve the following PDE using direct integration

1. $\frac{\partial z}{\partial x} = -\frac{x}{y}$

2. $\frac{\partial^2 z}{\partial x^2} = \sin y$

3. $\frac{\partial^2 z}{\partial x \partial y} = x^2 + y^2$

4. $\frac{\partial z}{\partial x} = 4x - 2y$; $\frac{\partial z}{\partial y} = -2x + 6y$

5. $\frac{\partial^2 z}{\partial y^2} - z = 0$; when $y=0$, $z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$

4. Solution of standard types of first order PDEs (non-linear)

A partial differential equation which involves only the first order partial derivatives $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$ is called a first order PDE.

The general form of 1^{st} order PDE is $f(x,y,z,p,q)=0$. We shall see some standard form of such equations and solve them by special methods.

4.1 $\boxed{\text{Type I}}$: $f(p,q)=0$ i.e. the equation contain p and q only

To find Complete integral

Given $f(p,q)=0$ --- ①

Let $z = ax+by+c$ be a trial solution of equation ①

Then $p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$

From ①, we get

$$f(a,b)=0$$

Hence the Complete integral of ① is

$$z = ax+by+c$$

Solving for b from $f(a,b)=0$, we get $b=\beta(a)$

$\therefore$ The Complete integral of ① is

$$z = ax+\beta(a)y+c \quad \text{--- ②}$$

Since [number of arbitrary constant (a,c) = number of independent variables $(x,y) = 2$].

To Find Singular Integral

To obtain the singular integral we have to eliminate a and c from equation $z = ax + \beta(a)y + c$, $\frac{\partial z}{\partial a} = 0$, $\frac{\partial z}{\partial c} = 0$

i.e. $z = ax + \beta(a)y + c$

$$x + \beta'(a)y = 0$$

$$1 = 0$$

The last equation being absurd and hence there is no singular integral

To find General Integral

Put $c = g(a)$ into complete integral ②, we get

$$z = ax + \beta(a)y + g(a) \quad \text{--- ③}$$

Differentiate ③ w.r.t. 'a', we get

$$z = x + \beta'(a)y + g'(a) \quad \text{--- ④}$$

Eliminating 'a' from ③ and ④, we get general integral of the given PDE.

Note: For the equation of the type $f(p,q) = 0$ there is No singular integral.

Example 4.1: Solve $pq = 1$

Solution:

To find Complete Integral

let $z = ax + by + c$ be a trial solution. Then

$$p = \frac{\partial z}{\partial x} = a \text{ and } q = \frac{\partial z}{\partial y} = b$$

$$\Rightarrow ab = 1 \Rightarrow b = \frac{1}{a}$$

Hence the Complete integral is

$$z = ax + \frac{1}{a}y + c$$

Since [number of arbitrary Constants (a, c) = number of independent Variables $(x, y) = 2$]

To find Singular Integral

we have to eliminate a and c from $z = ax + \frac{1}{a}y + c$, $\frac{\partial z}{\partial a} = 0$ and $\frac{\partial z}{\partial c} = 0$

i.e. $z = ax + \frac{1}{a}y + c$

$$x - \frac{1}{a^2}y = 0$$

$$1 = 0$$

The last equation being absurd and hence there is no Singular Integral.

To find General Integral

let $c = g(a)$, so

$$z = ax + \frac{1}{a}y + g(a)$$

Differentiating w.r.t. a, we get

$$z = x - \frac{1}{a^2}y + g'(a)$$

Eliminating a from the last two equations, we get general integral of the given PDE.

Example 4.2: Solve $p^2 + q^2 = 1$

Solution: Let $z = ax + by + c$, then $p = \frac{\partial z}{\partial x} = a$, $q = \frac{\partial z}{\partial y} = b$

$$\Rightarrow a^2 + b^2 = 1 \Rightarrow b = \sqrt{1 - a^2}$$

$\therefore$ The Complete integral is $z = ax + \sqrt{1 - a^2}\, y + c$

The Singular integral and general integrals are obtained as example 4.1

5. Charpit's Method

It is a general method due to Charpit for solving non-linear equations of first order. When it is difficult to solve such equations under any of the standard forms, then this method is employed to find the complete integral s.

Let the given equation be $f(x,y,z,p,q)=0$ --- (1)

If we succeed to find another relation

$$F(x,y,z,p,q)=0 \quad \text{--- (2)}$$

satisfied by p and q, then we can solve equation (1) and (2) for p and q.

Since z consists of two independent variables x and y

$$\therefore \quad dz = p\,dx + q\,dy \quad \text{--- (3)}$$

For determining F, differentiate (1) and (2) with respect to x and y respectively giving

$$\left.\begin{aligned} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0 \\ \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0 \end{aligned}\right\} \quad \text{--- (4)}$$

$$\left.\begin{aligned} \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0 \\ \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}q + \frac{\partial F}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial y} = 0 \end{aligned}\right\} \quad \text{--- (5)}$$

Eliminating $\frac{\partial p}{\partial x}$ from the first pair (4), we get

$$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right)p + \left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0 \quad \text{--- (6)}$$

Similarly, eliminating $\frac{\partial q}{\partial y}$ from the second pair (5), we get

$$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right)q + \left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0 \quad \text{--- (7)}$$

Since $\frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y}$, hence the last terms in ⑥ and ⑦ are the same with opposite signs.

Adding ⑥ and ⑦, we get

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z} + \left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y} = 0 \quad \text{--- ⑧}$$

Clearly this is Lagrange's equation (linear equation of first order) with x, y, z, p, q as independent variables and F as dependent variable. Thus, identical to section 2 its solution will depend on solution of subsidiary equations

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0} \quad \text{--- ⑨}$$

An integral of the above equation ⑨ which involves (p and q) or (p or q) may be taken as assumed relation ②.
The more simple the integrals involving p or q or both derived from ⑨, the more easy to solve them for p and q and the given equation ①

Example 5.1: Using Charpit's method find complete integral of $pxy + pq + qy = zy$

Solution: $f(x, y, z, p, q) = pxy + pq + qy - yz = 0 \quad \text{--- ①}$

then by Charpit's method, the auxiliary equations is

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{d\phi}{0}$$

or

$$\frac{dx}{-(xy+q)} = \frac{dy}{-(p+y)} = \frac{dz}{-p(xy+q) - q(p+y)} = \frac{dp}{py + p(-y)} = \frac{dq}{(px+q-z) + q(-y)}$$

implying $dp=0$ or $p=a$ --- (2)

Putting $p=a$ in (1), $axy+aq+qy=yz$

$$q(a+y) = y(z-ax)$$

$$q = \frac{y(z-ax)}{(a+y)} \quad \text{---- (3)}$$

Also we know that for $z(x,y)$,

$$dz = p\,dx + q\,dy \quad \text{---- (4)}$$

On substituting the values of p and q from (2) and (3)

$$dz = a\,dx + \frac{y(z-ax)}{a+y}dy$$

or
$$\frac{dz - a\,dx}{z-ax} = \left(1 - \frac{a}{a+y}\right)dy$$

Integrating,

$$\ln(z-ax) = y - a\ln(a+y) + C_1$$

$$\boxed{z - ax = b\,e^{y}(y+a)^{-a}}$$

Example 5.2: Find complete integral of the equation $p^2x+q^2y=z$

Solution: $f(x,y,z,p,q) = p^2x + q^2y - z = 0$ --- (1)

By Charpit's method, auxiliary equations are

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{d\phi}{0}$$

i.e.
$$\frac{dx}{-2px} = \frac{dy}{-2qy} = \frac{dz}{-2(p^2x+q^2y)} = \frac{dp}{-p+p^2} = \frac{dq}{-q+q^2}$$

From above, we have

$$\frac{(p^2dx + 2px\,dp)}{p^2x} = \frac{(q^2dy + 2py\,dq)}{q^2y}$$

or
$$\frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y} \Rightarrow p^2x = aq^2y \quad \text{--- (2)}$$

6. Homogeneous Linear Partial Differential Equation

A linear partial differential equation in which all partial derivatives are of the same order is called homogeneous linear partial differential equation. Otherwise it is called non-homogeneous linear partial equation.

A homogeneous Linear PDE of n^{th} order with constant coefficients is of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = F(x,y) \quad \cdots \text{①}$$

where $a_0, a_1, \cdots a_n$ are constants.

This equation also can be written in the form

$$(a_0 D^n + a_1 D^{n-1} D' + \cdots + a_n D'^n) z = F(x,y)$$

where $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$

i.e., $f(D, D') z = F(x,y) \quad \cdots$ ②

The particular solution of ② is called the particular integral function of ② given symbolically by

$$P.I. = \frac{F(x,y)}{f(D,D')}$$

Hence the complete solution z = Complementary function + Particular Integral.

i.e., Complete Solution $z = C.F. + P.I.$

6.1 To Find the Complementary function (C.F.)

Put $D = m$ and $D' = 1$ in $f(D, D') = 0$, then we get an equation, which is called the auxillary equation of ②

∴ The auxillary equation of ② is $f(m,1) = 0$

i.e., $a_0 m^n + a_1 m^{n-1} + \cdots + a_n = 0$

Let the roots of this equation be $m_1, m_2, \cdots, m_n$.

CASE I If the roots are real (or imaginary) and distinct. Then $C.F. = f_1(y+m_1x)+f_2(y+m_2x)+\cdots+f_n(y+m_nx)$

CASE II

(a) If any two roots are equal (i.e., $m_1=m_2=m$) and others are distinct. Then

$$C.F. = f_1(y+mx)+x f_2(y+mx)+f_3(y+m_3x)+\cdots+f_n(y+m_nx)$$

(b) If any three roots are equal (i.e., $m_1=m_2=m_3=m$) and others are distinct. Then

$$C.F. = f_1(y+mx)+x f_2(y+mx)+x^2 f_3(y+mx)+ f_4(y+m_4x)+\cdots+f_n(y+m_nx)$$

6.2 To Find Particular Integral (P.I.)

Type I If $F(x,y)=e^{ax+by}$, then

$$P.I. = \frac{1}{f(D,D')}e^{ax+by}$$
$$= \frac{1}{f(a,b)}e^{ax+by} \quad \text{provided } f(a,b)\neq 0$$

If $f(a,b)=0$, then

$$P.I. = x\frac{1}{f'(D,D')}e^{ax+by}$$

where $f'(D,D')$ is the partial derivative of $f(D,D')$ w.r.t. D

Type II If $F(x,y)=\sin(ax+by)$ then

$$P.I. = \frac{1}{f(D,D')}\sin(ax+by)$$

Replace D^2 by $-a^2$, D'^2 by $-b^2$ and DD' by $-ab$ in $f(D,D')$ provided the denominator is not equal to zero.
If the denominator is zero, then

$$P.I. = x \cdot \frac{1}{f'(D,D')} \sin(ax+by)$$

where $f'(D,D')$ is the partial derivative of $f(D,D')$ w.r.t. D.

__Note__: Similar formula for $F(x,y) = \cos(ax+by)$.

$\boxed{\text{Type III}}$ If $F(x,y) = x^m y^n$, then

$$P.I = \frac{1}{f(D,D')} x^m y^n$$

$$= [f(D,D')]^{-1} x^m y^n$$

Expand $[f(D,D')]^{-1}$ is ascending powers of D, D' and then operate on $x^m y^n$.

__Note__: In $x^m y^n$ if $m > n$, then try to write $f(D,D')$ as a function of $\frac{D'}{D}$ and if $m < n$, then try to write $f(D,D')$ as a function of $\frac{D}{D'}$.

__Note__: $\frac{1}{D}$ denotes integration w.r.t. 'x', $\frac{1}{D'}$ denotes integration w.r.t. 'y'.

$\boxed{\text{Type IV}}$ If $F(x,y) = e^{ax+by} \phi(x,y)$, then

$$P.I. = \frac{1}{f(D,D')} e^{ax+by} \phi(x,y)$$

$$= e^{ax+by} \frac{1}{f(D+a, D'+b)} \phi(x,y)$$

Exercises

Solve the following partial differential equations:

1) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = 0$

2) $(2D^2 - 2DD' + D'^2)z = 2e^{3y} + e^{x+y} + y^2$

3) $(D^2 - DD')z = \cos 5x \cos 2y$

9. Non – Homogeneous Linear P.D.E.

A linear partial differential equation in which all partial derivatives are not of the same order is called non-homogeneous linear partial differential equation.

* Finding P.I. of non-homogeneous P.D.E. is same as homogeneous P.D.E.

• To find the Complementary function (C.F.)

Case (1): If $(D - m_1 D' - c_1)z = 0$, then

$$C.F. = e^{c_1 x} f_1(y + m_1 x)$$

Case (2): If $(D' - m_1 D - c_1)z = 0$, then

$$C.F. = e^{c_1 y} f_1(x + m_1 y)$$

Case (3): If $(D + m_1 D' + c_1)z = 0 \Rightarrow [D - (-m_1)D' - (-c_1)]z = 0$

then $C.F. = e^{-c_1 x}(f_1(y - m_1 x))$

Case(4): If $(D'+m_1D+c_1)z=0 \Rightarrow [D'-(-m_1)D-(-c_1)]z=0$

then $\quad C.F. = e^{-c_1 y} f_1(x-m_1 y)$

Case(5): If $(D-m_1D'-c_1)(D-m_2D'-c_2)z=0$, then

$$C.F. = e^{c_1 x} f_1(y+m_1 x) + e^{c_2 x} f_2(y+m_2 x)$$

Case(6): If $(D-m_1D'-c_1)^2 z=0$, then

$$C.F. = e^{c_1 x} f_1(y+m_1 x) + x e^{c_1 x} f_2(y+m_1 x)$$

Case(7): Put $D=h$ and $D'=k$ in $f(D,D')=0$, then we get an Aux. eq.

$$f(h,k)=0$$

Solving this equation to find the value of h interm of k or k interm of h.

Let the n values of h be denoted by

$$f_1(k),\ f_2(k),\ f_3(k),\ \cdots\ f_n(k)$$

Then

$$C.F. = \sum c_1 e^{f_1(k)x+ky} + \sum c_2 e^{f_2(k)x+ky} + \sum c_3 e^{f_3(k)x+ky} + \cdots + \sum c_n e^{f_n(k)x+ky}$$

Example 9.1 Solve $(D-2D'-1)z=0$

$$C.F. = e^{x} f_1(y+2x)$$

$[\because c_1 = 1, m_1 = 2 \text{ Case(1)}]$

Example 9.2 Solve $(D'-2D-3)z=0$

$$C.F. = e^{3y} f_1(x+2y)$$

$[\because c_1 = 3, m_1 = 2 \text{ Case(2)}]$

Example 9.3 Solve $(D+3D'+1)z=0$

Given $(D+3D'+1)z=0$

$$\Rightarrow [D-(-3)D'-(-1)]z=0$$

$$C.F. = e^{-x} f_1(y-3x)$$

$[\because c_1 = 1, m_1 = 3 \text{ Case(3)}]$

Example 9.4 Solve $(D'+5D+6)z=0$

Given $(D'+5D+6)z=0$

$$\Rightarrow [D'-(-5)D-(-6)]z=0$$

$$C.F. = e^{-6y} f_1(x-5y)$$

$[\because c_1 = 6, m_1 = 5 \text{ Case(4)}]$

Example 9.5 Solve $(D-4D'-1)(D'+5D-3)z=0$

Given $(D-4D'-1)(D'+5D-3)=0$

$$\Rightarrow (D-4D'-1)(D'-(-5)D-3)z=0$$

$$C.F. = e^{x} f_1(y+4x) + e^{3y} f_2(x-5y)$$

10. Method of Separation of Variables

It is a Convenient and powerful Solution technique of PDEs. The main idea is to Convert the given PDE into Several ODEs and then obtain the Solutions by familiar Solution techniques.
The basic idea of this method is that the Solution is assumed to Consist of product of two or more functions Each function being the function of one independent Variable only.

10.1 Finite Vibrating String Problem

For a point x on the string we let

$u(x,t)$ = displacement of the point x at time t

Assuming small displacements this is well modeled by PDE Called the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad x \in [0, L],\ t \geq 0 \quad \text{---(1)}$$

with the boundary Conditions (BC)

$$u(0,t) = 0 \ , \ u(L,t) = 0 \quad \forall t \geq 0 \qquad \text{---(2)}$$

and initial Conditions (IC) for $0 \leq x \leq L$

$$u(x,0) = \phi(x) \ , \ u_t(x,0) = \psi(x) \qquad \text{---(3)}$$

Now, Assume that

$$u(x,t) = X(x)T(t) \qquad \text{---(4)}$$

here $X(x)$ is function of x alone and $T(t)$ is a function of t alone. Substituting (4) in equation (1)

$$XT'' = c^2 X'' T$$

Separating the variables

$$\frac{X''}{X} = \frac{T''}{c^2 T}$$

The left side is a function of x and the right side is a function of t.
The equality will hold only if both are equal to a Constant, Say k.

We get two differential equations as follows:

$$X'' - kX = 0 \qquad \text{---(5)}$$

$$T'' - c^2 kT = 0 \qquad \text{....(6)}$$

Since k is any constant,
- it can be zero, or
- it can be positive, or
- it can be negative.

Consider all the possibilities and examine what value(s) of k lead to a non-trivial Solution.

Case I $\quad k = 0$

In this Case equation (5) and (6) reduce to

$$X'' = 0 \quad \text{and} \quad T'' = 0$$

$$\Rightarrow \quad X(x) = Ax + B \qquad \text{and} \quad T(t) = Ct + D$$

But the Solution $u(x,t) = X(x)T(t)$ is a trivial Solution if it has to satisfy the boundary Conditions

$$u(0,t) = u(L,t) = 0$$

So, this Case is rejected Since it gives rise to trivial Solutions only.

Case II $\quad k > 0$, let $k = \lambda^2$ for Some $\lambda > 0$

$$\Rightarrow \quad X'' - \lambda^2 X = 0 \quad \text{and} \quad T'' - c^2\lambda^2 T = 0$$

Giving rise to Solutions

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x}$$
$$T(t) = Ce^{c\lambda t} + De^{-c\lambda t}$$

Therefore

$$u(x,t) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(Ce^{c\lambda t} + De^{-c\lambda t}\right)$$

Using boundary Condition $u(0,t)=0$

$$A+B=0 \Rightarrow A=-B$$

Using boundary Condition $u(L,t)=0$

$$\left(Ae^{\lambda L} - Ae^{-\lambda L}\right)\left(Ce^{c\lambda t} + De^{-c\lambda t}\right) = 0$$
$$A\left(Ae^{\lambda L} - e^{-\lambda L}\right)\left(Ce^{c\lambda t} + De^{-c\lambda t}\right) = 0$$

The t part of the Solution Cannot be Zero as it will lead to a trivial Solution. Then we must have

$$A\left(e^{\lambda L} - e^{-\lambda L}\right) = 0$$

which leads to $A=0$ as $\lambda \neq 0$

$K>0$ also gives rise to trivial Solution and also rejected.

Case III $\quad K<0$, let $K=-\lambda^2$ for Some $\lambda>0$

$$\Rightarrow \quad X'' + \lambda^2 X = 0 \quad \text{and} \quad T'' + c^2\lambda^2 T = 0$$

giving rise to Solutions

$$X(x) = A\cos\lambda x + B\sin\lambda x$$
$$T(t) = C\cos(c\lambda t) + D\sin(c\lambda t)$$

10.2 One Dimensional Heat Flow

Let $u(x,t)$ denote the temperature at position x and time t in a long, thin rod of length L that runs from $x=0$ to $x=L$. Assume that the sides of the rod are isulated so heat energy niether enters nor leaves the rod through its sides. Also assume that heat energy is neither created nor destroyed in the interior of the rod. Then $u(x,t)$ obeys the heat eq.

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{for all } 0<x<L, \text{ and } t>0$$

This equation is called (The Heat Equation (one Space Dim))

$$\text{B.C.} \begin{cases} u(0,t)=0 & \text{for all } t>0 \\ u(L,t)=0 & \text{for all } t>0 \end{cases}$$

$$\text{I.C.} \quad u(x,0)=f(x) \quad \text{for all } 0<x<L$$

<u>Example</u> Find the solution to the heat Conduction problem

$$4u_t = u_{xx}, \quad 0\leq x\leq 2 \quad t>0$$

$$u(0,t)=0$$

$$u(2,t)=0$$

$$u(x,0) = 2\sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4\sin(2\pi x) = f(x)$$

Solution

Let $u(x,t) = X(x)T(t)$

$$\Rightarrow \quad 4X(x)T'(t) = X''(x)T(t)$$

$$\Rightarrow \quad 4\frac{T'}{T} = \frac{X''}{X} = \lambda$$

where λ is Constant

$0 = u(0,t) = X(0)T(t) \Rightarrow X(0) = 0$

$0 = u(2,t) = X(2)T(t) \Rightarrow X(2) = 0$

(Since $T(t)$ won't be 0 for all t)

$\therefore\ X(0) = X(2) = 0$ are the BCs

Now, $T' = \frac{\lambda}{4} T \Rightarrow \frac{dT}{dt} = \frac{\lambda}{4} T \Rightarrow \frac{dT}{T} = \frac{\lambda}{4} dt$

Integrating both sides

$$\int \frac{dT}{T} = \frac{\lambda}{4} \int dt \Rightarrow \ln T = \frac{\lambda}{4} t + C_1$$

$$\Rightarrow T = C_1 e^{\lambda t/4}$$

and $X'' = \lambda X$ with $X(0) = X(2) = 0$

There are 3 cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$

① $\lambda > 0 \Rightarrow T(t) = C e^{\lambda t/4} \rightarrow \infty$ at $t \rightarrow \infty$

$\Rightarrow u(x,t) = X(x) e^{\lambda t/4} \rightarrow \infty$

and $\lambda > 0$ would suggest the the temperature $u \rightarrow \infty$ which doesn't make sense.

set $\lambda = k^2 > 0 \Rightarrow X'' - k^2 X = 0$

$\Rightarrow r^2 - k^2 = 0 \Rightarrow r_{1,2} = \mp k$

$\therefore\ X(x) = C_1 e^{kx} + C_2 e^{-kx}$

$X(0) = 0 \Rightarrow C_1 + C_2 = 0 \Rightarrow C_1 = -C_2$

$X(2) = 0 \Rightarrow C_1 e^{2k} + C_2 e^{-2k} = 0$

$C_1 (e^{2k} - e^{-2k}) = 0 \Rightarrow C_1 = 0 \Rightarrow C_2 = 0$

$\Rightarrow$ Trivial Solution

② $\lambda = 0$ $\quad X'' = 0 \Rightarrow X = Ax + B$

$X(0) = 0 \Rightarrow 0 = B \Rightarrow X(x) = Ax$

$X(2) = 0 \Rightarrow 0 = 2A \Rightarrow A = 0$

$\Rightarrow$ Trivial Solution

③ $\lambda < 0$, set $\lambda = -k^2 < 0$

$$X'' + k^2 X = 0 \Rightarrow m^2 + k^2 = 0 \Rightarrow m = \mp ik$$

$$\Rightarrow X(x) = C_1 e^{ikx} + C_2 e^{-ikx}$$

$$\text{or} \quad X(x) = C_1 \sin(kx) + C_2 \cos(kx)$$

$$X(0) = 0 \Rightarrow C_1 \sin(0) + C_2 \cos(0) = 0 \Rightarrow C_2 = 0$$

$$X(2) = 0 \Rightarrow C_1 \sin(2k) = 0$$

Since $\sin(\theta)$ has roots at $\theta = n\pi$, $n = 1, 2, 3, \ldots$ the second Condition tells us that $2k = n\pi$

$$\Rightarrow k = \frac{n\pi}{2}, \quad n = 1, 2, \ldots$$

Thus we have our eigenfunctions with eigenvalues $\lambda < 0$:

$$\lambda_n = -\left(\frac{n\pi}{2}\right)^2$$

$$X_n = \sin\left(\frac{n\pi x}{2}\right)$$

$$u_n(x,t) = X_n(x)\, T_n(t) = \sin\left(\frac{n\pi x}{2}\right) \exp\left(-\frac{n^2\pi^2}{16} t\right)$$

$$n = 1, 2, 3, \ldots$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) \exp\left(-\frac{n^2\pi^2}{16} t\right)$$

we solve for the b_n using the initial condition. That is, $u(x,0) = f(x)$ so

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

which is a Fourier Sine Series. we exploit orthogonality of the sines, that is

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & m \neq n \\ L/2 & m = n \end{cases}$$

where $L = 2$,

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \int_0^2 f(x) \sin\left(\frac{n\pi x}{2}\right) dx$$

11. Review of properties of Laplace Transform

Definition: Given a function $f(t)$, $t \geq 0$, its Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} \doteq \int_0^\infty e^{-st} f(t)\, dt \doteq \lim_{A\to\infty} \int_0^A e^{-st} f(t)\, dt$$

We say the transform converges if the limit exists, and diverges if not.

Next we will give examples on computing the Laplace transform of given functions by definition.

Example 1. $f(t) = 1$ for $t \geq 0$.

$$F(s) = \mathcal{L}\{f(t)\} = \lim_{A\to\infty} \int_0^A e^{-st} \cdot 1\, dt = \lim_{A\to\infty} -\frac{1}{s} e^{-st}\Big|_0^A = \lim_{A\to\infty} -\frac{1}{s}\left[e^{-sA} - 1\right] = \frac{1}{s}, \quad (s > 0)$$

Example 2. $f(t) = e^t$.

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} = \lim_{A\to\infty} \int_0^A e^{-st} e^{at} dt = \lim_{A\to\infty} \int_0^A e^{-(s-a)t} dt = \lim_{A\to\infty} -\frac{1}{s-a} e^{-(s-a)t}\Big|_0^A \\ &= \lim_{A\to\infty} -\frac{1}{s-a}\left(e^{-(s-a)A} - 1\right) = \frac{1}{s-a}, \quad (s > a) \end{aligned}$$

Example 3. $f(t) = t^n$, for $n \geq 1$ integer.

$$\begin{aligned} F(s) &= \lim_{A\to\infty} \int_0^A e^{-st} t^n dt = \lim_{A\to\infty} \left\{ t^n \frac{e^{-st}}{-s}\Big|_0^A - \int_0^A \frac{nt^{n-1} e^{-st}}{-s} dt \right\} \\ &= 0 + \frac{n}{s} \lim_{A\to\infty} \int_0^A e^{-st} t^{n-1} dt = \frac{n}{s} \mathcal{L}\{t^{n-1}\}. \end{aligned}$$

So we get a recursive relation

$$\mathcal{L}\{t^n\} = \frac{n}{s} \mathcal{L}\{t^{n-1}\}, \quad \forall n,$$

which means

$$\mathcal{L}\{t^{n-1}\} = \frac{n-1}{s} \mathcal{L}\{t^{n-2}\}, \quad \mathcal{L}\{t^{n-2}\} = \frac{n-2}{s} \mathcal{L}\{t^{n-3}\}, \cdots$$

By induction, we get

$$\begin{aligned} \mathcal{L}\{t^n\} &= \frac{n}{s} \mathcal{L}\{t^{n-1}\} = \frac{n}{s} \frac{(n-1)}{s} \mathcal{L}\{t^{n-2}\} = \frac{n}{s} \frac{(n-1)}{s} \frac{(n-2)}{s} \mathcal{L}\{t^{n-3}\} \\ &= \cdots = \frac{n}{s} \frac{(n-1)}{s} \frac{(n-2)}{s} \cdots \frac{1}{s} \mathcal{L}\{1\} = \frac{n!}{s^n} \frac{1}{s} = \frac{n!}{s^{n+1}}, \quad (s > 0) \end{aligned}$$

Example 4. Find the Laplace transform of $\sin at$ and $\cos at$.

Method 1. Compute by definition, with integration-by-parts, twice. (lots of work...)

Method 2. Use the Euler's formula

$$e^{iat} = \cos at + i\sin at, \qquad \Rightarrow \qquad \mathcal{L}\{e^{iat}\} = \mathcal{L}\{\cos at\} + i\mathcal{L}\{\sin at\}.$$

By Example 2 we have

$$\mathcal{L}\{e^{iat}\} = \frac{1}{s-ia} = \frac{1(s+ia)}{(s-ia)(s+ia)} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}.$$

Comparing the real and imaginary parts, we get

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, \qquad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}, \quad (s>0).$$

Remark: Now we will use $\int_0^\infty$ instead of $\lim_{A\to\infty}\int_0^A$, without causing confusion.

For piecewise continuous functions, Laplace transform can be computed by integrating each integral and add up at the end.

Example 5. Find the Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \le t < 2, \\ t-2, & 2 \le t. \end{cases}$$

We do this by definition:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t)\,dt = \int_0^2 e^{-st}dt + \int_2^\infty (t-2)e^{-st}dt \\ &= \frac{1}{-s}e^{-st}\Big|_{t=0}^2 + (t-2)\frac{1}{-s}e^{-st}\Big|_{t=2}^\infty - \int_2^A \frac{1}{-s}e^{-st}dt \\ &= \frac{1}{-s}(e^{-2s}-1) + (0-0) + \frac{1}{s}\frac{1}{-s}e^{-st}\Big|_{t=2}^\infty = \frac{1}{-s}(e^{-2s}-1) + \frac{1}{s^2}e^{-2s} \end{aligned}$$

Properties of Laplace transform:

1. Linearity: $\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1\mathcal{L}\{f(t)\} + c_2\mathcal{L}\{g(t)\}$.
2. First derivative: $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$.
3. Second derivative: $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$.
4. Higher order derivative:
$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$
5. $\mathcal{L}\{-tf(t)\} = F'(s)$ where $F(s) = \mathcal{L}\{f(t)\}$. This also implies $\mathcal{L}\{tf(t)\} = -F'(s)$.
6. $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ where $F(s) = \mathcal{L}\{f(t)\}$. This implies $e^{at}f(t) = \mathcal{L}^{-1}\{F(s-a)\}$.

By using these properties, we could find more easily Laplace transforms of many other functions.

Example 1.

$$\text{From} \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad \text{we get} \qquad \mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}.$$

Example 2.

$$\text{From} \qquad \mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}, \qquad \text{we get} \qquad \mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}.$$

Example 3.

$$\text{From} \qquad \mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}, \qquad \text{we get} \qquad \mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}.$$

Example 4.

$$\mathcal{L}\{t^3+5t-2\} = \mathcal{L}\{t^3\} + 5\mathcal{L}\{t\} - 2\mathcal{L}\{1\} = \frac{3!}{s^4} + 5\frac{1}{s^2} - 2\frac{1}{s}.$$

Example 5.

$$\mathcal{L}\{e^{2t}(t^3+5t-2)\} = \frac{3!}{(s-2)^4} + 5\frac{1}{(s-2)^2} - 2\frac{1}{s-2}.$$

Solving PDEs using Laplace Transforms

Given a function $u(x,t)$ defined for all $t>0$ and assumed to be bounded we can apply the Laplace transform in t considering x as a parameter.

$$L(u(x,t)) = \int_0^\infty e^{-st}u(x,t)\,dt \equiv U(x,s)$$

In applications to PDEs we need the following:

$$L(u_t(x,t) = \int_0^\infty e^{-st}u_t(x,t)\,dt = e^{-st}u(x,t)\Big|_0^\infty + s\int_0^\infty e^{-st}u(x,t)\,dt = sU(x,s) - u(x,0)$$

$u = e^{-st}$
$du = -se^{-st}\,dt$
$dv = u_t\,dt$
$v = u$

so we have

$$L(u_t(x,t) = sU(x,s) - u(x,0)$$

In exactly the same way we obtain

$$L(u_{tt}(x,t) = s^2U(x,s) - su(x,0) - u_t(x,0).$$

We also need the corresponding transforms of the x derivatives:

$$L(u_x(x,t)) = \int_0^\infty e^{-st}u_x(x,t)\,dt = U_x(x,s)$$

$$L(u_{xx}(x,t)) = \int_0^\infty e^{-st}u_{xx}(x,t)\,dt = U_{xx}(x,s)$$

Consider the following examples.

Example 1.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = x, \quad x>0, \quad t>0,$$

with boundary and initial condition

$$u(0,t) = 0 \quad t>0, \qquad \text{and} \quad u(x,0)=0, \quad x>0.$$

As above we use the notation $U(x,s) = L(u(x,t))(s)$ for the Laplace transform of u. Then applying the Laplace transform to this equation we have

$$\frac{dU}{dx}(x,s) + sU(x,s) - u(x,0) = \frac{x}{s} \quad \Rightarrow \quad \frac{dU}{dx}(x,s) + sU(x,s) = \frac{x}{s}.$$

This is a constant coefficient first order ODE. We solve it by finding the integrating factor

$$\mu = e^{\int s\,dx} = e^{sx}$$

Thus we have

$$\frac{d}{dx}\left[e^{sx}U(x,s)\right] = e^{sx}\frac{x}{s}.$$

We integrate both sides to get

$$U(x,s) = \frac{e^{-sx}}{s}\left(\int e^{sr}r\,dr\right) + Ce^{-sx}.$$

1st order linear ODE
$y' + a(x)y = f(x)$
$\mu(x) = e^{\int a(x)dx}$
$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x)$
$y = \frac{\int \mu(x)f(x)dx + C}{\mu(x)}$

We can use integration by parts to evaluate the integral:

$u = x$
$du = dx$
$dv = e^{sx}dx$
$v = \frac{1}{s}e^{sx}$

$$\int e^{sx} x\, dx = \int \left(\frac{e^{sx}}{s}\right)' x\, dx$$
$$= \frac{xe^{sx}}{s} - \int \left(\frac{e^{sx}}{s}\right) dx$$
$$\frac{xe^{sx}}{s} - \frac{e^{sx}}{s^2}.$$

So we have

$$U(x,s) = \frac{e^{-sx}}{s}\left(\frac{xe^{sx}}{s} - \frac{e^{sx}}{s^2}\right) + Ce^{-sx} = \frac{x}{s^2} - \frac{1}{s^3} + Ce^{-sx}.$$

We can evaluate the constant C using the boundary condition

$$0 = U(0,s) = -\frac{1}{s^3} + C \;\;\Rightarrow\;\; C = \frac{1}{s^3}$$

so we have

$$U(x,s) = \frac{x}{s^2} - \frac{1}{s^3} + \frac{e^{-sx}}{s^3}.$$

Taking the inverse Laplace transform we have

$$u(x,t) = xt - \frac{t^2}{2} + H(t-x)\frac{(t-x)^2}{2}$$

where H is the unit step function (or Heaviside function)

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}.$$

Example 2.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + u = 0, \quad x > 0, \;\; t > 0,$$

with boundary and initial condition

$$u(0,t) = 0 \;\; t > 0, \quad \text{and} \;\; u(x,0) = \sin(x), \;\; x > 0.$$

As above we use the notation $U(x,s) = L(u(x,t))(s)$ for the Laplace transform of u. Then applying the Laplace transform to this equation we have

$$\frac{dU}{dx}(x,s) + sU(x,s) - u(x,0) + U(x,s) = 0 \;\;\Rightarrow\;\; \frac{dU}{dx}(x,s) + (s+1)U(x,s) = \sin(x).$$

This is a constant coefficient first order linear ODE. We solve it by finding the integrating factor

$$\mu = e^{\int (s+1)dx} = e^{(s+1)x}$$

Thus we have

$$\frac{d}{dx}\left[e^{(s+1)x}U(x,s)\right] = e^{(s+1)x}\sin(x).$$

We integrate both sides to get

$$U(x,s) = e^{-(s+1)x}\left(\int e^{(s+1)r}\sin(r)\, dr\right) + Ce^{-(s+1)x}.$$

We can use integration by parts to evaluate the integral:

$$e^{-(s+1)x}\left(\int_0^x e^{(s+1)r}\sin(r)\,dr\right) = \frac{(s+1)\sin(x)-\cos(x)}{s^2+2s+2}.$$

So we have

$$U(x,s) = \frac{(s+1)\sin(x)-\cos(x)}{s^2+2s+2} + Ce^{-(s+1)x}.$$

We can evaluate the constant C using the boundary condition

$$0 = U(0,s) = \frac{-1}{s^2+2s+2} + C \quad \Rightarrow \quad C = \frac{1}{s^2+2s+2}.$$

So we have

$$U(x,s) = \frac{(s+1)\sin(x)-\cos(x)+e^{-(s+1)x)}}{s^2+2s+2}.$$

Taking the inverse Laplace transform we have

$$u(x,t) = e^{-t}\cos(t)\sin(x) - e^{-t}\sin(t)\cos(t) + e^{-t}H(t-x)\sin(t-x)\,e^{-x}$$

This can be written as

$$u(x,t) = e^{-t}\left[\sin(x-t) + e^{-x}H(t-x)\sin(t-x)\right].$$

Example 3.

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,t), \quad 0<x<2, \quad t>0,\\
u(0,t) &= 0, \quad u(2,t)=0\\
u(x,0) &= 3\sin(2\pi x).
\end{aligned}$$

Take the Laplace transform and apply the initial condition

$$\frac{d^2U}{dx^2}(x,s) = sU(x,s) - u(x,0) = sU(x,s) - 3\sin(2\pi x).$$

We write this equation as a non-homogeneous, second order linear constant coefficient equation for which we can apply the methods from Math 3354.

$$\frac{d^2U}{dx^2}(x,s) - sU(x,s) = -3\sin(2\pi x).$$

$m^2 - s = 0$

$m_{1,2} = \mp\sqrt{s}$

The general solution can be written as

$$U(x,s) = U_h(x,s) + U_p(x,s)$$

where $U_h(x,s)$ is the general solution of the homogeneous problem

$$U_h(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$$

and $U_p(x,s)$ is any particular solution of the non-homogeneous problem

$$U_p(x,s) = A\cos(2\pi x) + B\sin(2\pi x).$$