University of Baghdad
College of Science
Department of Mathematic

Theory of Differential Equations
نظرية المعادلات التفاضلية

Fourth Class
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THEORY OF DIFFERENTIAL EQUATIONS

Syllabus

- System of Differential Equations: The fundamental matrix of solution; Linear ordinary differential equations (Homogeneous, nonhomogeneous, with constant coefficients, with variables coefficients) systems
- Introduction to stability theory of the linear ordinary differential equations systems.
- The stability of the second order linear ordinary differential equations systems.
- Nonlinear ordinary differential equations with their stability and dynamics
- Lyapunov stability.
- Existence and uniqueness theorem.

The main textbook is

The Qualitative Theory of Ordinary Differential Equations: An Introduction

By Fred Brauer, John A. Nohel

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Linear Homogeneous Systems of Differential Equations with Constant Coefficients

An nth order linear system of differential equations with constant coefficients is written as

$$\frac{dx_i}{dt} = x_i' = \sum_{j=1}^{n} a_{ij} x_j(t) + f_i(t), \; i = 1, 2, \ldots, n,$$

where $x_1(t), x_2(t), \ldots, x_n(t)$ are unknown functions of the variable t, which often has the meaning of time, a_{ij} are certain constant coefficients, which can be either real or complex, $f_i(t)$ are given (in general case, complex-valued) functions of the variable t.

We assume that all these functions are continuous on an interval $[a, b]$ of the real number axis t.

By setting

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \; X'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}, \; f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix},$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

the system of differential equations can be written in matrix form:

$$X'(t) = AX(t) + f(t).$$

If the vector $f(t)$ is identically equal to zero: $f(t) \equiv 0$, then the system is said to be homogeneous:

$$X'(t) = AX(t).$$

Homogeneous systems of equations with constant coefficients can be solved in different ways. The following methods are the most commonly used:

- elimination method (the method of reduction of n equations to a single equation of the nth order);
- method of integrable combinations;
- method of eigenvalues and eigenvectors (including the method of undetermined coefficients or using the Jordan form in the case of multiple roots of the characteristic equation);
- method of the matrix exponential.

Below on this page we will discuss in detail the elimination method. Other methods for solving systems of equations are considered separately in the following pages.

Elimination Method

Using the method of elimination, a normal linear system of n equations can be reduced to a single linear equation of nth order. This method is useful for simple systems, especially for systems of order 2.

Consider a homogeneous system of two equations with constant coefficients:

$$\left\{ \begin{array}{l} x_1' = a_{11}x_1 + a_{12}x_2 \\ x_2' = a_{21}x_1 + a_{22}x_2 \end{array} \right. ,$$

where the functions x_1, x_2 depend on the variable t.

We differentiate the first equation and substitute the derivative x_2' from the second equation:

$$x_1'' = a_{11}x_1' + a_{12}x_2', \quad \Rightarrow x_1'' = a_{11}x_1' + a_{12}\left(a_{21}x_1 + a_{22}x_2\right),$$
$$\Rightarrow x_1'' = a_{11}x_1' + a_{12}a_{21}x_1 + a_{22}a_{12}x_2.$$

Now we substitute $a_{12}x_2$ from the first equation. As a result we obtain a second order linear homogeneous equation:

$$x_1'' = a_{11}x_1' + a_{12}a_{21}x_1 + a_{22}\left(x_1' - a_{11}x_1\right), \quad \Rightarrow x_1'' = a_{11}x_1' + a_{12}a_{21}x_1$$
$$+ a_{22}x_1' - a_{11}a_{22}x_1, \quad \Rightarrow x_1'' - \left(a_{11} + a_{22}\right)x_1' + \left(a_{11}a_{22} - a_{12}a_{21}\right)x_1 = 0.$$

It is easy to construct its solution, if we know the roots of the characteristic equation:

$$\lambda^2 - \left(a_{11} + a_{22}\right)\lambda + \left(a_{11}a_{22} - a_{12}a_{21}\right) = 0.$$

In the case of real coefficients a_{ij}, the roots can be both real (distinct or multiple) and complex. In particular, if the coefficients a_{12} and a_{21} have the same sign, then the discriminant of the characteristic equation will always be positive and, therefore, the roots will be real and distinct.

After the function $x_1(t)$ is determined, the other function $x_2(t)$ can be found from the first equation.

The elimination method can be applied not only to homogeneous linear systems. It can also be used for solving nonhomogeneous systems of differential equations or systems of equations with variable coefficients.

Example 1.

Solve the system of differential equations by elimination:

$$x_1' = 2x_1 + 3x_2, \quad x_2' = 4x_1 - 2x_2.$$

Solution.

We differentiate the first equation and then substitute the derivative x_2' from the second equation:

$$x_1'' = 2x_1' + 3x_2', \quad \Rightarrow x_1'' = 2x_1' + 3\left(4x_1 - 2x_2\right), \quad \Rightarrow x_1'' = 2x_1' + 12x_1 - 6x_2$$

Express $3x_2$ from the first equation:

$$3x_2 = x_1' - 2x_1.$$

Substituting this into the last equation, we get:

$$x_1'' = 2x_1' + 12x_1 - 2\left(x_1' - 2x_1\right), \quad \Rightarrow x_1'' = \cancel{2x_1'} + 12x_1 - \cancel{2x_1'} + 4x_1,$$
$$\Rightarrow x_1'' - 16x_1 = 0.$$

Find the roots of the characteristic equation:

$$\lambda^2 - 16 = 0, \quad \Rightarrow \lambda_{1,2} = \pm 4.$$

Hence, the general solution of the 2nd order equation for the variable x_1 is given by

$$x_1(t) = C_1 e^{4t} + C_2 e^{-4t},$$

where C_1, C_2 are arbitrary constants.

Now we compute the derivative x_1' and substitute the expressions for x_1, x_1' in the first equation of the original system:

$$x_1'(t) = 4C_1 e^{4t} - 4C_2 e^{-4t}, \quad \Rightarrow 4C_1 e^{4t} - 4C_2 e^{-4t} = 2C_1 e^{4t} + 2C_2 e^{-4t} + 3x_2,$$
$$\Rightarrow 3x_2 = 2C_1 e^{4t} - 6C_2 e^{-4t}, \quad \Rightarrow x_2 = \frac{2}{3} C_1 e^{4t} - 2C_2 e^{-4t}.$$

To keep integer coefficients, it is convenient to designate: $C_1 \to 3C_1$. As a result, we obtain the final solution in the following form:

$$\begin{cases} x_1(t) = 3C_1 e^{4t} + C_2 e^{-4t} \\ x_2(t) = 2C_1 e^{4t} - 2C_2 e^{-4t} \end{cases}.$$

Example 2

Solve the system by elimination:

$$x' = 6x - y, \quad y' = x + 4y.$$

Solution.

We convert this system to a single 2nd order equation for the function $x(t)$.
Differentiating the first equation and substituting y' from the second equation, we have:

$$x'' = 6x' - y', \quad \Rightarrow x'' = 6x' - (x + 4y), \quad \Rightarrow x'' = 6x' - x - 4y.$$

Express the variable y in terms of x and x' from the first equation:

$$y = 6x - x', \quad \Rightarrow x'' = 6x' - x - 4\left(6x - x'\right), \quad \Rightarrow x'' = 6x' - x - 24x + 4x',$$
$$\Rightarrow x'' - 10x' + 25x = 0.$$

Compute the roots of the auxiliary equation:

$$\lambda^2 - 10\lambda + 25 = 0, \quad D = 0, \quad \Rightarrow \lambda_1 = 5.$$

So, we have one root $\lambda = 5$ of multiplicity 2. Consequently, the general solution for the function $x(t)$ is written as

$$x(t) = (C_1 + C_2 t)\, e^{5t},$$

where C_1, C_2 are arbitrary numbers.

Find the derivative $x'(t)$ and substituting it in the first equation of the original system determine the function $y(t)$:

$$x'(t) = C_2 e^{5t} + (5C_1 + 5C_2 t)\, e^{5t} = (5C_1 + C_2 + 5C_2 t)\, e^{5t},$$
$$\Rightarrow (5C_1 + C_2 + 5C_2 t)\, e^{5t} = (6C_1 + 6C_2 t)\, e^{5t} - y, \quad \Rightarrow y = (C_1 - C_2 + C_2 t)\, e^{5t}.$$

Thus, the general solution is written as

$$\begin{cases} x(t) = (C_1 + C_2 t)\, e^{5t} \\ y(t) = (C_1 - C_2 + C_2 t)\, e^{5t} \end{cases}.$$

Example 3

Find the general solution of the system

$$x_1' = 5x_1 + 2x_2, \quad x_2' = -4x_1 + x_2.$$

Linear Systems of Differential Equations with Variable Coefficients

A normal linear system of differential equations with variable coefficients can be written as

$$\frac{dx_i}{dt} = x_i' = \sum_{j=1}^{n} a_{ij}(t)\, x_j(t) + f_i(t)\,,\;\; i = 1, 2, \ldots, n,$$

where $x_i(t)$ are unknown functions, which are continuous and differentiable on an interval $[a, b]$. The coefficients $a_{ij}(t)$ and the free terms $f_i(t)$ are continuous functions on the interval $[a, b]$.

Using vector-matrix notation, this system of equations can be written as

$$\mathbf{X}'(t) = A(t)\,\mathbf{X}(t) + \mathbf{f}(t)\,,$$

where

$$\mathbf{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix},\; A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \vdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \vdots & a_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(t) & a_{n2}(t) & \vdots & a_{nn}(t) \end{bmatrix},\; \mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}.$$

In the general case, the matrix $A(t)$ and the vector functions $\mathbf{X}(t)$, $\mathbf{f}(t)$ can take both real and complex values.

The corresponding homogeneous system with variable coefficients in vector form is given by

$$\mathbf{X}'(t) = A(t)\,\mathbf{X}(t)\,.$$

Fundamental System of Solutions and Fundamental Matrix

The vector functions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ are linearly dependent on the interval $[a, b]$, if there are numbers $c_1, c_2, \ldots, c_n$, not all zero, such that the following identity holds:

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) \equiv 0, \;\; \forall t \in [a, b].$$

If this identity is satisfied only if

$$c_1 = c_2 = \cdots = c_n = 0,$$

the vector functions $\mathbf{x}_i(t)$ are called linearly independent on the given interval.

Any system of n linearly independent solutions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ is called a fundamental system of solutions.

A square matrix $\Phi(t)$ whose columns are formed by linearly independent solution $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$, is called the fundamental matrix of the system of equations. It has the following form:

$$\Phi(t) = \begin{bmatrix} x_{11}(t) & x_{12}(t) & \vdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \vdots & x_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1}(t) & x_{n2}(t) & \vdots & x_{nn}(t) \end{bmatrix},$$

where $x_{ij}(t)$ are the coordinates of the linearly independent vector solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$.

Note that the fundamental matrix $\Phi(t)$ is nonsingular, i.e. there always exists the inverse matrix $\Phi^{-1}(t)$. Since the fundamental matrix has n linearly independent solutions, after its substitution into the homogeneous system we obtain the identity

$$\Phi'(t) \equiv A(t)\,\Phi(t)\,.$$

We multiply this equation on the right by the inverse function $\Phi^{-1}(t)$:

$$\Phi'(t)\,\Phi^{-1}(t) \equiv A(t)\,\Phi(t)\,\Phi^{-1}(t)\,, \quad \Rightarrow A(t) \equiv \Phi'(t)\,\Phi^{-1}(t)\,.$$

The resulting relation uniquely defines a homogeneous system of equations, given the fundamental matrix.

The general solution of the homogeneous system is expressed in terms of the fundamental matrix in the form

$$\mathbf{X}_0(t) = \Phi(t)\,\mathbf{C},$$

where $\mathbf{C}$ is an n-dimensional vector consisting of arbitrary numbers.

Let us mention an interesting special case of homogeneous systems. It turns out that if the product of the matrix $A(t)$ and the integral of this matrix is commutative, that is

$$A(t) \cdot \int\limits_a^t A(\tau)\,dt = \int\limits_a^t A(\tau)\,dt \cdot A(t)\,,$$

the fundamental matrix $\Phi(t)$ for such a system of equations is given by

$$\Phi(t) = e^{\int\limits_a^t A(\tau)d\tau}.$$

Such property is satisfied in the case of symmetric matrices and, in particular, in the case of diagonal matrices.

Wronskian and Liouville's Formula

The determinant of the fundamental matrix $\Phi(t)$ is called the Wronskian of the system of solutions $\mathbf{x}_1(t)\,, \mathbf{x}_2(t)\,, \ldots, \mathbf{x}_n(t)$:

$$W(t) = W[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] = \begin{vmatrix} x_{11}(t) & x_{12}(t) & \vdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \vdots & x_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1}(t) & x_{n2}(t) & \vdots & x_{nn}(t) \end{vmatrix}.$$

The Wronskian is useful to check the linear independence of solutions. The following rules apply:

- The solutions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ of the homogeneous system form a fundamental system if and only if the corresponding Wronskian is not zero at any point t of the interval $[a, b]$.
- The solutions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ are linearly dependent on the interval $[a, b]$ if and only if the Wronskian is identically zero on this interval.

The Wronskian of the solutions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ is given by Liouville's formula:

$$W(t) = e^{\int\limits_a^t \operatorname{tr}(A(\tau))d\tau},$$

where $\operatorname{tr}(A(\tau))$ is the trace of the matrix $A(\tau)$, i.e. the sum of all diagonal elements:

$$\operatorname{tr}(A(\tau)) = a_{11}(\tau) + a_{22}(\tau) + \cdots + a_{nn}(\tau).$$

Liouville's formula can be used to construct the general solution of the homogeneous system if a particular solution is known.

Method of Variation of Constants (Lagrange Method)

Now we consider the nonhomogeneous systems that can be written in the vector-matrix form as

$$\mathbf{X}'(t) = A(t)\mathbf{X}(t) + \mathbf{f}(t).$$

The general solution of such a system is the sum of the general solution $\mathbf{X}_0(t)$ of the corresponding homogeneous system and a particular solution $\mathbf{X}_1(t)$ of the nonhomogeneous system, that is

$$\mathbf{X}(t) = \mathbf{X}_0(t) + \mathbf{X}_1(t) = \Phi(t)\,\mathbf{C} + \mathbf{X}_1(t),$$

where $\Phi(t)$ is a fundamental matrix, $\mathbf{C}$ is an arbitrary vector.

The most common method for solving the nonhomogeneous systems is the method of variation of constants (Lagrange method). With this method, instead of the constant vector $\mathbf{C}$ we consider the vector $\mathbf{C}(t)$ whose components are continuously differentiable functions of the independent variable t, that is we assume

$$\mathbf{X}(t) = \Phi(t)\,\mathbf{C}(t).$$

Substituting this into the nonhomogeneous system, we find the unknown vector $\mathbf{C}(t)$:

$$\mathbf{X}'(t) = A(t)\,\mathbf{X}(t) + \mathbf{f}(t), \quad \Rightarrow \cancel{\Phi'(t)\,\mathbf{C}(t)} + \Phi(t)\,\mathbf{C}'(t)$$
$$= \cancel{A(t)\,\Phi(t)\,\mathbf{C}(t)} + \mathbf{f}(t), \quad \Rightarrow \Phi(t)\,\mathbf{C}'(t) = \mathbf{f}(t).$$

Given that the matrix $\Phi(t)$ is nonsingular, we multiply this equation on the left by $\Phi^{-1}(t)$:

$$\Phi^{-1}(t)\,\Phi(t)\,\mathbf{C}'(t) = \Phi^{-1}(t)\,\mathbf{f}(t), \quad \Rightarrow \mathbf{C}'(t) = \Phi^{-1}(t)\,\mathbf{f}(t).$$

After integration we obtain the vector $\mathbf{C}(t)$.

Example 1.

Write the linear system of equations with the following solutions:

$$\mathbf{x}_1(t) = \begin{bmatrix} 2 \\ t \end{bmatrix}, \ \mathbf{x}_2(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}, \ t \neq 0.$$

Solution.

In this problem the fundamental matrix of the system is known:

$$\Phi(t) = \begin{bmatrix} 2 & t \\ t & t^2 \end{bmatrix}.$$

We compute the inverse matrix $\Phi^{-1}(t)$:

$$\Delta(\Phi) = \begin{vmatrix} 2 & t \\ t & t^2 \end{vmatrix} = 2t^2 - t^2 = t^2, \quad \Rightarrow \Phi^{-1}(t) = \frac{1}{\Delta(\Phi)} C_{ij}^T = \frac{1}{t^2} \begin{bmatrix} t^2 & -t \\ -t & 2 \end{bmatrix}^T$$

$$= \frac{1}{t^2} \begin{bmatrix} t^2 & -t \\ -t & 2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{t} \\ -\frac{1}{t} & \frac{2}{t^2} \end{bmatrix}.$$

Here C_{ij} denote the cofactors of the corresponding elements of the fundamental matrix $\Phi(t)$.

The coefficient matrix of the system of equations is given by

$$A(t) = \Phi'(t)\,\Phi^{-1}(t).$$

The derivative of the fundamental matrix (it is calculated element by element) is equal to

$$\Phi'(t) = \begin{bmatrix} 0 & 1 \\ 1 & 2t \end{bmatrix}.$$

Hence, we obtain:

$$A(t) = \begin{bmatrix} 0 & 1 \\ 1 & 2t \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{t} \\ -\frac{1}{t} & \frac{2}{t^2} \end{bmatrix} = \begin{bmatrix} 0 - \frac{1}{t} & 0 + \frac{2}{t^2} \\ 1 - 2 & -\frac{1}{t} + \frac{4}{t} \end{bmatrix} = \begin{bmatrix} -\frac{1}{t} & \frac{2}{t^2} \\ -1 & \frac{3}{t} \end{bmatrix}.$$

Thus, the system of equations whose solutions are $\mathbf{x}_1(t), \mathbf{x}_2(t)$, can be written as

$$\frac{dx}{dt} = -\frac{x}{t} + \frac{2y}{t^2}, \quad \frac{dy}{dt} = -x + \frac{3y}{t}.$$

Example 2.

Find a fundamental matrix of the system of differential equations

$$\frac{dx}{dt} = x + ty,\ \ \frac{dy}{dt} = tx + y,$$

making sure that the coefficient matrix $A\left(t\right)$ commutes with its integral.

Solution.

We first show that the multiplication of the matrix $A\left(t\right)$ by its integral is commutative. The original matrix is

$$A\left(t\right) = \begin{bmatrix} 1 & t \\ t & 1 \end{bmatrix}.$$

The integral of the matrix $A\left(t\right)$ is found by elementwise integration. For simplicity, we take the lower boundary of integration to be zero. Then

$$\int\limits_0^t A\left(\tau\right) d\tau = \begin{bmatrix} t & \frac{t^2}{2} \\ \frac{t^2}{2} & t \end{bmatrix}.$$

As a result, we have

$$A\left(t\right) \cdot \int\limits_0^t A\left(\tau\right) d\tau = \begin{bmatrix} 1 & t \\ t & 1 \end{bmatrix} \begin{bmatrix} t & \frac{t^2}{2} \\ \frac{t^2}{2} & t \end{bmatrix} = \begin{bmatrix} t + \frac{t^3}{2} & \frac{t^2}{2} + t^2 \\ t^2 + \frac{t^2}{2} & \frac{t^3}{2} + t \end{bmatrix} = \begin{bmatrix} t + \frac{t^3}{2} & \frac{3t^2}{2} \\ \frac{3t^2}{2} & t + \frac{t^3}{2} \end{bmatrix},$$

$$\int\limits_0^t A\left(\tau\right) d\tau \cdot A\left(t\right) = \begin{bmatrix} t & \frac{t^2}{2} \\ \frac{t^2}{2} & t \end{bmatrix} \begin{bmatrix} 1 & t \\ t & 1 \end{bmatrix} = \begin{bmatrix} t + \frac{t^3}{2} & t^2 + \frac{t^2}{2} \\ \frac{t^2}{2} + t^2 & \frac{t^3}{2} + t \end{bmatrix} = \begin{bmatrix} t + \frac{t^3}{2} & \frac{3t^2}{2} \\ \frac{3t^2}{2} & t + \frac{t^3}{2} \end{bmatrix}.$$

So, the commutative property of the matrix product is true. Therefore, the fundamental matrix is given by

$$\Phi\left(t\right) = e^{\int\limits_0^t A\left(\tau\right) d\tau} = e^{\begin{bmatrix} t & \frac{t^2}{2} \\ \frac{t^2}{2} & t \end{bmatrix}}.$$

We compute the matrix exponential by converting the matrix to diagonal form. In this case, the eigenvalues depend on the variable t and can be expressed as follows:

$$\begin{vmatrix} t-\lambda & \frac{t^2}{2} \\ \frac{t^2}{2} & t-\lambda \end{vmatrix} = 0, \quad \Rightarrow (t-\lambda)^2 - \left(\frac{t^2}{2}\right)^2 = 0, \quad \Rightarrow |\lambda - t| = \pm\frac{t^2}{2}, \quad \Rightarrow \lambda_{1,2} = t \pm \frac{t^2}{2}.$$

For each eigenvalue, we find the corresponding eigenvector. For λ_1 we obtain:

$$\lambda_1 = t + \frac{t^2}{2}, \quad \Rightarrow (A - \lambda_1 I)\,\mathbf{V}_1 = \mathbf{0}, \quad \Rightarrow \begin{bmatrix} t - \left(t + \frac{t^2}{2}\right) & \frac{t^2}{2} \\ \frac{t^2}{2} & t - \left(t + \frac{t^2}{2}\right) \end{bmatrix}.$$

$$\begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \mathbf{0}, \quad \Rightarrow -\frac{t^2}{2}V_{11} + \frac{t^2}{2}V_{21} = 0, \quad \Rightarrow \mathbf{V}_1 = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Similarly, we find the eigenvector $\mathbf{V}_2 = (V_{12}, V_{22})^T$ for the eigenvalue λ_2 :

$$\lambda_2 = t - \frac{t^2}{2}, \quad \Rightarrow (A - \lambda_2 I)\,\mathbf{V}_2 = \mathbf{0}, \Rightarrow \begin{bmatrix} t - \left(t - \frac{t^2}{2}\right) & \frac{t^2}{2} \\ \frac{t^2}{2} & t - \left(t - \frac{t^2}{2}\right) \end{bmatrix}.$$

$$\begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \mathbf{0}, \quad \Rightarrow \frac{t^2}{2}V_{12} + \frac{t^2}{2}V_{22} = 0, \quad \Rightarrow \mathbf{V}_2 = \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Then the matrix of reduction to diagonal form (more precisely to Jordan form) is given by

$$H = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

We compute the inverse matrix H^{-1} :

$$\Delta(H) = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 + 1 = 2, \quad \Rightarrow H^{-1} = \frac{1}{\Delta(H)} H_{ij}^T = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^T = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Hence, the Jordan form J is as follows:

$$J = H^{-1}\left[\int_0^t A(\tau)\,d\tau\right] H = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} t & \frac{t^2}{2} \\ \frac{t^2}{2} & t \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} t + \frac{t^2}{2} & \frac{t^2}{2} + t \\ -t + \frac{t^2}{2} & -\frac{t^2}{2} + t \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} t + \frac{t^2}{2} & 0 \\ 0 & t - \frac{t^2}{2} \end{bmatrix}.$$

The exponential of the matrix J is given by

$$e^J = \begin{bmatrix} e^{t+\frac{t^2}{2}} & 0 \\ 0 & e^{t-\frac{t^2}{2}} \end{bmatrix} = e^t \begin{bmatrix} e^{\frac{t^2}{2}} & 0 \\ 0 & e^{-\frac{t^2}{2}} \end{bmatrix}.$$

We can now calculate the fundamental matrix $\Phi(t)$:

$$\Phi(t) = e^{\int\limits_0^t A(\tau)d\tau} = He^J H^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot e^t \begin{bmatrix} e^{\frac{t^2}{2}} & 0 \\ 0 & e^{-\frac{t^2}{2}} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{e^t}{2} \begin{bmatrix} e^{\frac{t^2}{2}} + 0 & 0 - e^{-\frac{t^2}{2}} \\ e^{\frac{t^2}{2}} + 0 & 0 + e^{-\frac{t^2}{2}} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{e^t}{2} \begin{bmatrix} e^{\frac{t^2}{2}} & -e^{-\frac{t^2}{2}} \\ e^{\frac{t^2}{2}} & e^{-\frac{t^2}{2}} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{e^t}{2} \begin{bmatrix} e^{\frac{t^2}{2}} + e^{-\frac{t^2}{2}} & e^{\frac{t^2}{2}} - e^{-\frac{t^2}{2}} \\ e^{\frac{t^2}{2}} - e^{-\frac{t^2}{2}} & e^{\frac{t^2}{2}} + e^{-\frac{t^2}{2}} \end{bmatrix} = e^t \begin{bmatrix} \cosh\frac{t^2}{2} & \sinh\frac{t^2}{2} \\ \sinh\frac{t^2}{2} & \cosh\frac{t^2}{2} \end{bmatrix}.$$

Example 3.

Find the general solution of the system

$$\frac{dx}{dt} = -tx + y, \ \frac{dy}{dt} = \left(1 - t^2\right)x + ty, \ x > 0,$$

if one solution is known:

$$\mathbf{X}_1(t) = \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \begin{bmatrix} 1 \\ t \end{bmatrix}.$$

Method of Matrix Exponential

Definition and Properties of the Matrix Exponential

Consider a square matrix A of size $n \times n$, elements of which may be either real or complex numbers. Since the matrix A is square, the operation of raising to a power is defined, i.e. we can calculate the matrices

$$A^0 = I, \;\; A^1 = A, \;\; A^2 = A \cdot A, \;\; A^3 = A^2 \cdot A, \;\; \ldots, A^k = \underbrace{A \cdot A \cdots A}_{\text{k times}},$$

where I denotes a unit matrix of order n.

We form the infinite matrix power series

$$I + \frac{t}{1!}A + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots + \frac{t^k}{k!}A^k + \cdots$$

The sum of the infinite series is called the matrix exponential and denoted as e^{tA} :

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k.$$

This series is absolutely convergent.

In the limiting case, when the matrix consists of a single number a, i.e. has a size of 1×1, this formula is converted into a known formula for expanding the exponential function e^{at} in a Maclaurin series:

$$e^{at} = 1 + at + \frac{a^2 t^2}{2!} + \frac{a^3 t^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{a^k t^k}{k!}.$$

The matrix exponential has the following main properties:

- If A is a zero matrix, then $e^{tA} = e^0 = I$; (I is the identity matrix);
- If $A = I$, then $e^{tI} = e^t I$;
- If A has an inverse matrix A^{-1}, then $e^A e^{-A} = I$;
- $e^{mA} e^{nA} = e^{(m+n)A}$, where m, n are arbitrary real or complex numbers;
- The derivative of the matrix exponential is given by the formula

$$\frac{d}{dt}\left(e^{tA}\right) = Ae^{tA}.$$

- Let H be a nonsingular linear transformation. If $A = HMH^{-1}$, then $e^{tA} = He^{tM}H^{-1}$.

The Use of the Matrix Exponential for Solving Homogeneous Linear Systems with Constant Coefficients

The matrix exponential can be successfully used for solving systems of differential equations. Consider a system of linear homogeneous equations, which in matrix form can be written as follows:

$$\mathbf{X}'(t) = A\mathbf{X}(t).$$

The general solution of this system is represented in terms of the matrix exponential as

$$\mathbf{X}(t) = e^{tA}\mathbf{C},$$

where $\mathbf{C} = (C_1, C_2, \ldots, C_n)^T$ is an arbitrary n-dimensional vector. The symbol ^{T} denotes transposition. In this formula, we cannot write the vector $\mathbf{C}$ in front of the matrix exponential as the matrix product $\underset{[n\times 1]}{\mathbf{C}} \underset{[n\times n]}{e^{tA}}$ is not defined.

For an initial value problem (Cauchy problem), the components of $\mathbf{C}$ are expressed in terms of the initial conditions. In this case, the solution of the homogeneous system can be written as

$$\mathbf{X}(t) = e^{tA}\mathbf{X}_0, \text{ where } \mathbf{X}_0 = \mathbf{X}(t = t_0).$$

Thus, the solution of the homogeneous system becomes known, if we calculate the corresponding matrix exponential. To calculate it, we can use the infinite series, which is contained in the definition of the matrix exponential. Often, however, this allows us to find the matrix exponential only approximately. To solve the problem, one can also use an algebraic method based on the latest property listed above. Consider this method and the general pattern of solution in more detail.

Algorithm for Solving the System of Equations Using the Matrix Exponential

1. We first find the eigenvalues λ_i of the matrix (linear operator) A;
2. Calculate the eigenvectors and (in the case of multiple eigenvalues) generalized eigenvectors;
3. Construct the nonsingular linear transformation matrix H using the found regular and generalized eigenvectors. Compute the corresponding inverse matrix H^{-1};
4. Find the Jordan normal form J for the given matrix A, using the formula

$$J = H^{-1}AH.$$

Note: In the process of finding the regular and generalized eigenvectors, the structure of each Jordan block often becomes clear. This allows to write the Jordan form without calculation by the above formula.

5. Knowing the Jordan form J, we compose the matrix e^{tJ}. The corresponding formulas for this conversion are derived from the definition of the matrix exponential. The matrices e^{tJ} for some simple Jordan forms are shown in the following table:

Jordan Form J	Matrix e^{tJ}
$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$	$\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$
$\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$	$\begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} \end{bmatrix} = e^{\lambda_1 t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$
$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$	$\begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix}$
$\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$	$\begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{t^2}{2}e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_1 t} \end{bmatrix} = e^{\lambda_1 t} \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$

Figure 1.

6 Compute the matrix exponential e^{tA} by the formula

$$e^{tA} = He^{tJ}H^{-1}.$$

7 Write the general solution of the system:

$$\mathbf{X}(t) = e^{tA}\mathbf{C}.$$

For a second order system, the general solution is given by

$$\mathbf{X}(t) = \begin{bmatrix} x \\ y \end{bmatrix} = e^{tA} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$$

Example 1.

Find the general solution of the system, using the matrix exponential:

$$\frac{dx}{dt} = 2x + 3y, \quad \frac{dy}{dt} = 3x + 2y.$$

Solution.

We solve this system by following the algorithm described above. Calculate the eigenvalues of the matrix A :

$$\det(A-\lambda I)=\begin{vmatrix}2-\lambda & 3\\ 3 & 2-\lambda\end{vmatrix}=0,\quad \Rightarrow (2-\lambda)^2-9=0,$$
$$\Rightarrow 4-4\lambda+\lambda^2-9=0,\quad \Rightarrow \lambda^2-4\lambda-5=0,\quad \Rightarrow \lambda_1=5,\ \lambda_2=-1.$$

We find the corresponding eigenvectors for each of the eigenvalues. For the number $\lambda_1=5$ we have:

$$\begin{bmatrix}2-5 & 3\\ 3 & 2-5\end{bmatrix}\begin{bmatrix}V_{11}\\ V_{21}\end{bmatrix}=\mathbf{0},\quad \Rightarrow \begin{bmatrix}-3 & 3\\ 3 & -3\end{bmatrix}\begin{bmatrix}V_{11}\\ V_{21}\end{bmatrix}=\mathbf{0},\quad \Rightarrow 3V_{11}-3V_{21}=0,$$
$$\Rightarrow V_{11}-V_{21}=0.$$

By setting $V_{21}=t$, we find the eigenvector $\mathbf{V}_1=(V_{11},V_{21})^T$:

$$V_{21}=t,\quad \Rightarrow V_{11}=V_{21}=t,\quad \Rightarrow \mathbf{V}_1=\begin{bmatrix}V_{11}\\ V_{21}\end{bmatrix}=\begin{bmatrix}t\\ t\end{bmatrix}=t\begin{bmatrix}1\\ 1\end{bmatrix}\sim\begin{bmatrix}1\\ 1\end{bmatrix}.$$

Similarly, we find the eigenvector $\mathbf{V}_2=(V_{12},V_{22})^T$ associated with the eigenvalue $\lambda_2=-1$:

$$\begin{bmatrix}2-(-1) & 3\\ 3 & 2-(-1)\end{bmatrix}\begin{bmatrix}V_{12}\\ V_{22}\end{bmatrix}=0,\quad \Rightarrow \begin{bmatrix}3 & 3\\ 3 & 3\end{bmatrix}\begin{bmatrix}V_{12}\\ V_{22}\end{bmatrix}=0,\quad \Rightarrow 3V_{12}+3V_{22}=0,$$
$$\Rightarrow V_{12}+V_{22}=0.$$

Let $V_{22}=t$. Then $V_{12}=-V_{22}=-t$. Hence,

$$\mathbf{V}_2=\begin{bmatrix}V_{12}\\ V_{22}\end{bmatrix}=\begin{bmatrix}-t\\ t\end{bmatrix}=t\begin{bmatrix}-1\\ 1\end{bmatrix}\sim\begin{bmatrix}-1\\ 1\end{bmatrix}.$$

We compose the matrix H of the found eigenvectors $\mathbf{V}_1$ and $\mathbf{V}_2$:

$$H=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}.$$

Next, we compute the inverse matrix H^{-1} :

$$\Delta(H) = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 + 1 = 2,$$

$$H^{-1} = \frac{1}{\Delta(H)} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}^T = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^T = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

As in this example the eigenvalues are simple roots of the characteristic equation, we can immediately write down the Jordan form, which will have a simple diagonal form:

$$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}.$$

We verify this by using the formula of the transition from the original matrix A to the Jordan normal form J :

$$J = H^{-1}AH = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 5 & 5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} 5+5 & -5+5 \\ 1-1 & -1-1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 10 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} = J.$$

Now we form the matrix e^{tJ} (it can also be called the matrix exponential):

$$e^{tJ} = \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Compute the matrix exponential e^{tA} :

$$e^{tA} = He^{tJ}H^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{-t} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{5t} & -e^{-t} \\ e^{5t} & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} e^{5t} + e^{-t} & e^{5t} - e^{-t} \\ e^{5t} - e^{-t} & e^{5t} + e^{-t} \end{bmatrix}.$$

The general solution of the system can be written as

$$\mathbf{X}(t) = \begin{bmatrix} x \\ y \end{bmatrix} = e^{tA} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{5t} + e^{-t} & e^{5t} - e^{-t} \\ e^{5t} - e^{-t} & e^{5t} + e^{-t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$$

where C_1, C_2 are arbitrary numbers.

This answer can also be expressed in another form:

$$\mathbf{X}(t) = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} C_1 e^{5t} + C_1 e^{-t} + C_2 e^{5t} - C_2 e^{-t} \\ C_1 e^{5t} - C_1 e^{-t} + C_2 e^{5t} + C_2 e^{-t} \end{bmatrix} = \frac{1}{2} \cdot$$
$$\begin{bmatrix} e^{5t}(C_1 + C_2) + e^{-t}(C_1 - C_2) \\ e^{5t}(C_1 + C_2) - e^{-t}(C_1 - C_2) \end{bmatrix} = \frac{1}{2}(C_1 + C_2) e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}(C_1 + C_2) e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= B_1 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + B_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where B_1 and B_2 denote arbitrary constants associated with C_1, C_2.

Example 2.

Solve the system of equations by the method of matrix exponential:

$$\frac{dx}{dt} = 4x, \ \frac{dy}{dt} = x + 4y.$$

Solution.

We solve the auxiliary equation and find the eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 4 - \lambda & 0 \\ 1 & 4 - \lambda \end{vmatrix} = 0, \ \Rightarrow (4 - \lambda)^2 = 0, \ \Rightarrow \lambda_1 = 4.$$

So, we have one eigenvalue $\lambda_1 = 4$ of multiplicity 2. Determine the eigenvector $\mathbf{V}_1 = (V_{11}, V_{21})^T$:

$$\begin{bmatrix} 4 - 4 & 0 \\ 1 & 4 - 4 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \mathbf{0}, \ \Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \mathbf{0}, \ \Rightarrow 1 \cdot V_{11} + 0 \cdot V_{21} = 0.$$

It follows that the coordinate $V_{11}=0$, and the coordinate V_{21} can be any number. We choose for simplicity $V_{21}=1$. Hence, the eigenvector $\mathbf{V}_1$ is equal: $\mathbf{V}_1=(0,1)^T$.

The second linearly independent vector is defined as the generalized eigenvector $\mathbf{V}_2=(V_{12},V_{22})^T$, connected to $\mathbf{V}_1$. It can be found from the equation

$$(A-\lambda_1 I)\,\mathbf{V}_2=\mathbf{V}_1, \quad \Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Rightarrow \begin{cases} 0\cdot V_{12}+0\cdot V_{22}=0 \\ 1\cdot V_{12}+0\cdot V_{22}=1 \end{cases}.$$

Here the coordinate V_{22} can be any number. We choose $V_{22}=0$. Then we obtain $V_{11}=1$. Thus, the generalized eigenvector is $\mathbf{V}_2=(1,0)^T$.

Now, using the basis vectors, we form the matrix H – the transition matrix from A to the Jordan canonical form J :

$$H=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Calculate the inverse matrix H^{-1} :

$$\Delta(H)=\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}=0-1=-1,\ H^{-1}=\frac{1}{\Delta(H)}\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}^T.$$

Here H_{ij} denote the cofactors of the elements of the matrix H. In the result of the calculations we find:

$$H^{-1}=\frac{1}{(-1)}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}^T=(-1)\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Interestingly, in this case the inverse matrix H^{-1} coincides with the initial matrix H. Such an effect is possible, if the square of the original matrix is the identity matrix:

$$H^2=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I.$$

The Jordan form J of the matrix A is

$$J = H^{-1}AH = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+1 & 0+4 \\ 4+0 & 0+0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 4 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+4 & 1+0 \\ 0+0 & 4+0 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}.$$

We see that the Jordan form J consists of a single block of size 2.

Compose the matrix e^{tJ} :

$$e^{tJ} = \begin{bmatrix} e^{4t} & te^{4t} \\ 0 & e^{4t} \end{bmatrix} = e^{4t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Calculate the matrix exponential e^{tA} :

$$e^{tA} = He^{tJ}H^{-1} = e^{4t} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = e^{4t} \begin{bmatrix} 0+0 & 0+1 \\ 1+0 & t+0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= e^{4t} \begin{bmatrix} 0 & 1 \\ 1 & t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = e^{4t} \begin{bmatrix} 0+1 & 0+0 \\ 0+t & 1+0 \end{bmatrix} = e^{4t} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}.$$

The general solution is written as

$$\mathbf{X}(t) = \begin{bmatrix} x \\ y \end{bmatrix} = e^{tA}\mathbf{C} = e^{4t} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$$

where C_1, C_2 are arbitrary constants.

Example 3.

Solve the system of equations using the matrix exponential:

$$\frac{dx}{dt} = x + y, \ \frac{dy}{dt} = -x + y.$$

Solution.

In this case, the coefficient matrix A is

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Calculate its eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0, \quad \Rightarrow (1-\lambda)^2 + 1 = 0, \quad \Rightarrow (\lambda - 1)^2 = -1,$$
$$\Rightarrow \lambda - 1 = \pm i, \quad \Rightarrow \lambda_{1,2} = 1 \pm i.$$

Thus, the matrix A has a pair of complex conjugate eigenvalues. For each eigenvalue, we find the corresponding eigenvector (it can have complex coordinates).

Let $\mathbf{V}_1 = (V_{11}, V_{21})^T$ be an eigenvector, associated with the eigenvalue $\lambda_1 = 1 + i$. The coordinates of this vector satisfy the following matrix-vector equation:

$$\begin{bmatrix} 1-(1+i) & 1 \\ -1 & 1-(1+i) \end{bmatrix} \cdot \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \mathbf{0}, \quad \Rightarrow \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \mathbf{0},$$
$$\Rightarrow \begin{cases} -iV_{11} + V_{21} = 0 \\ -V_{11} - iV_{21} = 0 \end{cases}, \quad \Rightarrow \left\{ \begin{array}{l} V_{11} + iV_{21} = 0 \\ iV_{11} - V_{21} = 0 \end{array} \right| {\scriptstyle R_2 - iR_1}, \quad \Rightarrow \begin{cases} V_{11} + iV_{21} = 0 \\ 0 = 0 \end{cases},$$
$$\Rightarrow V_{11} + iV_{21} = 0.$$

We set $V_{21} = t$. Then $V_{11} = -it$. Hence, the eigenvector $\mathbf{V}_1$ is given by

$$\mathbf{V}_1 = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix} \sim \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Similarly, we find the eigenvector $\mathbf{V}_2 = (V_{12}, V_{22})^T$, associated with the number $\lambda_2 = 1 - i$:

$$\begin{bmatrix} 1-(1-i) & 1 \\ -1 & 1-(1-i) \end{bmatrix} \cdot \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \mathbf{0}, \quad \Rightarrow \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \mathbf{0},$$
$$\Rightarrow \begin{cases} iV_{12} + V_{22} = 0 \\ -V_{12} + iV_{22} = 0 \end{cases}, \quad \Rightarrow \left\{ \begin{array}{l} V_{12} - iV_{22} = 0 \\ iV_{12} + V_{22} = 0 \end{array} \right| {\scriptstyle R_2 - iR_1}, \quad \Rightarrow \begin{cases} V_{12} - iV_{22} = 0 \\ 0 = 0 \end{cases},$$
$$\Rightarrow V_{12} - iV_{22} = 0.$$

Here we set $V_{22} = t$. Therefore, $V_{12} = it$. The vector $\mathbf{V}_2$ is equal to:

$$\mathbf{V}_2 = \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix} \sim \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

We form the matrix H of the found eigenvectors $\mathbf{V}_1$ and $\mathbf{V}_2$:

$$H = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}.$$

Calculate the inverse matrix H^{-1} by the formula

$$H^{-1} = \frac{1}{\Delta(H)} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}^T,$$

where $\Delta(H)$ is the determinant of the matrix H, and H_{ij} are cofactors of the elements of the matrix H. As a result, we obtain:

$$\Delta(H) = \begin{vmatrix} -i & i \\ 1 & 1 \end{vmatrix} = -i - i = -2i,$$

$$H^{-1} = \frac{1}{(-2i)} \begin{bmatrix} 1 & -1 \\ -i & -i \end{bmatrix}^T = \frac{1}{(-2i)} \begin{bmatrix} 1 & -i \\ -1 & -i \end{bmatrix} = \frac{1}{2i} \begin{bmatrix} -1 & i \\ 1 & i \end{bmatrix}.$$

Now we find the Jordan form J by the formula

$$J = H^{-1} A H.$$

Performing calculations, we find:

$$J = \frac{1}{2i} \begin{bmatrix} -1 & i \\ 1 & i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} = \frac{1}{2i} \begin{bmatrix} -1-i & -1+i \\ 1-i & 1+i \end{bmatrix} \cdot \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2i} \begin{bmatrix} 2i-2 & 0 \\ 0 & 2i+2 \end{bmatrix} = \begin{bmatrix} \frac{i-1}{i} & 0 \\ 0 & \frac{i+1}{i} \end{bmatrix} = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}.$$

Generally speaking, we can immediately write the Jordan form J, which in this case is diagonal (as the eigenvalues λ_1, λ_2 have multiplicity 1). We will assume that the computation is done to test the Jordan form J, and the matrices H and H^{-1} are needed to define the matrix exponential.

Now we form the matrix e^{tJ} :

$$e^{tJ} = \begin{bmatrix} e^{(1+i)t} & 0 \\ 0 & e^{(1-i)t} \end{bmatrix} = \begin{bmatrix} e^{t}e^{it} & 0 \\ 0 & e^{t}e^{-it} \end{bmatrix} = e^{t} \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}.$$

Compute the matrix exponential e^{tA} :

$$e^{tA} = He^{tJ}H^{-1} = \frac{e^{t}}{2i} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} \cdot \begin{bmatrix} -1 & i \\ 1 & i \end{bmatrix}.$$

The exponential functions e^{it}, e^{-it}, can be expanded by Euler's formula:

$$e^{it} = \cos t + i \sin t, \;\; e^{-it} = \cos t - i \sin t.$$

We get the following result:

$$e^{tA} = \frac{e^{t}}{2i} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos t + i \sin t & 0 \\ 0 & \cos t - i \sin t \end{bmatrix} \cdot \begin{bmatrix} -1 & i \\ 1 & i \end{bmatrix} = \frac{e^{t}}{2i} \cdot$$
$$\begin{bmatrix} -i \cos t + \sin t & i \cos t + \sin t \\ \cos t + i \sin t & \cos t - i \sin t \end{bmatrix} \cdot \begin{bmatrix} -1 & i \\ 1 & i \end{bmatrix} = \frac{e^{t}}{2i} \begin{bmatrix} 2i \cos t & 2i \sin t \\ -2i \sin t & 2i \cos t \end{bmatrix}$$
$$= e^{t} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

The general solution of the system of differential equations is given by

$$\mathbf{X}(t) = \begin{bmatrix} x \\ y \end{bmatrix} = e^{tA}\mathbf{C} = e^{t} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$$

where $C = (C_1, C_2)^T$ is an arbitrary vector.

Higher Order Linear Homogeneous Differential Equations with Constant Coefficients

The linear homogeneous differential equation of the nth order with constant coefficients can be written as

$$y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots + a_{n-1} y'(x) + a_n y(x) = 0,$$

where $a_1, a_2, \ldots, a_n$ are constants which may be real or complex.

Using the linear differential operator $L(D)$, this equation can be represented as

$$L(D)\, y(x) = 0,$$

where

$$L(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n.$$

For each differential operator with constant coefficients, we can introduce the characteristic polynomial

$$L(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n.$$

The algebraic equation

$$L(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

is called the characteristic equation of the differential equation.

According to the fundamental theorem of algebra, a polynomial of degree n has exactly n roots, counting multiplicity. In this case the roots can be both real and complex (even if all the coefficients of $a_1, a_2, \ldots, a_n$ are real).

Let us consider in more detail the different cases of the roots of the characteristic equation and the corresponding formulas for the general solution of differential equations.

Case 1. All Roots of the Characteristic Equation are Real and Distinct

We assume that the characteristic equation $L(\lambda) = 0$ has n roots $\lambda_1, \lambda_2, \ldots, \lambda_n$. In this case the general solution of the differential equation is written in a simple form:

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \cdots + C_n e^{\lambda_n x},$$

where $C_1, C_2, \ldots, C_n$ are constants depending on initial conditions.

Case 2. The Roots of the Characteristic Equation are Real and Multiple

Let the characteristic equation $L(\lambda) = 0$ of degree n have m roots $\lambda_1, \lambda_2, \ldots, \lambda_m$, the multiplicity of which, respectively, is equal to $k_1, k_2, \ldots, k_m$. It is clear that the following condition holds:

$$k_1 + k_2 + \cdots + k_m = n.$$

Then the general solution of the homogeneous differential equations with constant coefficients has the form

$$y(x) = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x} + \cdots + C_{k_1} x^{k_1 - 1} e^{\lambda_1 x} + \cdots + C_{n-k_m+1} e^{\lambda_m x}$$
$$+ C_{n-k_m+2} x e^{\lambda_m x} + \cdots + C_n x^{k_m - 1} e^{\lambda_m x}.$$

It is seen that the formula of the general solution has exactly k_i terms corresponding to each root λ_i of multiplicity k_i. These terms are formed by multiplying x to a certain degree by the exponential function $e^{\lambda_i x}$. The degree of x varies in the range from 0 to $k_i - 1$, where k_i is the multiplicity of the root λ_i.

Case 3. The Roots of the Characteristic Equation are Complex and Distinct

If the coefficients of the differential equation are real numbers, the complex roots of the characteristic equation will be presented in the form of conjugate pairs of complex

$$\lambda_{1,2} = \alpha \pm i\beta, \;\; \lambda_{3,4} = \gamma \pm i\delta, \;\; \ldots$$

In this case the general solution is written as

$$y(x) = e^{\alpha x}\left(C_1 \cos \beta x + C_2 \sin \beta x\right) + e^{\gamma x}\left(C_3 \cos \delta x + C_4 \sin \delta x\right) + \cdots$$

Case 4. The Roots of the Characteristic Equation are Complex and Multiple

Here, each pair of complex conjugate roots $\alpha \pm i\beta$ of multiplicity k produces $2k$ particular solutions:

$$e^{\alpha x} \cos \beta x,\ e^{\alpha x} \sin \beta x,\ e^{\alpha x} x \cos \beta x,\ e^{\alpha x} x \sin \beta x, \ldots,\ e^{\alpha x} x^{k-1} \cos \beta x,\ e^{\alpha x} x^{k-1} \sin \beta x.$$

Then the part of the general solution of the differential equation corresponding to a given pair of complex conjugate roots is constructed as follows:

$$\begin{aligned} y(x) &= e^{\alpha x}\left(C_1 \cos \beta x + C_2 \sin \beta x\right) + x e^{\alpha x}\left(C_3 \cos \beta x + C_4 \sin \beta x\right) + \cdots \\ &+ x^{k-1} e^{\alpha x}\left(C_{2k-1} \cos \beta x + C_{2k} \sin \beta x\right). \end{aligned}$$

In general, when the characteristic equation has both real and complex roots of arbitrary multiplicity, the general solution is constructed as the sum of the above solutions of the form $1 - 4$.

Example 1

Solve the differential equation $y''' + 2y'' - y' - 2y = 0$.

Solution.

Write the corresponding characteristic equation:

$$\lambda^3 + 2\lambda^2 - \lambda - 2 = 0.$$

Solving it, we find the roots:

$$\begin{aligned} &\lambda^2(\lambda + 2) - (\lambda + 2) = 0, \quad \Rightarrow (\lambda + 2)\left(\lambda^2 - 1\right) = 0, \\ &\Rightarrow (\lambda + 2)(\lambda - 1)(\lambda + 1) = 0, \quad \Rightarrow \lambda_1 = -2,\ \lambda_2 = 1,\ \lambda_3 = -1. \end{aligned}$$

It is seen that all three roots are real. Therefore, the general solution of the differential equations can be written as

$$y(x) = C_1 e^{-2x} + C_2 e^{x} + C_3 e^{-x},$$

where C_1, C_2, C_3 are arbitrary constants.

Example 2.

Solve the equation $y''' - 7y'' + 11y' - 5y = 0$.

Solution.

The corresponding characteristic equation is

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0.$$

It is easy to see that one of the roots is the number $\lambda = 1$. Then, factoring the term $(\lambda - 1)$ from the equation, we obtain

$$\lambda^3 - \lambda^2 - 6\lambda^2 + 6\lambda + 5\lambda - 5 = 0, \quad \Rightarrow \lambda^2 (\lambda - 1) - 6\lambda (\lambda - 1) + 5 (\lambda - 1) = 0,$$
$$\Rightarrow (\lambda - 1) \cdot \left(\lambda^2 - 6\lambda + 5\right) = 0, \quad \Rightarrow (\lambda - 1) \cdot (\lambda - 1) \cdot (\lambda - 5) = 0,$$
$$\Rightarrow (\lambda - 1)^2 (\lambda - 5) = 0.$$

Thus, the equation has two roots $\lambda_1 = 1$, $\lambda_2 = 5$, the first of which has multiplicity 2. Then the general solution of differential equations can be written as follows:

$$y(x) = (C_1 + C_2 x) e^{x} + C_3 e^{5x},$$

where C_1, C_2, C_3 are arbitrary numbers.

Example 3.

Solve the equation $y^{IV} - y''' + 2y' = 0$.

Solution.

Write the characteristic equation:

$$\lambda^4 - \lambda^3 + 2\lambda = 0.$$

Factor the left side and find the roots:

$$\lambda\left(\lambda^3 - \lambda^2 + 2\right) = 0.$$

Note that one of the roots of the cubic polynomial is the number $\lambda = -1$. Therefore, we divide $\lambda^3 - \lambda^2 + 2$ by $\lambda + 1$:

$$\frac{\lambda^3 - \lambda^2 + 2}{\lambda + 1} = \lambda^2 - 2\lambda + 2.$$

As a result, the characteristic equation takes the following form:

$$\lambda\left(\lambda + 1\right) \cdot \left(\lambda^2 - 2\lambda + 2\right) = 0.$$

We find the roots of the quadratic equation:

$$\lambda^2 - 2\lambda + 2 = 0, \quad \Rightarrow D = 4 - 8 = -4, \quad \Rightarrow \lambda = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i.$$

Thus, the characteristic equation has four distinct roots, two of which are complex:

$$\lambda_1 = 0, \;\; \lambda_2 = -1, \;\; \lambda_{3,4} = 1 \pm i.$$

The general solution of the differential equation can be represented as

$$y\left(x\right) = C_1 + C_2 e^{-x} + e^x\left(C_3 \cos x + C_4 \sin x\right),$$

where $C_1, \ldots, C_4$ are arbitrary constants.

Example 4.

Solve the equation $y^V + 18y''' + 81y' = 0.$

Solution.

The characteristic equation can be written as

$$\lambda^5 + 18\lambda^3 + 81\lambda = 0.$$

Factor the left side and calculate the roots:

$$\lambda\left(\lambda^4 + 18\lambda^2 + 81\right) = 0, \quad \Rightarrow \lambda\left(\lambda^2 + 9\right)^2 = 0.$$

As it can be seen, the equation has the following roots:

$$\lambda_1 = 0, \;\; \lambda_{2,3} = \pm 3i,$$

and imaginary roots have multiplicity 2. In accordance with the rules set out above, we write the general solution in the form

$$y\left(x\right) = C_1 + \left(C_2 + C_3 x\right)\cos 3x + \left(C_4 + C_5 x\right)\sin 3x,$$

where $C_1, \ldots, C_5$ are arbitrary numbers.

Example 5.

Solve the differential equation $y^{IV} - 4y''' + 5y'' - 4y' + 4y = 0.$

Higher Order Linear Nonhomogeneous Differential Equations with Constant Coefficients

These equations have the form

$$y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots + a_{n-1} y'(x) + a_n y(x) = f(x),$$

where $a_1, a_2, \ldots, a_n$ are real or complex numbers, and the right-hand side $f(x)$ is a continuous function on some interval $[a, b]$.

Using the linear differential operator $L(D)$ equal to

$$L(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n,$$

the nonhomogeneous differential equation can be written as

$$L(D)\, y(x) = f(x).$$

The general solution $y(x)$ of the nonhomogeneous equation is the sum of the general solution $y_0(x)$ of the corresponding homogeneous equation and a particular solution $y_1(x)$ of the nonhomogeneous equation:

$$y(x) = y_0(x) + y_1(x).$$

For an arbitrary right side $f(x)$, the general solution of the nonhomogeneous equation can be found using the method of variation of parameters. If the right-hand side is the product of a polynomial and exponential functions, it is more convenient to seek a particular solution by the method of undetermined coefficients.

Method of Variation of Parameters

We assume that the general solution of the homogeneous differential equation of the n th order is known and given by

$$y_0(x) = C_1 Y_1(x) + C_2 Y_2(x) + \cdots + C_n Y_n(x).$$

According to the method of variation of constants (or Lagrange method), we consider the functions $C_1(x), C_2(x), \ldots, C_n(x)$ instead of the regular numbers $C_1, C_2, \ldots, C_n$. These functions are chosen so that the solution

$$y = C_1(x) Y_1(x) + C_2(x) Y_2(x) + \cdots + C_n(x) Y_n(x)$$

satisfies the original nonhomogeneous equation.

The derivatives of n unknown functions $C_1(x), C_2(x), \ldots, C_n(x)$ are determined from the system of n equations:

$$\begin{cases} C_1'(x) Y_1(x) + C_2'(x) Y_2(x) + \cdots + C_n'(x) Y_n(x) = 0 \\ C_1'(x) Y_1'(x) + C_2'(x) Y_2'(x) + \cdots + C_n'(x) Y_n'(x) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ C_1'(x) Y_1^{(n-1)}(x) + C_2'(x) Y_2^{(n-1)}(x) + \cdots + C_n'(x) Y_n^{(n-1)}(x) = f(x) \end{cases}$$

The determinant of this system is the Wronskian of $Y_1, Y_2, \ldots, Y_n$ forming a fundamental system of solutions. By the linear independence of these functions, the determinant is not zero and the system is uniquely solvable. The final expressions for the functions $C_1(x), C_2(x), \ldots, C_n(x)$ can be found by integration.

Method of Undetermined Coefficients

If the right-hand side $f(x)$ of the differential equation is a function of the form

$$P_n(x) e^{\alpha x} \quad \text{or} \quad [P_n(x) \cos \beta x \, + \, Q_m(x) \sin \beta x] \, e^{\alpha x},$$

where $P_n(x), Q_m(x)$ are polynomials of degree n and m, respectively, then the method of undetermined coefficients may be used to find a particular solution.

In this case, we seek a particular solution in the form corresponding to the structure of the right-hand side of the equation. For example, if the function has the form

$$f(x) = P_n(x) e^{\alpha x},$$

the particular solution is given by

$$y_1(x) = x^s A_n(x) e^{\alpha x},$$

where $A_n(x)$ is a polynomial of the same degree n as $P_n(x)$. The coefficients of the polynomial $A_n(x)$ are determined by direct substitution of the trial solution $y_1(x)$ in the nonhomogeneous differential equation.

In the so-called resonance case, when the number of α in the exponential function coincides with a root of the characteristic equation, an additional factor x^s, where s is the multiplicity of the root, appears in the particular solution. In the *non-resonance case*, we set $s = 0$.

The same algorithm is used when the right-hand side of the equation is given in the form

$$f(x) = [P_n(x) \cos \beta x + Q_m(x) \sin \beta x] e^{\alpha x}.$$

Here the particular solution has a similar structure and can be written as

$$y_1(x) = x^s [A_n(x) \cos \beta x + B_n(x) \sin \beta x] e^{\alpha x},$$

where $A_n(x)$, $B_n(x)$ are polynomials of degree n (for $n \geq m$), and the degree s in the additional factor x^s is equal to the multiplicity of the complex root $\alpha \pm \beta i$ in the resonance case (i.e. when the numbers α and β coincide with the complex root of the characteristic equation), and accordingly, $s = 0$ in the non-resonance case.

Superposition Principle

The superposition principle is stated as follows. Let the right-hand side $f(x)$ be the sum of two functions:

$$f(x) = f_1(x) + f_2(x).$$

Suppose that $y_1(x)$ is a solution of the equation

$$L(D)\,y(x) = f_1(x),$$

and the function $y_2(x)$ is, accordingly, a solution of the second equation

$$L(D)\,y(x) = f_2(x).$$

Then the sum of the functions

$$y(x) = y_1(x) + y_2(x)$$

will be a solution of the linear nonhomogeneous equation

$$L(D)\,y(x) = f(x) = f_1(x) + f_2(x).$$

Example 1.

Find the general solution of the differential equation $y''' + 3y'' - 10y' = x - 3.$

Solution.

First we find the general solution of the homogeneous equation

$$y''' + 3y'' - 10y' = 0.$$

Calculate the roots of the characteristic equation:

$$\lambda^3 + 3\lambda^2 - 10\lambda = 0, \quad \Rightarrow \lambda\left(\lambda^2 + 3\lambda - 10\right) = 0, \quad \Rightarrow \lambda(\lambda - 2)(\lambda + 5) = 0.$$

Hence,

$$\lambda_1 = 0, \ \lambda_2 = 2, \ \lambda_3 = -5.$$

So the general solution of the homogeneous equation is given by

$$y_0(x) = C_1 + C_2 e^{2x} + C_3 e^{-5x},$$

where C_1, C_2, C_3 are arbitrary numbers.

The right side of the equation contains only a polynomial. However, if we take into account that $e^0 = 1$, we see that in fact we have the resonance case (in disguised form) as one of the roots of the characteristic equation is also zero: $\lambda_1 = 0$. Therefore, we will seek a particular solution in the form

$$y_1(x) = x(Ax + B) = Ax^2 + Bx.$$

Substitute the derivatives

$$y_1' = 2Ax + B, \ y_1'' = 2A, \ y_1''' = 0.$$

into the nonhomogeneous equation and determine the coefficients A, B :

$$0 + 3 \cdot 2A - 10(2Ax + B) = x - 3, \quad \Rightarrow 6A - 20Ax - 10B = x - 3,$$
$$\Rightarrow \begin{cases} -20A = 1 \\ 6A - 10B = -3 \end{cases}, \quad \Rightarrow \begin{cases} A = -\frac{1}{20} \\ B = \frac{27}{100} \end{cases}, \quad \Rightarrow \begin{cases} A = -\frac{5}{100} \\ B = \frac{27}{100} \end{cases}.$$

The particular solution y_1 is written as

$$y_1(x) = x\left(-\frac{5}{100}x + \frac{27}{100}\right) = \frac{x}{100}(27 - 5x).$$

Thus, the general solution of nonhomogeneous differential equation is given by

$$y(x) = y_0(x) + y_1(x) = C_1 + C_2 e^{2x} + C_3 e^{-5x} + \frac{x}{100}(27 - 5x).$$

Example 2

Solve the differential equation $y''' - y' = \sin 3x$.

Solution.

We construct the general solution of the homogeneous equation

$$y''' - y' = 0.$$

The roots of the characteristic equation are

$$\lambda^3 - \lambda = 0, \quad \Rightarrow \lambda\left(\lambda^2 - 1\right) = 0, \quad \Rightarrow \lambda(\lambda - 1)(\lambda + 1) = 0, \quad \Rightarrow \lambda_1 = 0, \;\; \lambda_2 = 1, \;\; \lambda_3 = -1.$$

Consequently, the general solution of the homogeneous equation can be written as

$$y_0(x) = C_1 + C_2 e^x + C_3 e^{-x},$$

where C_1, C_2, C_3 are arbitrary numbers.

Based on the structure of the right-hand side, we seek a particular solution in the form of trial function

$$y_1(x) = A \sin 3x + B \cos 3x.$$

The derivatives of this function are as follows:

$$y_1' = 3A \cos 3x - 3B \sin 3x,$$

$$y_1'' = -9A \sin 3x - 9B \cos 3x,$$

$$y_1''' = -27A \cos 3x + 27B \sin 3x.$$

Substituting these derivatives into the equation, we obtain

$$-27A \cos 3x + 27B \sin 3x - 3A \cos 3x + 3B \sin 3x = \sin 3x,$$

$$\Rightarrow -30A \cos 3x + 30B \sin 3x = \sin 3x, \quad \Rightarrow \begin{cases} -30A = 0 \\ 30B = 1 \end{cases}, \quad \Rightarrow \begin{cases} A = 0 \\ B = \frac{1}{30} \end{cases}.$$

Thus, a particular solution can be written as

$$y_1(x) = \frac{1}{30}\cos 3x.$$

Accordingly, the general solution of the nonhomogeneous equation is described by

$$y(x) = y_0(x) + y_1(x) = C_1 + C_2 e^{x} + C_3 e^{-x} + \frac{1}{30}\cos 3x.$$

Example 3.

Solve the differential equation $y^{IV} - y = 2\cos x$.

Solution.

We first consider the homogeneous equation

$$y^{IV} - y = 0$$

and construct its general solution. The characteristic equation

$$\lambda^4 - 1 = 0$$

has the following roots:

$$\left(\lambda^2 - 1\right)\left(\lambda^2 + 1\right) = 0, \quad \Rightarrow (\lambda - 1)(\lambda + 1)\left(\lambda^2 + 1\right) = 0, \quad \Rightarrow \lambda_1 = 1, \ \lambda_2 = -1, \ \lambda_{3,4} = \pm i.$$

Consequently, the general solution of the homogeneous equation has the form:

$$y_0(x) = C_1 e^{x} + C_2 e^{-x} + C_3 \cos x + C_4 \sin x,$$

where $C_1, \ldots, C_4$ are arbitrary numbers.

Now we find a particular solution of the nonhomogeneous equation. Here we have the resonance case, since the expression in the right side corresponds to one of the roots of characteristic equation. Hence, we seek a particular solution in the form

$$y_1(x) = x\left(A\cos x + B\sin x\right).$$

The derivatives of this function are

$$y_1' = A\cos x + B\sin x + x\left(-A\sin x + B\cos x\right),$$

$$y_1'' = -A\sin x + B\cos x + \left(-A\sin x + B\cos x\right) + x\left(-A\cos x - B\sin x\right)$$
$$= -2A\sin x + 2B\cos x - x\left(A\cos x + B\sin x\right),$$

$$y_1''' = -2A\cos x - 2B\sin x - \left(A\cos x + B\sin x\right) - x\left(-A\sin x + B\cos x\right)$$
$$= -3A\cos x - 3B\sin x + x\left(A\sin x - B\cos x\right),$$

$$y^{IV} = 3A\sin x - 3B\cos x + \left(A\sin x - B\cos x\right) + x\left(A\cos x + B\sin x\right)$$
$$= 4A\sin x - 4B\cos x + x\left(A\cos x + B\sin x\right).$$

Substitute the derivatives in the nonhomogeneous equation and determine the coefficients A, B :

$$4A\sin x - 4B\cos x + \cancel{x\left(A\cos x + B\sin x\right)} - \cancel{x\left(A\cos x + B\sin x\right)} = 2\cos x,$$
$$\Rightarrow \begin{cases} 4A = 0 \\ -4B = 2 \end{cases}, \quad \Rightarrow \begin{cases} A = 0 \\ B = -\frac{1}{2} \end{cases}.$$

Thus, a particular solution is expressed as

$$y_1(x) = -\frac{x}{2}\sin x.$$

Then the general solution of the original nonhomogeneous equation can be written as

$$y(x) = y_0(x) + y_1(x) = C_1e^x + C_2e^{-x} + C_3\cos x + C_4\sin x - \frac{x}{2}\sin x.$$

Example 4.

Solve the equation $y^{IV} + y''' - 3y'' - 5y' - 2y = e^{2x} - e^{-x}$.

Solution.

First we find the general solution of the homogeneous equation

$$y^{IV} + y''' - 3y'' - 5y' - 2y = 0.$$

Write the characteristic equation and find its roots:

$$\lambda^4 + \lambda^3 - 3\lambda^2 - 5\lambda - 2 = 0, \quad \Rightarrow \lambda^4 - 2\lambda^3 + 3\lambda^3 - 6\lambda^2 + 3\lambda^2 - 6\lambda + \lambda - 2 = 0,$$
$$\Rightarrow \lambda^3(\lambda - 2) + 3\lambda^2(\lambda - 2) + 3\lambda(\lambda - 2) + \lambda - 2 = 0,$$
$$\Rightarrow \left(\lambda^3 + 3\lambda^2 + 3\lambda + 1\right) \cdot (\lambda - 2) = 0, \quad \Rightarrow (\lambda + 1)^3 (\lambda - 2) = 0.$$

It is seen that the equation has two roots:

$$\lambda_1 = -1, \ \lambda_2 = 2,$$

and the multiplicity of the first root is 3.

Then the general solution of the homogeneous equation can be written as

$$y_0(x) = \left(C_1 + C_2 x + C_3 x^2\right) e^{-x} + C_4 e^{2x},$$

where $C_1, \ldots, C_4$ are as usual arbitrary numbers.

We now construct a particular solution of the nonhomogeneous equation. Using the superposition principle, it is convenient to consider two nonhomogeneous equations of the form

1 $y^{IV} + y''' - 3y'' - 5y' - 2y = e^{2x};$

2 $y^{IV} + y''' - 3y'' - 5y' - 2y = -e^{-x}.$

The sum of the right sides of these equations corresponds to the right side of the original nonhomogeneous equation.

Note that we have the resonance cases in both equations. In the first equation the number 2 in the exponential function coincides with the root $\lambda_2 = 2$ of multiplicity 2. In the second equation the number -1 in the exponential function coincides with another root $\lambda_1 =$

the multiplicity of which is equal to 3. With this in mind, we seek particular solutions y_1, y_2, respectively, for Equations 1 and 2 in the form

$$y_1 = Axe^{2x},\ \ y_2 = Bx^3e^{-x}.$$

The derivatives for the trial solution y_1 have the form

$$y_1' = A\left(e^{2x} + 2xe^{2x}\right) = A\left(2x+1\right)e^{2x},$$

$$y_1'' = A\left[2e^{2x} + \left(4x+2\right)e^{2x}\right] = A\left(4x+4\right)e^{2x},$$

$$y_1''' = A\left[4e^{2x} + \left(8x+8\right)e^{2x}\right] = A\left(8x+12\right)e^{2x},$$

$$y_1^{IV} = A\left[8e^{2x} + \left(16x+24\right)e^{2x}\right] = A\left(16x+32\right)e^{2x}.$$

Substituting this into the first equation, we find the coefficient A :

$$A\left(16x+32\right)e^{2x} + A\left(8x+12\right)e^{2x} - 3A\left(4x+4\right)e^{2x} - 5A\left(2x+1\right)e^{2x} - 2Axe^{2x} = e^{2x},$$

$$\Rightarrow A\left(\cancel{16x} + \cancel{8x} - \cancel{12x} - \cancel{10x} - \cancel{2x}\right)e^{2x} + A\left(32 + \cancel{12} - \cancel{12} - 5\right)e^{2x} = e^{2x},$$

$$\Rightarrow 27A = 1, \quad \Rightarrow A = \frac{1}{27}.$$

Therefore, the particular solution y_1 is given by

$$y_1\left(x\right) = \frac{x}{27}e^{2x}.$$

Similarly, we find the particular solution y_2. The derivatives of the trial function y_2 are

$$y_2' = B\left(3x^2e^{-x} - x^3e^{-x}\right) = B\left(-x^3 + 3x^2\right)e^{-x},$$

$$y_2'' = B\left[\left(-3x^2 + 6x\right)e^{-x} - \left(-x^3 + 3x^2\right)e^{-x}\right] = B\left(x^3 - 6x^2 + 6x\right)e^{-x},$$

$$y_2''' = B\left[\left(3x^2 - 12x + 6\right)e^{-x} - \left(x^3 - 6x^2 + 6x\right)e^{-x}\right] = B\left(-x^3 + 9x^2 - 18x + 6\right)e^{-x},$$

$$y_2^{IV} = B\left[\left(-3x^2+18x-18\right)e^{-x} - \left(-x^3+9x^2-18x+6\right)e^{-x}\right] = B\big(x^3-12x^2 + 36x-24\big)e^{-x}.$$

Substituting these derivatives into the second equation, we calculate the coefficient B :

$$B\left(x^3-12x^2+36x-24\right)e^{-x} + B\left(-x^3+9x^2-18x+6\right)e^{-x} - 3B\left(x^3-6x^2+6x\right)e^{-x} - 5B\left(-x^3+3x^2\right)e^{-x} - 2Bx^3e^{-x} = -e^{-x},$$

$$\Rightarrow B\left(\cancel{x^3} - \cancel{x^3} - \cancel{3x^3} + \cancel{5x^3} - \cancel{2x^3}\right)e^{-x} + B\left(-\cancel{12x^2} + \cancel{9x^2} + \cancel{18x^2} - \cancel{15x^2}\right)e^{-x} + B\left(\cancel{36x} - \cancel{18x} - \cancel{18x}\right)e^{-x} + B\left(-24+6\right)e^{-x} = -e^{-x},$$

$$\Rightarrow -18B = -1, \quad \Rightarrow B = \frac{1}{18}.$$

We obtain the solution y_2 as follows:

$$y_2\left(x\right) = \frac{x^3}{18}e^{-x}.$$

In accordance with the principle of superposition, a particular solution of the original nonhomogeneous equation is represented as

$$y_{\text{p}} = y_1\left(x\right) + y_2\left(x\right) = \frac{x}{27}e^{2x} + \frac{x^3}{18}e^{-x}.$$

Finally, the general solution is given by

$$y\left(x\right) = \left(C_1 + C_2x + C_3x^2\right)e^{-x} + C_4e^{2x} + \frac{x}{27}e^{2x} + \frac{x^3}{18}e^{-x}$$
$$= \left(C_1 + C_2x + C_3x^2 + \frac{x^3}{18}\right)e^{-x} + \left(C_4 + \frac{x}{27}\right)e^{2x}.$$

Example 5.

Find the general solution of the equation $y''' + y' = \frac{1}{\cos x}$ using the method of variation of constants.

Basic Concepts of Stability Theory

Suppose that a phenomenon is described by the system of n differential equations

$$\frac{dx_i}{dt} = f_i\left(t, x_1, x_2, \ldots, x_n\right), \;\; i = 1, 2, \ldots, n$$

with initial conditions

$$x_i\left(t_0\right) = x_{i0}, \;\; i = 1, 2, \ldots, n.$$

We assume that the functions $f_i\left(t, x_1, x_2, \ldots, x_n\right)$ are defined and continuous together with its partial derivatives on the set $\left\{t \in \left[t_0, +\infty\right), x_i \in \Re^n\right\}$. Then without loss of generality we may assume that the initial time is zero: $t_0 = 0$.

It is convenient to write the system of differential equations in vector form:

$$\mathbf{X}' = \mathbf{f}\left(t, \mathbf{X}\right), \;\; \text{where} \;\; \mathbf{X} = \left(x_1, x_2, \ldots, x_n\right), \;\; \mathbf{f} = \left(f_1, f_2, \ldots, f_n\right).$$

In real systems, the initial conditions are specified with some precision. This raises the obvious question: how small changes in initial conditions affect the behavior of solutions for large time – in the extreme case when $t \to \infty$?

If the trajectory of the system varies little under small perturbations of the initial position, we say that the motion of the system is stable.

A mathematically rigorous definition of stability using $\varepsilon - \delta$-notation was proposed in 1892 by the Russian mathematician A.M.Lyapunov ($1857 - 1918$). Let us consider in more detail the concept of stability introduced by Lyapunov.

Lyapunov Stability

The solution $\boldsymbol{\varphi}\left(t\right)$ of the system of differential equations

$$\mathbf{X}' = \mathbf{f}\left(t, \mathbf{X}\right)$$

with initial conditions

$$\mathbf{X}\left(0\right) = \mathbf{X}_0$$

is stable (in the sense of Lyapunov) if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$, such that if

$$|\mathbf{X}(0) - \boldsymbol{\varphi}(0)| < \delta, \text{ then } |\mathbf{X}(t) - \boldsymbol{\varphi}(t)| < \varepsilon$$

for all values $t \geq 0$. Otherwise, the solution $\boldsymbol{\varphi}(t)$ is said to be unstable.

As the norm for measuring the distance between two points one can use, for example, the Euclidean metric $\|\mathbf{x}_e\|$ or Manhattan metric $\|\mathbf{x}_m\|$:

$$\|\mathbf{x}_e\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}, \quad \|\mathbf{x}_m\| = \sum_{i=1}^{n} |x_i|.$$

In the case $n = 2$, Lyapunov stability means that any trajectory $\mathbf{X}(t)$, which starts at $\delta(\varepsilon)$-neighborhood of the point $\boldsymbol{\varphi}(0)$, remains inside the tube with a maximum radius ε for all $t \geq 0$ (Figure 1).

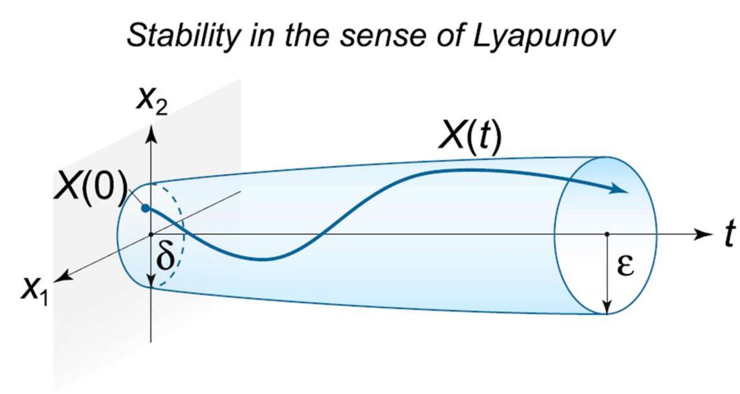

Figure 1.

Asymptotic and Exponential Stability

If the solution $\boldsymbol{\varphi}(t)$ of the system of differential equations is not only stable in the sense of Lyapunov, but also satisfies the relationship

$$\lim_{t \to \infty} |\mathbf{X}(t) - \boldsymbol{\varphi}(t)| = 0$$

provided that

$$|\mathbf{X}(0) - \boldsymbol{\varphi}(0)| < \delta,$$

then we say that the solution $\boldsymbol{\varphi}(t)$ is asymptotically stable.

In this case, all solutions that are sufficiently close to $\boldsymbol{\varphi}(0)$ at the initial time, gradually converge to $\boldsymbol{\varphi}(t)$ with increasing t. Schematically, this is shown in Figure 2.

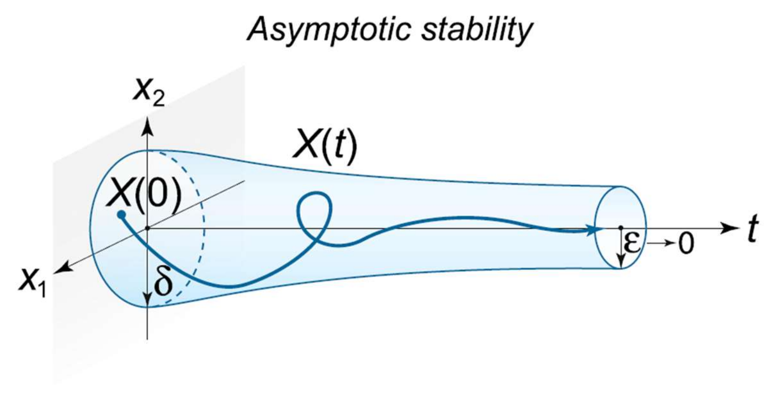

Figure 2.

If the solution $\boldsymbol{\varphi}(t)$ is asymptotically stable and, in addition, from the condition

$$|\mathbf{X}(0) - \boldsymbol{\varphi}(0)| < \delta$$

it follows that

$$|\mathbf{X}(t) - \boldsymbol{\varphi}(t)| \leq \alpha\,|\mathbf{X}(0) - \boldsymbol{\varphi}(0)|\, e^{-\beta t}$$

for all $t \geq 0$, we say that the solution $\boldsymbol{\varphi}(t)$ is exponentially stable. In this case all solutions that are close to $\boldsymbol{\varphi}(0)$ at the initial time converge to $\boldsymbol{\varphi}(t)$ with the rate (greater than or equal), which is determined by an exponential function with parameters α, β (Figure 3).

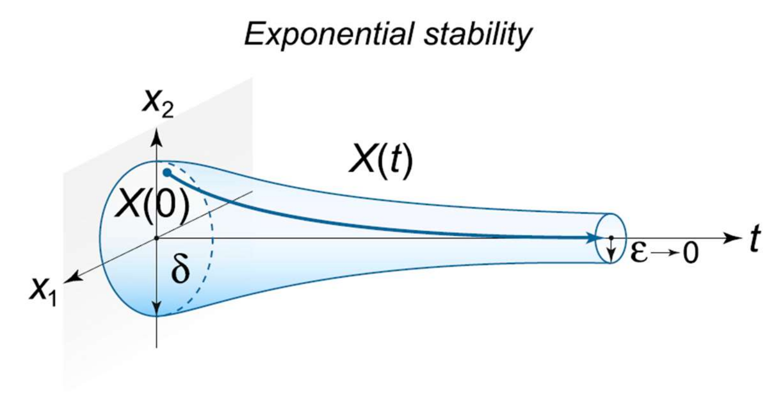

Figure 3.

The general theory of stability, in addition to stability in the sense of Lyapunov, contains many other concepts and definitions of stable movement. In particular, the concepts of orbital and structural stability are important.

Orbital Stability

Orbital stability describes the behavior of a closed trajectory (orbit) under the action of small external perturbations.

Consider the autonomous system

$$\frac{dx_i}{dt} = f_i\left(x_1, x_2, \ldots, x_n\right), \;\; x_i\left(t_0\right) = x_{i0}, \;\; i = 1, 2, \ldots, n,$$

that is the system of equations, the right hand side of which does not contain the independent variable t. In vector form, the autonomous system is written as

$$\mathbf{X}'\left(t\right) = \mathbf{f}\left(\mathbf{X}\right), \quad \text{where} \;\; \mathbf{X} = \left(x_1, x_2, \ldots, x_n\right), \;\; \mathbf{f} = \left(f_1, f_2, \ldots, f_n\right).$$

Let $\boldsymbol{\varphi}\left(t\right)$ be a periodic solution of the given autonomous system, that is has the form of a closed trajectory (orbit).

If for any $\varepsilon > 0$ there is a constant $\delta = \delta(\varepsilon) > 0$ such that the trajectory of any solution $\mathbf{X}(t)$ starting at the δ-neighborhood of the trajectory $\boldsymbol{\varphi}(t)$ remains in the ε-neighborhood of the trajectory $\boldsymbol{\varphi}(t)$ for all $t \geq 0$, then the trajectory $\boldsymbol{\varphi}(t)$ is called orbitally stable (Figure 4).

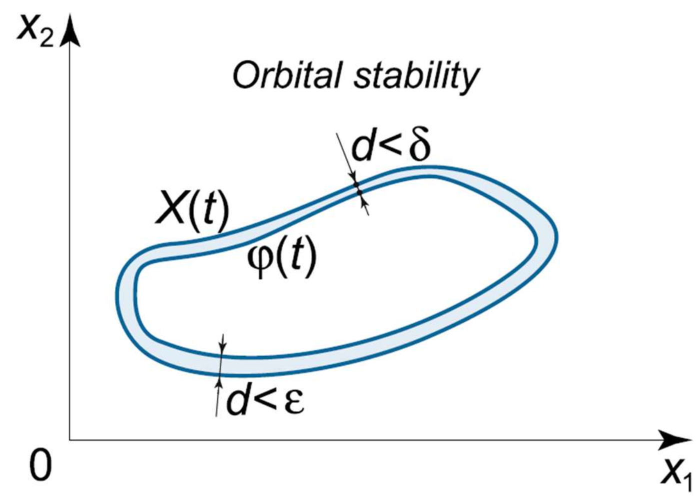

Figure 4.

By analogy with the asymptotic stability in the sense of Lyapunov, one can also introduce the concept of asymptotic orbital stability. This type of motion occurs, for example, in systems with a limit cycle.

Structural Stability

Suppose that we have two autonomous systems with similar properties – in the sense that their phase portraits have the same singular points and geometrically similar trajectories. Such systems can be called structurally stable.

In the strict definition, it is required that these systems are orbitally topologically equivalent, i.e. there must be a homeomorphism (this terrible word means one-to-one continuous mapping), which converts the family of trajectories of the first system into the family of trajectories of the second system while preserving the direction of motion.

In these terms, the structural stability is defined as follows.

Consider an autonomous system, which in the unperturbed and perturbed state is described, respectively, by two equations:

$$\mathbf{X}' = \mathbf{f}(\mathbf{X}),$$
$$\mathbf{X}' = \mathbf{f}(\mathbf{X}) + \varepsilon \mathbf{g}(\mathbf{X}).$$

If for any bounded and continuously differentiable vector function $\mathbf{g}(\mathbf{X})$ there exists a number $\varepsilon > 0$ such that the trajectories of the unperturbed and perturbed systems are orbitally topologically equivalent, then the system is called structurally stable.

Reduction to the Problem of Stability of the Zero Solution

Let an arbitrary non-autonomous system

$$\mathbf{X}' = \mathbf{f}(t, \mathbf{X})$$

be given with the initial condition $\mathbf{X}(0) = \mathbf{X}_0$ (an IVP or Cauchy problem). Here the vector-valued function $\mathbf{f}$ is defined on the set $\{t \in [t_0, +\infty), x_i \in \mathfrak{R}^n\}$.

Suppose that the system has a solution $\boldsymbol{\varphi}(t)$, the stability of which is to be examined. The stability analysis is simplified if we consider perturbations

$$\mathbf{Z}(t) = \mathbf{X}(t) - \boldsymbol{\varphi}(t),$$

for which we obtain the differential equation

$$\mathbf{Z}'(t) = \mathbf{f}(t, \mathbf{Z}).$$

Obviously, the last equation is satisfied by the trivial solution

$$\mathbf{Z}(t, \mathbf{0}) \equiv \mathbf{0},$$

which corresponds to the identity

$$\mathbf{X}(t) \equiv \boldsymbol{\varphi}(t).$$

Thus, the study of stability of the solution $\boldsymbol{\varphi}(t)$ can be replaced by the study of stability of the function $\mathbf{Z}(t)$ near the point $\mathbf{Z} = \mathbf{0}$.

Thus, the study of stability of the solution $\varphi(t)$ can be replaced by the study of stability of the function $\mathbf{Z}(t)$ near the point $\mathbf{Z} = \mathbf{0}$.

Stability of Linear Systems:

The linear system

$$\mathbf{X}' = A(t)\,\mathbf{X} + \mathbf{f}(t)$$

is said to be stable if all its solutions are stable in the sense of Lyapunov.

It turns out that the non-homogeneous linear system is stable with any free term $\mathbf{f}(t)$ if the zero solution of the associated homogeneous system

$$\mathbf{X}' = A(t)\,\mathbf{X}$$

is stable. Therefore, when investigating stability in the class of linear systems, it is sufficient to analyze the homogeneous differential systems. In the simplest case, when the coefficient matrix A is constant, the stability conditions are formulated in terms of the eigenvalues of the matrix A.

Consider the homogeneous linear system

$$\mathbf{X}' = A\mathbf{X},$$

where A is a constant matrix of size $n \times n$. Such a system (which is also autonomous) has the zero solution $\mathbf{X}(t) = \mathbf{0}$. The stability of this solution is determined by the following theorems.

Let λ_i be the eigenvalues of A.

Theorem 1.

A linear homogeneous system with constant coefficients is stable in the sense of Lyapunov if and only if all eigenvalues λ_i of A satisfy the condition

$$\operatorname{Re}[\lambda_i] \le 0 \quad (i = 1, 2, \ldots, n),$$

If the real part of an eigenvalue is equal to zero, the algebraic and geometric multiplicity of the eigenvalue must be the same (i.e. the corresponding Jordan block must be of size 1×1).

Theorem 2.

A linear homogeneous system with constant coefficients is asymptotically stable if and only if all eigenvalues λ_i have negative real parts:

$$\operatorname{Re}[\lambda_i] < 0 \quad (i = 1, 2, \ldots, n).$$

Theorem 3.

A linear homogeneous system with constant coefficients is unstable if at least one of the conditions is satisfied:

- The matrix A has an eigenvalue λ_i with a positive real part;
- The matrix A has an eigenvalue λ_i with zero real part, and the geometric multiplicity of the eigenvalue λ_i is less than its algebraic multiplicity.

The above theorems allow us to study the stability of linear systems with constant coefficients knowing the eigenvalues and eigenvectors.

However, in many cases, the character of stability can be determined by using a criteria of stability without solving the system of equations. One of these is the Routh-Hurwitz stability criterion. It allows to judge the stability of a system knowing only the coefficients of the characteristic equation of the matrix A.

Stability in the First Approximation

Consider a nonlinear autonomous system

$$\mathbf{X}' = f(\mathbf{X}).$$

Suppose that that the system has the trivial solution $\mathbf{X} = \mathbf{0}$, which we will investigate for stability.

Assuming that the functions $f_i(\mathbf{X})$ are twice continuously differentiable in a neighborhood of the origin, we can expand the right side in a Maclaurin series:

$$\frac{dx_1}{dt} = \frac{\partial f_1}{\partial x_1}(0)\, x_1 + \frac{\partial f_1}{\partial x_2}(0)\, x_2 + \cdots + \frac{\partial f_1}{\partial x_n}(0)\, x_n + R_1\left(x_1, x_2, \ldots, x_n\right),$$

$$\frac{dx_2}{dt} = \frac{\partial f_2}{\partial x_1}(0)\, x_1 + \frac{\partial f_2}{\partial x_2}(0)\, x_2 + \cdots + \frac{\partial f_2}{\partial x_n}(0)\, x_n + R_2\left(x_1, x_2, \ldots, x_n\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{dx_n}{dt} = \frac{\partial f_n}{\partial x_1}(0)\, x_1 + \frac{\partial f_n}{\partial x_2}(0)\, x_2 + \cdots + \frac{\partial f_n}{\partial x_n}(0)\, x_n + R_n\left(x_1, x_2, \ldots, x_n\right).$$

where the terms R_i describe the terms of the second (and higher) order of smallness with respect to the coordinate functions $x_1, x_2, \ldots, x_n$.
Returning to vector-matrix form, we obtain:

$$\mathbf{X}' = J\mathbf{X} + \mathbf{R}\left(\mathbf{X}\right),$$

where the Jacobian J is given by

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \vdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \vdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \vdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \vdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

The values of the partial derivatives in this matrix are calculated at the series expansion point, i.e. in this case, at zero.

In many cases, instead of the original nonlinear autonomous system, we can consider and investigate for stability the corresponding linearized system or the system of equations of the first approximation. The stability of such a system is determined by the following rules:

- If all eigenvalues of the Jacobian J have negative real parts, then the zero solution $\mathbf{X} = \mathbf{0}$ of the original and linearized systems is asymptotically stable.
- If at least one eigenvalue of the Jacobian J has a positive real part, then the zero solution $\mathbf{X} = \mathbf{0}$ of the original and linearized systems is unstable.

In critical cases, when the eigenvalues have a real part equal to zero, one should use other methods of stability analysis. The problems on stability in the first approximation are given here.

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Lyapunov Functions

One of the powerful tools for stability analysis of systems of differential equations, including nonlinear systems, are Lyapunov functions. This technique is discussed in detail in the separate web page "Method of Lyapunov Functions".

Example 1.

Using the definition of Lyapunov stability, show that the zero solution is stable.

$$\frac{dx}{dt} = -x - y, \quad \frac{dy}{dt} = -x + y.$$

Solution.

First we find the general solution. The eigenvalues λ_i of the coefficient matrix A are

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \ \det(A - \lambda I) = 0, \ \Rightarrow \begin{vmatrix} -1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0,$$
$$\Rightarrow (-1-\lambda)^2 + 1 = 0, \ \Rightarrow \lambda^2 + 2\lambda + 2 = 0, \ \Rightarrow \lambda_{1,2} = -1 \pm i.$$

Determine the eigenvector $\mathbf{V}_1 = (V_{11}, V_{21})^T$ for the eigenvalue $\lambda_1 = -1 + i$:

$$(A - \lambda_1 I)\,\mathbf{V}_1 = \mathbf{0}, \ \Rightarrow \begin{bmatrix} -1-(-1+i) & -1 \\ 1 & -1-(-1+i) \end{bmatrix} \cdot \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \mathbf{0},$$
$$\Rightarrow \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \mathbf{0}, \ \Rightarrow \begin{cases} -iV_{11} - V_{21} = 0 \\ V_{11} - iV_{21} = 0 \end{cases}.$$

The resulting equations are linearly dependent. Therefore, by setting $V_{21} = t$, we find from the second equation:

$$V_{11} = iV_{21} = it.$$

Hence, the eigenvector $\mathbf{V}_1$ is

$$\mathbf{V}_1 = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix} \sim \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

Then the solution $\mathbf{X}_1(t)$, corresponding to the complex eigenvalue $\lambda_1 = -1 + i$ is given by

$$\mathbf{X}_1(t) = \begin{bmatrix} x \\ y \end{bmatrix} = e^{\lambda_1 t}\mathbf{V}_1 = e^{(-1+i)t} \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

We expand the exponential function by Euler's formula:

$$e^{(-1+i)t} = e^{-t}e^{it} = e^{-t}(\cos t + i \sin t).$$

As a result, we have

$$\mathbf{X}_1(t) = \begin{bmatrix} x \\ y \end{bmatrix} = e^{-t}(\cos t + i \sin t) \begin{bmatrix} i \\ 1 \end{bmatrix} = e^{-t} \begin{bmatrix} (\cos t + i \sin t)\, i \\ \cos t + i \sin t \end{bmatrix}$$

$$= e^{-t} \begin{bmatrix} -\sin t + i \cos t \\ \cos t + i \sin t \end{bmatrix} = e^{-t} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + ie^{-t} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

This shows that the real and imaginary parts of the solution are equal:

$$\operatorname{Re}[\mathbf{X}_1(t)] = e^{-t} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}, \quad \operatorname{Im}[\mathbf{X}_1(t)] = e^{-t} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

Hence, the general solution is expressed by the formula

$$\mathbf{X}(t) = \begin{bmatrix} x \\ y \end{bmatrix} = C_1 e^{-t} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

or

$$\begin{cases} x(t) = -C_1 e^{-t} \sin t + C_2 e^{-t} \cos t \\ y(t) = C_1 e^{-t} \cos t + C_2 e^{-t} \sin t \end{cases}.$$

Assuming that at the initial time $t = 0$ the system is at the point (x_0, y_0), we obtain:

$$\begin{cases} x(0) = C_2 = x_0 \\ y(0) = C_1 = y_0 \end{cases}.$$

Taking this into account, the solution can be written as follows:

$$\begin{cases} x(t) = -y_0 e^{-t} \sin t + x_0 e^{-t} \cos t \\ y(t) = y_0 e^{-t} \cos t + x_0 e^{-t} \sin t \end{cases}.$$

The trajectory of this solution goes from the point (x_0, y_0).

Now we study the stability of the zero solution, which we denote as $\boldsymbol{\varphi}(t) \equiv 0$. According to the definition of Lyapunov stability, we introduce a number $\varepsilon > 0$ and find the corresponding number $\delta = \delta(\varepsilon) > 0$ such that if

$$|\mathbf{X}(0) - \boldsymbol{\varphi}(0)| < \delta,$$

the relationship

$$|\mathbf{X}(t) - \boldsymbol{\varphi}(t)| < \varepsilon$$

will hold for all $t \geq 0$.

Suppose that the perturbed solution $\mathbf{X}(t)$ at the initial moment has the coordinates (x_0, y_0). Assuming that the deviation from zero for each of the coordinates does not exceed $\frac{\delta}{2}$ and applying the triangle inequality, we can write:

$$\|\mathbf{X}(0) - \boldsymbol{\varphi}(0)\| = \sqrt{|x_0|^2 + |y_0|^2} \leq |x_0| + |y_0| = \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

We have used here the usual Euclidean metric as the norm.

Next, we establish a relationship between the numbers δ and ε. Substituting the known expressions for the solution $\mathbf{X}(t) = (x(t), y(t))$, we get:

$$\|\mathbf{X}(t) - \boldsymbol{\varphi}(t)\| = e^{-t}\sqrt{|-y_0 \sin t + x_0 \cos t|^2 + |y_0 \cos t + x_0 \sin t|^2}$$

$$\leq e^{-t}\sqrt{(|y_0| + |x_0|)^2 + (|y_0| + |x_0|)^2} = e^{-t}\sqrt{2(|y_0| + |x_0|)^2} = e^{-t}\sqrt{2\left(\frac{\delta}{2} + \frac{\delta}{2}\right)^2}$$

$$= e^{-t}\sqrt{2\delta^2} = \sqrt{2}e^{-t}\delta \leq \sqrt{2}\delta = \varepsilon.$$

So, if we set $\delta = \frac{\varepsilon}{\sqrt{2}}$, all perturbed trajectories emanating from the point (x_0, y_0), provided that $|x_0| < \frac{\delta}{2},\ |y_0| < \frac{\delta}{2}$, will remain in the tube with radius ε. Thus, the system is stable in the sense of Lyapunov.

Note that a stronger condition is satisfied here:

$$\lim_{t\to\infty} |\mathbf{X}(t) - \boldsymbol{\varphi}(t)| = \lim_{t\to\infty} \left(\sqrt{2}e^{-t}\delta\right) = 0.$$

that is the system is asymptotically stable as well.

Example 2.

Investigate the stability of the zero solution of the system, the general solution of which is given by

$$\begin{cases} x(t) = 3C_1 + C_2 e^{-t} \\ y(t) = 2C_1 t^2 e^{-t} - C_2 \cos t \end{cases}.$$

Solution.

Let the initial conditions be given as $x(0) = x_0, y(0) = y_0$. Express the general solution in terms of the coordinates x_0, y_0 :

$$\begin{cases} x(0) = 3C_1 = x_0 \\ y(0) = -C_2 = y_0 \end{cases}, \Rightarrow \begin{cases} C_1 = \frac{x_0}{3} \\ C_2 = -y_0 \end{cases}.$$

Then

$$\begin{cases} x(t) = x_0 - y_0 e^{-t} \\ y(t) = \frac{2}{3} x_0 t^2 e^{-t} + y_0 \cos t \end{cases}.$$

Suppose that the solutions $x(t), y(t)$ for all values $t \geq 0$ satisfy the relations

$$|x(t)| < \varepsilon, \quad |y(t)| < \varepsilon,$$

where ε is a positive number. According to the definition of Lyapunov stability, we try to choose a number $\delta(\varepsilon)$ depending on ε such that the following inequalities hold:

$$|x(0)| = |x_0| < \delta(\varepsilon), \quad |y(0)| = |y_0| < \delta(\varepsilon).$$

The result is

$$|x(t)| = \left|x_0 - y_0 e^{-t}\right| \leq |x_0| + |y_0| < \varepsilon,$$

$$|y(t)| = \left|\frac{2}{3} x_0 t^2 e^{-t} + y_0 \cos t\right| \leq \frac{2}{3}|x_0| t^2 e^{-t} + |y_0| < \varepsilon.$$

We take into account that the function $g(t) = t^2 e^{-t}$ is bounded. Indeed,

$$g'(t) = \left(t^2 e^{-t}\right)' = 2te^{-t} - t^2 e^{-t} = \left(2t - t^2\right) e^{-t}$$
$$= t(2-t) e^{-t}, \Rightarrow g'(t) = 0 \text{ at } t = 0, 2.$$

At $t = 2$, the function $g(t) = t^2 e^{-t}$ has a maximum equal

$$g_{\max} = g(t=2) = 2^2 e^{-2} \approx 0.54 < 1.$$

Then the inequality for $|y(t)|$ can be written as

$$|y(t)| \leq \frac{2}{3}|x_0| + |y_0| < \varepsilon.$$

If we now choose $\delta = \frac{\varepsilon}{2}$, so that

$$|x_0| < \delta = \frac{\varepsilon}{2} \text{ and } |y_0| < \delta = \frac{\varepsilon}{2},$$

the inequalities

$$|x(t)| = |x_0| + |y_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \ |y(t)| \leq \frac{2}{3}|x_0| + |y_0| < \frac{2}{3} \cdot \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \frac{5\varepsilon}{6} < \varepsilon.$$

are obviously satisfied. Hence, the zero solution of the given system of equations is stable.

It follows from the formulas for $x(t), y(t)$ that the system is not asymptotically stable, since the values of $x(t), y(t)$ do not tend to zero as $t \to \infty$.

Example 3.

Determine the values of the parameters a, b for which the zero solution of the system is asymptotically stable.

$$\frac{dx}{dt} = ax + y, \quad \frac{dy}{dt} = x + by$$

Solution.

We calculate the eigenvalues λ_i of the coefficient matrix A :

$$A = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}, \ \det(A - \lambda I) = 0, \ \Rightarrow \begin{vmatrix} a-\lambda & 1 \\ 1 & b-\lambda \end{vmatrix} = 0,$$
$$\Rightarrow (a-\lambda)(b-\lambda) - 1 = 0, \ \Rightarrow \lambda^2 - (a+b)\lambda + ab - 1 = 0.$$

Solve the resulting quadratic equation with the parameters a, b.

$$D = (a+b)^2 - 4(ab-1) = a^2 + 2ab + b^2 - 4ab + 4 = a^2 - 2ab + b^2 + 4$$
$$= (a-b)^2 + 4 > 0.$$

As it can be seen, the discriminant is always positive. Therefore, the eigenvalues are real numbers and are defined by

$$\lambda_{1,2} = \frac{a + b \pm \sqrt{(a-b)^2 + 4}}{2}.$$

We find the set of values of the numbers a, b at which the eigenvalues λ_1, λ_2 are negative (this means that the system is asymptotically stable):

$$\begin{cases} \lambda_1 < 0 \\ \lambda_2 < 0 \end{cases}, \quad \Rightarrow \begin{cases} a + b + \sqrt{(a-b)^2 + 4} < 0 \\ a + b - \sqrt{(a-b)^2 + 4} < 0 \end{cases}.$$

Adding the two inequalities, we get $a + b < 0$. In this case, the second inequality

$$\sqrt{(a-b)^2 + 4} > a + b$$

holds for all a, b, satisfying $a + b < 0$.

We solve the first inequality:

$$\begin{cases} \sqrt{(a-b)^2 + 4} < -(a+b) \\ a + b < 0 \end{cases}, \quad \Rightarrow \begin{cases} (a-b)^2 + 4 < (a+b)^2 \\ a + b < 0 \end{cases},$$

$$\Rightarrow \begin{cases} (a+b)^2 - (a+b)^2 > 4 \\ a + b < 0 \end{cases}, \quad \Rightarrow \begin{cases} (a + b - a + b)(a + b + a - b) > 4 \\ a + b < 0 \end{cases}$$

$$\Rightarrow \begin{cases} 4ab > 4 \\ a + b < 0 \end{cases}, \quad \Rightarrow \begin{cases} ab > 1 \\ a + b < 0 \end{cases}.$$

The solutions of both elementary inequalities are shown graphically in Figure 5.

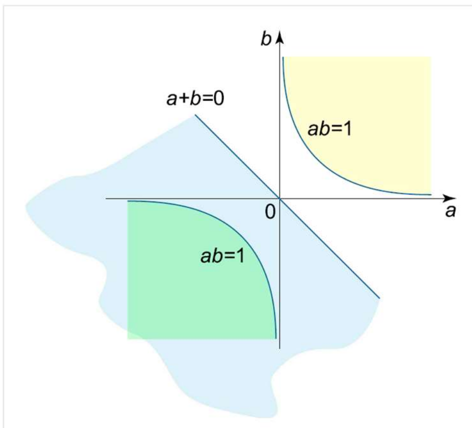


Figure 5.

The common solution is the region (shaded in green) below the hyperbola $ab = 1$ in the left half-plane. For all values of a, b from this region, the solution of the system is asymptotically stable.

Equilibrium Points of Linear Autonomous Systems

Types of Equilibrium Points

Let a second order linear homogeneous system with constant coefficients be given:

$$\begin{cases} \frac{dx}{dt} = a_{11}x + a_{12}y \\ \frac{dy}{dt} = a_{21}x + a_{22}y \end{cases}.$$

This system of equations is autonomous since the right hand sides of the equations do not explicitly contain the independent variable t.

In matrix form, the system of equations can be written as

$$\mathbf{X}' = A\mathbf{X}, \quad \text{where } \mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

The equilibrium positions can be found by solving the stationary equation

$$A\mathbf{X} = \mathbf{0}.$$

This equation has the unique solution $\mathbf{X} = \mathbf{0}$ if the matrix A is nonsingular, i.e. provided that $\det A \neq 0$. In the case of a singular matrix, the system has an infinite number of equilibrium points.

Classification of equilibrium points is determined by the eigenvalues λ_1, λ_2 of the matrix A. The numbers λ_1, λ_2 can be found by solving the auxiliary equation

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0.$$

In general, when the matrix A is nonsingular, there are 4 different types of equilibrium points:

#	Equilibrium Point	Eigenvalues λ_1, λ_2
1	Node	λ_1, λ_2 are real numbers of the same sign $(\lambda_1 \cdot \lambda_2 > 0)$
2	Saddle	λ_1, λ_2 are real numbers and non-zero of opposite sign $(\lambda_1 \cdot \lambda_2 < 0)$
3	Focus	λ_1, λ_2 are complex numbers, the real parts are equal and non-zero $(\text{Re}\,\lambda_1 = \text{Re}\,\lambda_2 \neq 0)$
4	Center	λ_1, λ_2 are purely imaginary numbers $(\text{Re}\,\lambda_1 = \text{Re}\,\lambda_2 = 0)$

Figure 1.

The stability of equilibrium points is determined by the general theorems on stability. So, if the real eigenvalues (or real parts of complex eigenvalues) are negative, then the equilibrium point is asymptotically stable. Examples of such equilibrium positions are stable node and stable focus.

If the real part of at least one eigenvalue is positive, the corresponding equilibrium point is unstable. For example, it may be a saddle.

Finally, in the case of purely imaginary roots (when the equilibrium point is a center), we are dealing with the classical stability in the sense of Lyapunov.

Our next goal is to study the behavior of solutions near the equilibrium positions. For second order systems, it is convenient to do this graphically using the phase portrait, which is a set of phase trajectories in the coordinate plane. The arrows on the phase trajectories show the direction of movement of the point (i.e., a particular state of the system) over time.

Let's discuss each type of equilibrium point and the corresponding phase portraits.

Stable and Unstable Node

The eigenvalues λ_1, λ_2 of the points of type "node" satisfy the conditions:

$$\lambda_1, \lambda_2 \in \Re, \quad \lambda_1 \cdot \lambda_2 > 0.$$

The following particular cases may arise here.

The roots λ_1, λ_2 are distinct $(\lambda_1 \neq \lambda_2)$ and negative $(\lambda_1 < 0, \lambda_2 < 0)$.

Draw a schematic phase portrait for this system. Suppose for definiteness that $|\lambda_1| < |\lambda_2|$. The general solution has the form

$$\mathbf{X}(t) = C_1 e^{\lambda_1 t}\mathbf{V}_1 + C_2 e^{\lambda_2 t}\mathbf{V}_2,$$

where $\mathbf{V}_1 = (V_{11}, V_{21})^T$, $\mathbf{V}_2 = (V_{12}, V_{22})^T$ are eigenvectors corresponding to the eigenvalues λ_1, λ_2 and C_1, C_2 are arbitrary constants.

Since both eigenvalues are negative, then the solution $\mathbf{X} = \mathbf{0}$ is asymptotically stable. Such an equilibrium point is called stable node. As $t \to \infty$, the phase curves tend to the origin $\mathbf{X} = \mathbf{0}$.

Specify the direction of the phase trajectories. Since

$$x(t) = C_1 V_{11} e^{\lambda_1 t} + C_2 V_{12} e^{\lambda_2 t}, \;\; y(t) = C_1 V_{21} e^{\lambda_1 t} + C_2 V_{22} e^{\lambda_2 t},$$

the derivative $\frac{dy}{dx}$ is

$$\frac{dy}{dx} = \frac{C_1 V_{21} \lambda_1 e^{\lambda_1 t} + C_2 V_{22} \lambda_2 e^{\lambda_2 t}}{C_1 V_{11} \lambda_1 e^{\lambda_1 t} + C_2 V_{12} \lambda_2 e^{\lambda_2 t}}.$$

Divide the numerator and denominator by $e^{\lambda_1 t}$:

$$\frac{dy}{dx} = \frac{C_1 V_{21} \lambda_1 + C_2 V_{22} \lambda_2 e^{(\lambda_2 - \lambda_1)t}}{C_1 V_{11} \lambda_1 + C_2 V_{12} \lambda_2 e^{(\lambda_2 - \lambda_1)t}}.$$

Now we consider the behavior of the phase trajectories as $t \to -\infty$. Obviously, the coordinates $x(t), y(t)$ tend to infinity, and the derivative $\frac{dy}{dx}$ at $C_2 \neq 0$ takes the following form:

In this case, $\lambda_2 - \lambda_1 < 0$. Therefore, the terms with the exponential function tend to zero as $t \to \infty$. As a result, at $C_1 \neq 0$, we obtain:

$$\lim_{t \to \infty} \frac{dy}{dx} = \frac{V_{21}}{V_{11}}.$$

that is the phase trajectories become parallel to the eigenvector $\mathbf{V}_1$ as $t \to \infty$.

If $C_1 = 0$, the derivative at any t equals

$$\frac{dy}{dx} = \frac{V_{22}}{V_{12}},$$

i.e. the phase trajectory lies on a line directed along the eigenvector $\mathbf{V}_2$.

$$\frac{dy}{dx} = \frac{C_1 V_{21} \lambda_1 e^{(\lambda_1 - \lambda_2)t} + C_2 V_{22} \lambda_2}{C_1 V_{11} \lambda_1 e^{(\lambda_1 - \lambda_2)t} + C_2 V_{12} \lambda_2} = \frac{V_{22}}{V_{12}},$$

that is the phase curves at the points at infinity become parallel to the vector $\mathbf{V}_2$.

Accordingly, when $C_2 = 0$, the derivative is

$$\frac{dy}{dx} = \frac{V_{21}}{V_{11}}.$$

In this case, the phase trajectory is determined by the direction of the eigenvector $\mathbf{V}_1$.

Given the above properties of the phase trajectories, the phase portrait of a stable node is shown schematically in Figure 2.

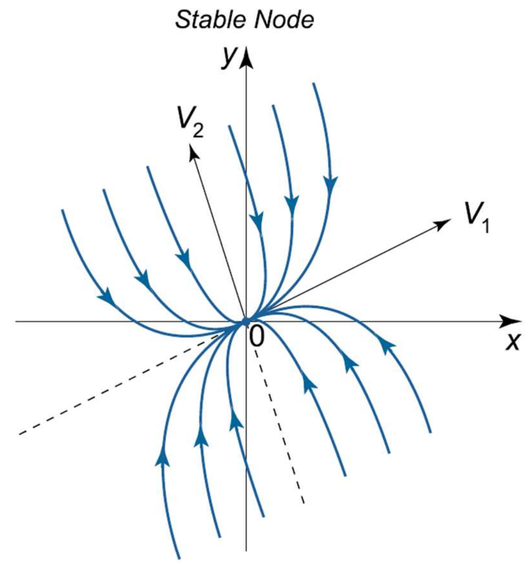


Figure 2.

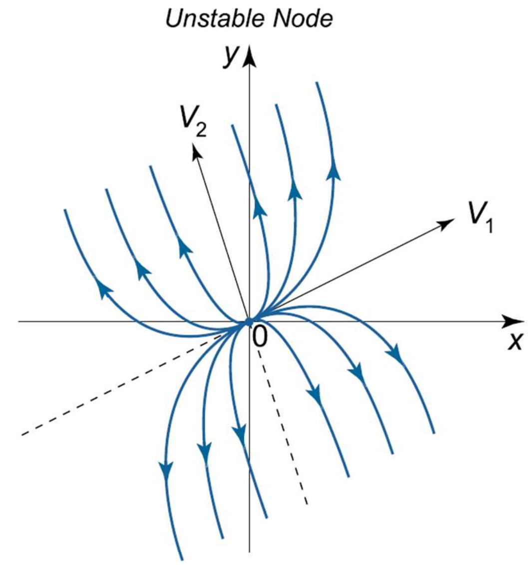


Figure 3.

Similarly, we can study the behavior of the phase trajectories for other types of equilibrium points. Furthermore, omitting the detailed analysis, we consider basic qualitative characteristics of the other equilibrium points.

The roots λ_1, λ_2 are distinct $(\lambda_1 \neq \lambda_2)$ and positive $(\lambda_1 > 0, \lambda_2 > 0)$.

In this case, the point $\mathbf{X} = \mathbf{0}$ is an unstable node. Its phase portrait is shown in Figure 3.

Note that in the case of both stable and unstable node, the phase trajectories touch the line, which is directed along the eigenvector corresponding to the smallest (in absolute value) eigenvalue λ.

Dicritical Node

Let the auxiliary equation have one zero root of multiplicity 2, i.e. consider the case $\lambda_1 = \lambda_2 = \lambda \neq 0$. The system has a basis of two eigenvectors, i.e. the geometric multiplicity of the eigenvalue λ is 2. In terms of the linear algebra, this means that the dimension of the eigenspace of A is equal to 2 : $\dim \ker A = 2$. This situation occurs in systems of the form

$$\frac{dx}{dt} = \lambda x, \;\; \frac{dy}{dt} = \lambda y.$$

The direction of the phase trajectories depends on the sign of λ. Here the following two cases can arise:

Case $\lambda_1 = \lambda_2 = \lambda < 0.$

Such an equilibrium position is called a stable dicritical node (Figure 4).

Case $\lambda_1 = \lambda_2 = \lambda > 0.$

This combination of eigenvalues corresponds to an unstable dicritical node (Figure 5).

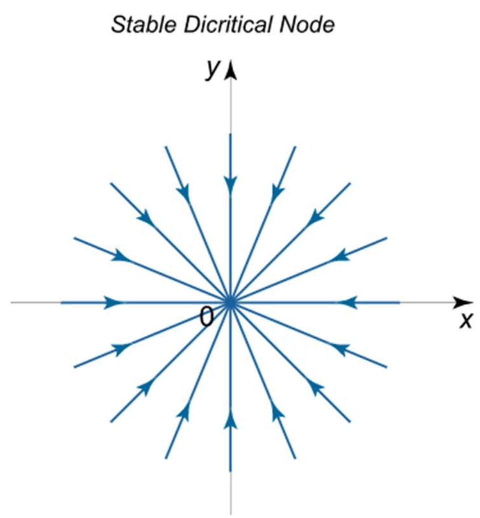


Figure 4.

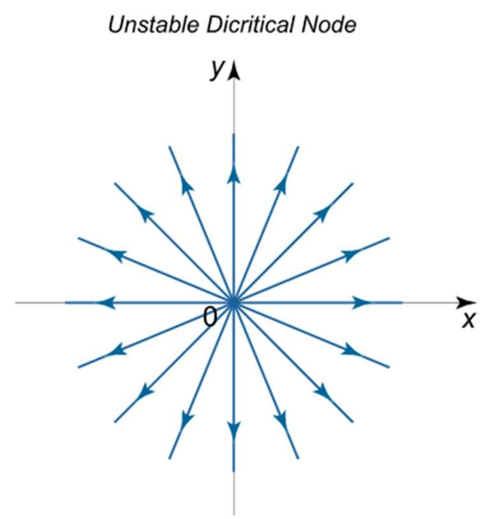


Figure 5.

Singular Node

Let the eigenvalues of A be again coincident: $\lambda_1 = \lambda_2 = \lambda \neq 0$. Unlike the previous case, we assume that the geometric multiplicity of the eigenvalue (or in other words, the dimension of the eigenspace) is now 1. This means that the matrix A has only one eigenvector $\mathbf{V}_1$. The second linearly independent vector required for the basis is defined as a generalized eigenvector $\mathbf{W}_1$ connected to $\mathbf{V}_1$.

Case $\lambda_1 = \lambda_2 = \lambda < 0.$

The equilibrium point is called stable singular node (Figure 6).

Case $\lambda_1 = \lambda_2 = \lambda > 0.$

The equilibrium position is called unstable singular node (Figure 7).

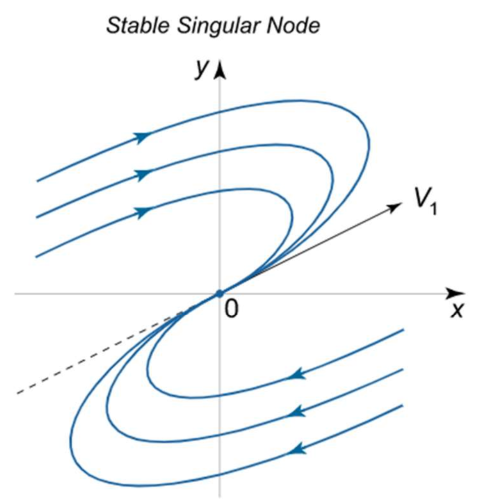


Figure 6.

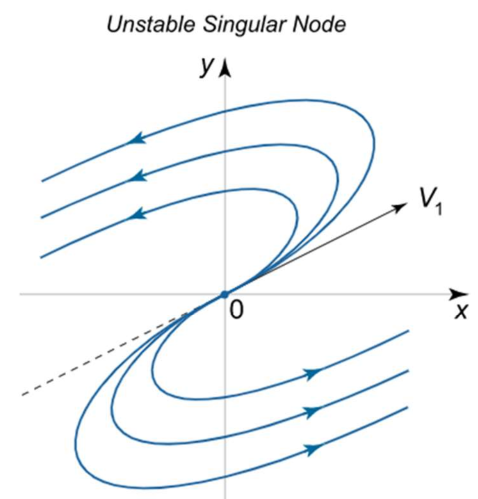


Figure 7.

Saddle

The equilibrium point is a saddle under the following condition:

$$\lambda_1, \lambda_2 \in \Re, \quad \lambda_1 \cdot \lambda_2 < 0.$$

Since one of the eigenvalues is positive, the saddle is an unstable equilibrium point. Suppose, for example, $\lambda_1 < 0, \lambda_2 > 0$. The eigenvalues λ_1 and λ_2 are associated with the corresponding eigenvectors $\mathbf{V}_1$ and $\mathbf{V}_2$. The straight lines directed along the eigenvectors $\mathbf{V}_1$, $\mathbf{V}_2$, are called separatrices. These are the asymptotes of other phase trajectories that have the form of a hyperbola. Each of the separatrices can be associated with a certain direction of motion.

If the separatrix is associated with a negative eigenvalue $\lambda_1 < 0$, i.e. in our case is directed along the vector $\mathbf{V}_1$, the movement along it occurs towards the equilibrium point $\mathbf{X} = \mathbf{0}$. And conversely, at $\lambda_2 > 0$, i.e. for the separatrix associated with the vector $\mathbf{V}_2$, the movement is directed from the origin. The phase portrait of the saddle is shown schematically in Figure 8.

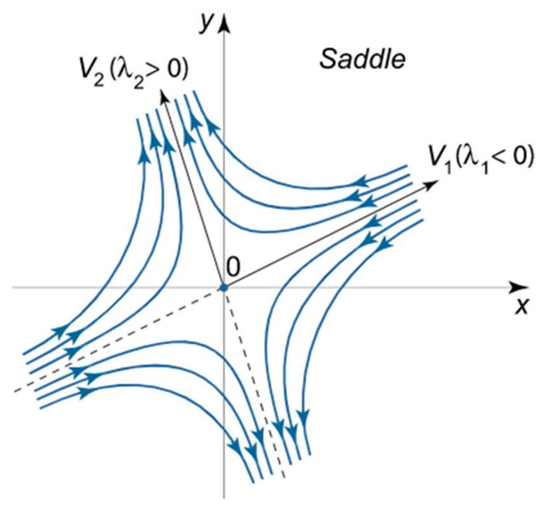


Figure 8.

Stable and Unstable Focus

Now suppose that the eigenvalues λ_1, λ_2 are complex numbers whose real parts are non-zero. If the matrix A is composed of real numbers, the complex roots will be presented in the form of complex conjugate numbers:

$$\lambda_{1,2} = \alpha \pm i\beta.$$

Find out what kind of phase trajectories are in the neighborhood of the origin. Construct a complex solution $\mathbf{X}_1(t)$ corresponding to the eigenvalue $\lambda_1 = \alpha + i\beta$:

$$\mathbf{X}_1(t) = e^{\lambda_1 t}\mathbf{V}_1 = e^{(\alpha+i\beta)t}(\mathbf{U} + i\mathbf{W}),$$

where $\mathbf{V}_1 = \mathbf{U} + i\mathbf{W}$ is the complex-valued eigenvector associated with the eigenvalue λ_1, $\mathbf{U}$ and $\mathbf{W}$ are real vector functions. As a result, we obtain:

$$\mathbf{X}_1(t) = e^{\alpha t}e^{i\beta t}(\mathbf{U} + i\mathbf{W}) = e^{\alpha t}(\cos\beta t + i\sin\beta t)(\mathbf{U} + i\mathbf{W})$$
$$= e^{\alpha t}(\mathbf{U}\cos\beta t + i\mathbf{U}\sin\beta t + i\mathbf{W}\cos\beta t - \mathbf{W}\sin\beta t) = e^{\alpha t}(\mathbf{U}\cos\beta t + -\mathbf{W}\sin\beta t)$$
$$+ ie^{\alpha t}(\mathbf{U}\sin\beta t + \mathbf{W}\cos\beta t).$$

The real and imaginary parts in the last expression form the general solution of the type

$$\mathbf{X}(t) = C_1 \mathrm{Re}\left[\mathbf{X}_1(t)\right] + C_2 \mathrm{Im}\left[\mathbf{X}_1(t)\right] = e^{\alpha t}\left[C_1\left(\mathbf{U}\cos\beta t - \mathbf{W}\sin\beta t\right)\right.$$
$$\left. + \, C_2\left(\mathbf{U}\sin\beta t + \mathbf{W}\cos\beta t\right)\right] = e^{\alpha t}\left[\mathbf{U}\left(C_1\cos\beta t + C_2\sin\beta t\right)\right.$$
$$\left. + \, \mathbf{W}\left(C_2\cos\beta t - C_1\sin\beta t\right)\right].$$

We represent the constant C_1, C_2 as

$$C_1 = C\sin\delta, \quad C_2 = C\cos\delta,$$

where δ is an auxiliary angle. Then the solution is written as

$$\mathbf{X}(t) = Ce^{\alpha t}\left[\mathbf{U}\left(\sin\delta\cos\beta t + \cos\delta\sin\beta t\right) + \mathbf{W}\left(\cos\delta\cos\beta t - \sin\delta\sin\beta t\right)\right]$$
$$= Ce^{\alpha t}\left[\mathbf{U}\sin(\beta t + \delta) + \mathbf{W}\cos(\beta t + \delta)\right].$$

Thus, the solution $\mathbf{X}(t)$ can be expanded in the basis of the vectors $\mathbf{U}$ and $\mathbf{W}$:

$$\mathbf{X}(t) = \mu(t)\,\mathbf{U} + \eta(t)\,\mathbf{W},$$

where the coefficients $\mu(t), \eta(t)$ are given by

$$\mu(t) = Ce^{\alpha t}\sin(\beta t + \delta), \;\; \eta(t) = Ce^{\alpha t}\cos(\beta t + \delta).$$

This shows that the phase trajectories are spirals. When $\alpha < 0$, the spirals twist approaching the origin. Such an equilibrium position is called stable focus. Accordingly, when $\alpha > 0$, we have an unstable focus.

The direction of twist can be identified by the sign of the coefficient a_{21} in the original matrix A. Indeed, consider the derivative $\frac{dy}{dt}$, for example, at the point $(1, 0)$:

$$\frac{dy}{dt}(1, 0) = a_{21} \cdot 1 + a_{22} \cdot 0 = a_{21}$$

The positive coefficient $a_{21} > 0$ corresponds to the twist counterclockwise as shown in Figure 9. When $a_{21} < 0$, the spirals will twist in a clockwise direction (Figure 10).

Thus, taking into account the direction of twist, there are only 4 different types of focus. Schematically, they are shown in Figures $9 - 12$.

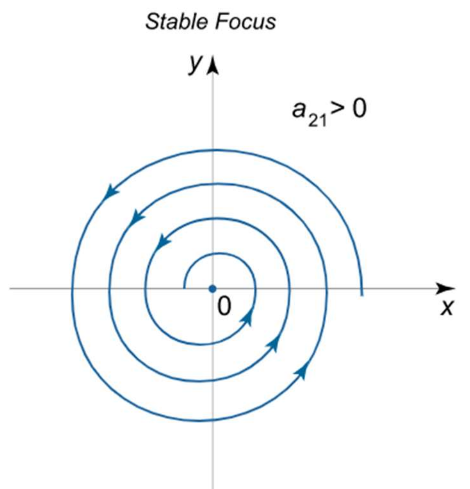


Figure 9.

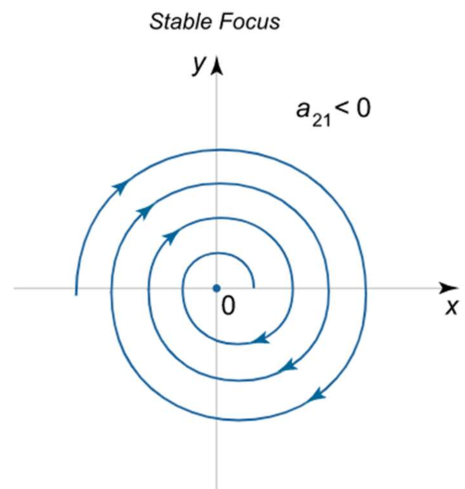


Figure 10.

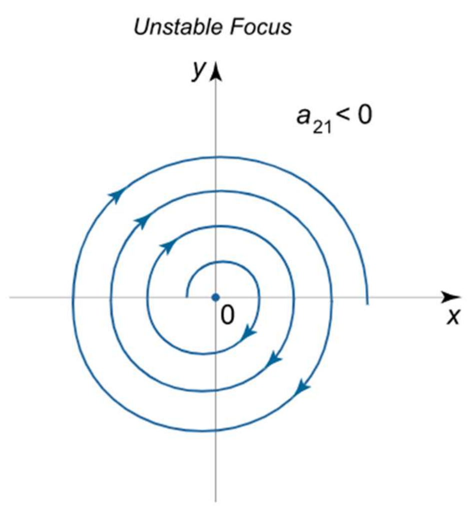


Figure 11.

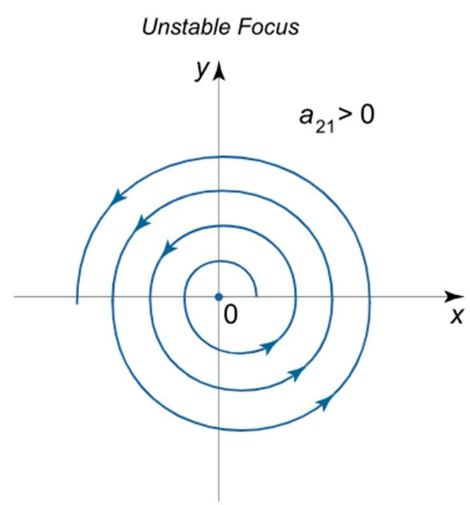


Figure 12.

Center

If the eigenvalues of the matrix A are purely imaginary numbers, then this equilibrium point is called a center. For a matrix with real elements, the imaginary eigenvalues are complex conjugate pairs. In the case of a center, the phase trajectories are formally obtained from the equation of spirals at $\alpha = 0$ and are ellipses, i.e. they describe periodic motion of a point in the phase space. A center equilibrium position is stable in the sense of Lyapunov.

There are two types of centers, which differ in the direction of movement of the points (Figures 13, 14). As in the case of focus, the direction of movement can be determined by the sign of the derivative $\frac{dy}{dt}$ at some point. If we take the point $(1, 0)$, then

$$\frac{dy}{dt}(1, 0) = a_{21}.$$

that is the direction of rotation is determined by the sign of the coefficient a_{21}.

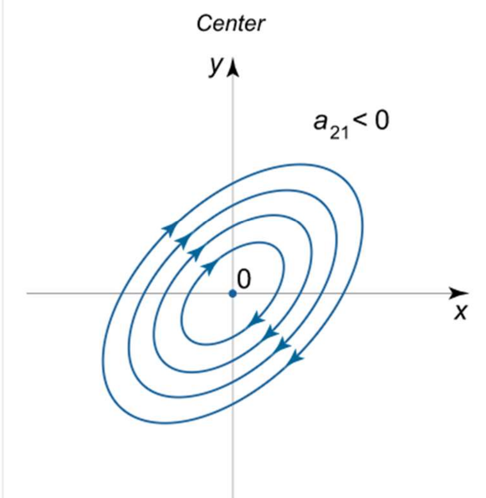


Figure 13.

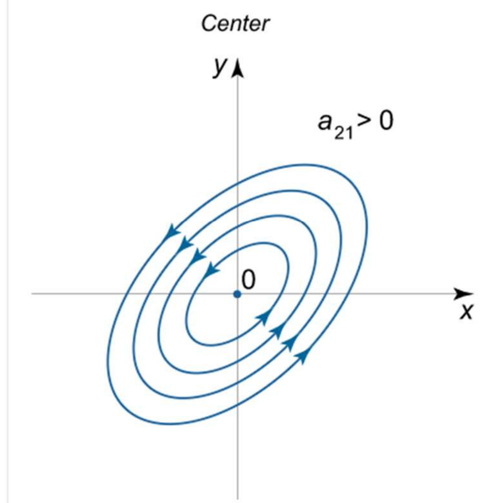


Figure 14.

Thus, we have considered different types of equilibrium points in the case of a non-singular matrix A $(\det A \neq 0)$. Taking into account the direction of phase trajectories, there are total 13 different phase portraits (shown, respectively, in Figures $2 - 14$).

We now turn to the case of a singular matrix A.

Singular Matrix

If the matrix is singular, then it has one or both eigenvalues equal to zero. In this case, there are the following special cases:

Case $\lambda_1 \neq 0, \lambda_2 = 0.$

Here, the general solution has the form

$$\mathbf{X}(t) = C_1 e^{\lambda_1 t}\mathbf{V}_1 + C_2\mathbf{V}_2,$$

where $\mathbf{V}_1 = (V_{11}, V_{21})^T$, $\mathbf{V}_2 = (V_{12}, V_{22})^T$, are the eigenvectors corresponding to the eigenvalues λ_1 and λ_2. It turns out that in this case the whole line passing through the origin and directed along the vector $\mathbf{V}_2$ consists of the equilibrium points (these points do not have a special name). The phase trajectories are rays parallel to the other eigenvector $\mathbf{V}_1$. Depending on the sign of λ_1, the motion at $t \to \infty$ occurs either in the direction of the line $\mathbf{V}_2$ (Figure 15), or away from it (Figure 16).

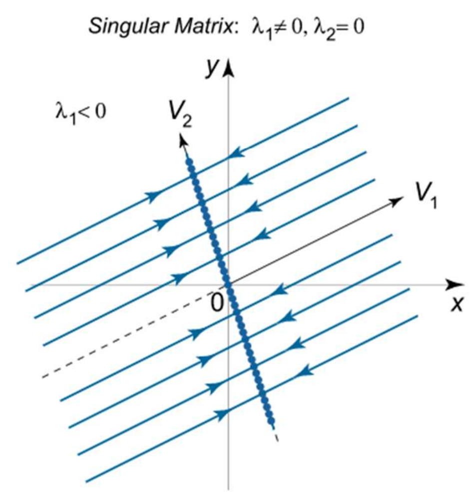


Figure 15.

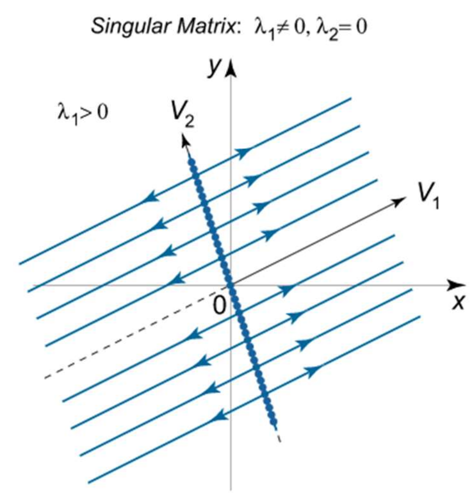


Figure 16.

Case $\lambda_1 = \lambda_2 = 0, \dim\ker A = 2.$

In this case, the dimension of the eigenspace of the matrix is equal to 2 and, therefore, there are two eigenvectors $\mathbf{V}_1$ and $\mathbf{V}_2$. This may happen when A is the zero matrix. The general solution is given by

$$\mathbf{X}(t) = C_1\mathbf{V}_1 + C_2\mathbf{V}_2.$$

It follows that every point in the plane is an equilibrium position of the system.

Case $\lambda_1 = \lambda_2 = 0, \dim\ker A = 1.$

This case is different from the previous one in that there is only one eigenvector (the matrix A will then be non-zero). To construct a basis, we can take the generalized eigenvector $\mathbf{W}_1$ connected to $\mathbf{V}_1$ as a second linearly independent vector. The general solution can be written as

$$\mathbf{X}(t) = (C_1 + C_2 t)\,\mathbf{V}_1 + C_2 \mathbf{W}_1.$$

Here, all points of the straight line passing through the origin and directed along the eigenvector $\mathbf{V}_1$ are unstable equilibrium positions. The phase trajectories are straight lines parallel to $\mathbf{V}_1$. The direction of movement along these lines as $t \to \infty$ depends on the constant C_2 : with $C_2 < 0$, the motion is from left to right, and with $C_2 > 0$ – in the opposite direction (Figure 17).

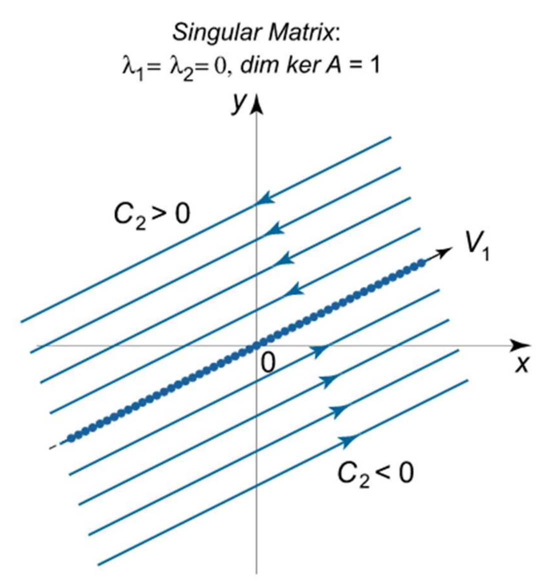


Figure 17.

As seen, there are 4 different phase portraits in the case of a singular matrix. Therefore, the linear second order autonomous system allows total 17 different phase portraits.

Bifurcation Diagram

In the above, we have reviewed the classification of equilibrium points of a linear system based on the eigenvalues. However, the type of an equilibrium point can be determined without computing the eigenvalues λ_1, λ_2, knowing only the determinant of the matrix $\det A$ and its trace $\operatorname{tr} A$.

Recall that the trace of the matrix is the number equal to the sum of the diagonal elements:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \ \operatorname{tr} A = a_{11} + a_{22}, \ \det A = a_{11}a_{22} - a_{12}a_{21}.$$

Indeed, the auxiliary equation of the matrix is

$$\lambda^2 - (a_{11} + a_{22})\,\lambda + a_{11}a_{22} - a_{12}a_{21} = 0.$$

It can be written in terms of the determinant and the trace of the matrix:

$$\lambda^2 - \operatorname{tr} A \cdot \lambda + \det A = 0.$$

The discriminant of this quadratic equation is given by

$$D = (\operatorname{tr} A)^2 - 4 \det A.$$

Thus, the bifurcation curve delineating the different stability modes is a parabola in the plane $(\operatorname{tr} A, \det A)$ (Figure 18):

$$\det A = \left(\frac{\operatorname{tr} A}{2}\right)^2.$$

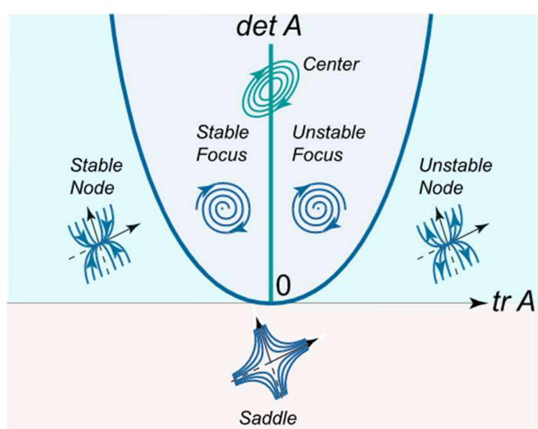


Figure 18.

The equilibrium points of the type "focus" and "center" are above the parabola. The points of the type "center" are located on the positive y-axis, i.e. provided that $\text{tr}\, A = 0$. The "nodes" and "saddles" are below the parabola. The parabola itself contains dicritical or singular nodes.

Stable modes of motion exist in the upper left quadrant of the bifurcation diagram. The other three quadrants correspond to unstable equilibrium positions.

How to Sketch a Phase Portrait

To draw the phase portrait of a second order linear autonomous system with constant coefficients

$$\mathbf{X}' = A\mathbf{X},\;\; A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\; \mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix},$$

it is necessary to do the following steps:

1. Find the eigenvalues of the matrix by solving the auxiliary equation

$$\lambda^2 - (a_{11} + a_{22})\,\lambda + a_{11}a_{22} - a_{12}a_{21} = 0.$$

2. Determine the type of the equilibrium point and the character of stability.

Hint:
The type of the equilibrium position can also be determined based on the bifurcation diagram (Figure 18), knowing the trace and the determinant of the matrix:

$$\text{tr}\, A = a_{11} + a_{22},\;\; \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

3. Find the equations of the isoclines:

$$\frac{dx}{dt} = a_{11}x + a_{12}y \;\; (\text{vertical isocline})\,,$$

$$\frac{dy}{dt} = a_{21}x + a_{22}y \;\; (\text{horizontal isocline})\,.$$

4. If the equilibrium position is a node or a saddle, it is necessary to compute the eigenvectors and draw the asymptotes parallel to the eigenvectors and passing through the origin.

5 Schematically draw the phase portrait.

6 Show the direction of motion along the phase trajectories (this depends on the stability or instability of the equilibrium point).In the case of a focus, one should determine the direction of trajectories twisting. This can be done by calculating the velocity vector $\left(\frac{dx}{dt}, \frac{dy}{dt}\right)$ at any point, for example, at the point $(1, 0)$. Similarly, we can determine the direction of movement if the equilibrium position is a center.

The algorithm described here is not a rigid scheme. In the study of a particular system, other tricks and techniques are acceptable in order to draw up the phase portrait.

5 Schematically draw the phase portrait.

6 Show the direction of motion along the phase trajectories (this depends on the stability or instability of the equilibrium point).In the case of a focus, one should determine the direction of trajectories twisting. This can be done by calculating the velocity vector $\left(\frac{dx}{dt}, \frac{dy}{dt}\right)$ at any point, for example, at the point $(1, 0)$. Similarly, we can determine the direction of movement if the equilibrium position is a center.

The algorithm described here is not a rigid scheme. In the study of a particular system, other tricks and techniques are acceptable in order to draw up the phase portrait.

Example 1.

Investigate the equilibrium positions of the linear autonomous system and draw its phase portrait.

$$\frac{dx}{dt} = -x, \quad \frac{dy}{dt} = 2x - 2y.$$

Solution.

1 We write the matrix of the system and compute its determinant:

$$A = \begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix}, \ \det A = \begin{vmatrix} -1 & 0 \\ 2 & -2 \end{vmatrix} = 2 \neq 0.$$

As $\det A \neq 0$, the system has the unique equilibrium point $\mathbf{X} = \mathbf{0}$. We find the eigenvalues of the matrix A :

$$\det(A - \lambda I) = 0, \quad \Rightarrow \begin{vmatrix} -1-\lambda & 0 \\ 2 & -2-\lambda \end{vmatrix} = 0, \quad \Rightarrow (\lambda + 1)(\lambda + 2) = 0,$$
$$\Rightarrow \lambda_1 = -1, \ \lambda_2 = -2.$$

2 Both eigenvalues are real and negative, so the equilibrium point $\mathbf{X} = \mathbf{0}$ is a stable node.

3 We derive the isocline equations, i.e. the lines which are tangent to the phase trajectories. The vertical isocline is given by

$$\frac{dx}{dt} = -x = 0 \ \text{ or } \ x = 0.$$

The horizontal isocline is written as

$$\frac{dy}{dt} = 2x - 2y = 0 \;\text{ or }\; y = x.$$

4 Find the equations of the asymptotes. This can be done by calculating the eigenvectors $\mathbf{V}_1, \mathbf{V}_2$ of the matrix A :

$$(A - \lambda_1 I)\,\mathbf{V}_1 = \mathbf{0}, \;\Rightarrow \begin{bmatrix} -1+1 & 0 \\ 2 & -2+1 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \mathbf{0},$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \mathbf{0}, \;\Rightarrow 2V_{11} - V_{21} = 0, \;\Rightarrow V_{11} = 1,\; V_{21} = 2,$$

$$\Rightarrow \mathbf{V}_1 = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix};$$

$$(A - \lambda_2 I)\,\mathbf{V}_2 = \mathbf{0}, \;\Rightarrow \begin{bmatrix} -1+2 & 0 \\ 2 & -2+2 \end{bmatrix} \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \mathbf{0},$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \mathbf{0}, \;\Rightarrow \begin{cases} 1 \cdot V_{12} + 0 \cdot V_{22} = 0 \\ 2 \cdot V_{12} + 0 \cdot V_{22} = 0 \end{cases}, \;\Rightarrow V_{12} = 0,\; V_{22} = 1,$$

$$\Rightarrow \mathbf{V}_2 = \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

5 Draw an xy-plane and show the eigenvectors $\mathbf{V}_1, \mathbf{V}_2$, the horizontal isocline $y = x$ and sketch the phase portrait of the system (Figure 19). The phase trajectories approach zero touching the line directed along the vector $\mathbf{V}_1$, as this eigenvector corresponds to the smallest (in absolute value) eigenvalue: $|\lambda_1| = 1$.

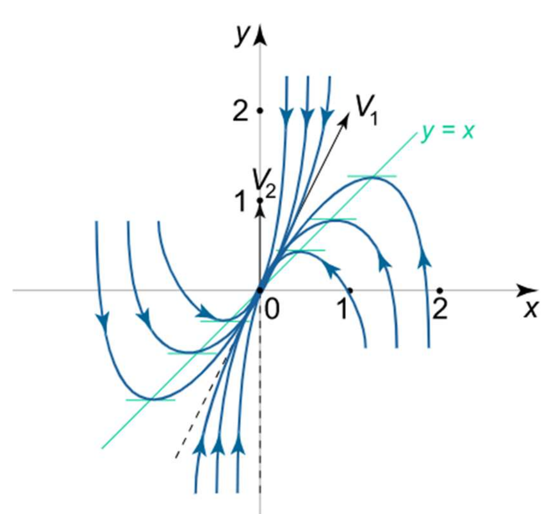


Figure 19.

Example 2.

Investigate the equilibrium positions of the dynamic system and sketch its phase portrait.

$$\frac{dx}{dt} = x + 3y, \quad \frac{dy}{dt} = 2x.$$

Solution.

We first make sure that the determinant is not zero:

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}, \quad \det A = \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = -6 \neq 0.$$

Hence, the system has the unique equilibrium point at the origin.

1 We solve this problem without computing the eigenvalues and eigenvectors.

2 As the determinant $\det A < 0$, then the zero equilibrium point is a saddle. This follows from the bifurcation diagram in Figure 18.

3 Define the equations of isoclines. The vertical isocline is described by the linear function:

$$\frac{dx}{dt} = x + 3y = 0, \quad \Rightarrow y = -\frac{x}{3}.$$

The equation of the horizontal isocline is

$$\frac{dy}{dt} = 2x = 0, \quad \Rightarrow x = 0 \;\; (y - \text{axis}).$$

4 Find the equations of the separatrices which have the form $y = kx$. Substituting this into the original system, we obtain a quadratic equation for the coefficient k :

$$\left\{ \begin{array}{l} \frac{dx}{dt} = x + 3y \\ \frac{dy}{dt} = 2x \end{array} \right., \quad \Rightarrow \left\{ \begin{array}{l} \frac{dx}{dt} = x + 3kx \\ \frac{kdx}{dt} = 2x \end{array} \right., \quad \Rightarrow 2x = k\left(x + 3kx\right),$$
$$\Rightarrow 3k^2x + kx - 2x = 0, \quad \Rightarrow 3k^2 + k - 2 = 0,$$
$$\Rightarrow D = 24, \;\; k_{1,2} = \frac{-1 \pm 5}{6} = -1, \;\; \frac{2}{3}.$$

Thus, the equations of the separatrices are as follows:

$$y = -x, \quad y = \frac{2}{3}x.$$

5 Draw the isoclines and separatrices on the phase plane and sketch the phase trajectories (Figure 20).

6 Determine the direction of motion along the phase trajectories. Take, for example, the point $(1, 0)$ and calculate the derivative $\frac{dy}{dt}$ at this point:

$$\frac{dy}{dt}(1,0) = 2 \cdot 1 = 2 > 0.$$

$$\frac{dy}{dt}(1,0) = 2 \cdot 1 = 2 > 0.$$

As the derivative $\frac{dy}{dt} > 0$, then the point crosses the x-axis in the upward direction with increasing time t. We mark this on the phase plane. Then, using symmetry, we can easy specify the direction of movement for the other phase trajectories (Figure 20).

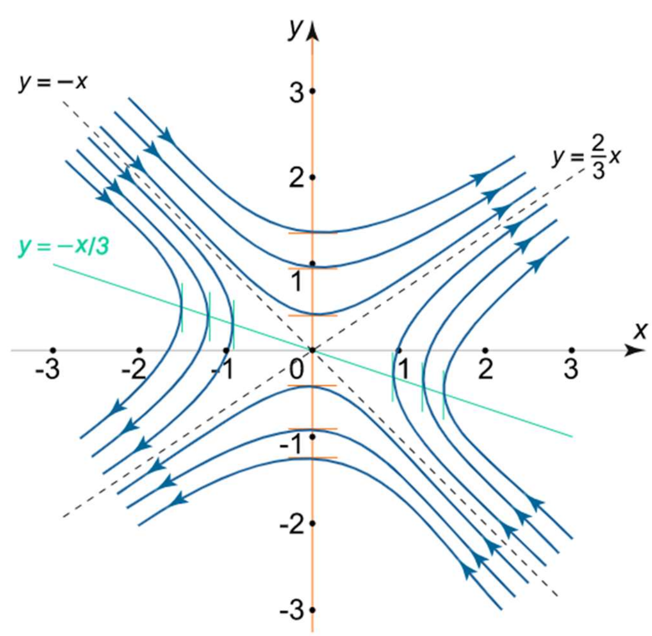


Figure 20.

Example 3.

Investigate the equilibrium points and sketch the phase portrait of the following system:

$$\frac{dx}{dt} = 3x - 4y, \quad \frac{dy}{dt} = 2x - y$$

Solution.

The determinant of this system is

$$A = \begin{bmatrix} 3 & -4 \\ 2 & -1 \end{bmatrix}, \ \det A = \begin{vmatrix} 3 & -4 \\ 2 & -1 \end{vmatrix} = 5 \neq 0.$$

Hence, the system has the unique equilibrium point $(0, 0)$.

1 We calculate the eigenvalues of A :

$$\begin{vmatrix} 3-\lambda & -4 \\ 2 & -1-\lambda \end{vmatrix} = 0, \quad \Rightarrow (\lambda - 3)(\lambda + 1) + 8 = 0,$$

$$\Rightarrow \lambda^2 - 2\lambda + 5 = 0, \ D = -16, \quad \Rightarrow \lambda_{1,2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

2 The eigenvalues λ_1, λ_2 are complex conjugate numbers with a positive real part. Therefore, the equilibrium position at the origin is an unstable focus.

3 Find the equations of isoclines. The vertical isocline is described by the following equation:

$$\frac{dx}{dt} = 3x - 4y = 0, \quad \Rightarrow y = \frac{3}{4}x.$$

The horizontal isocline is defined by the equation:

$$\frac{dy}{dt} = 2x - y = 0, \quad \Rightarrow y = 2x.$$

4 Find out the direction of twisting calculating the derivative $\frac{dy}{dt}$ at the point $(1, 0)$:

$$\frac{dy}{dt}(1, 0) = 2 \cdot 1 - 0 = 2 > 0.$$

Thus, the spirals twist counterclockwise.

5 Given the data found, we can construct a schematic phase portrait of the system (Figure 21).

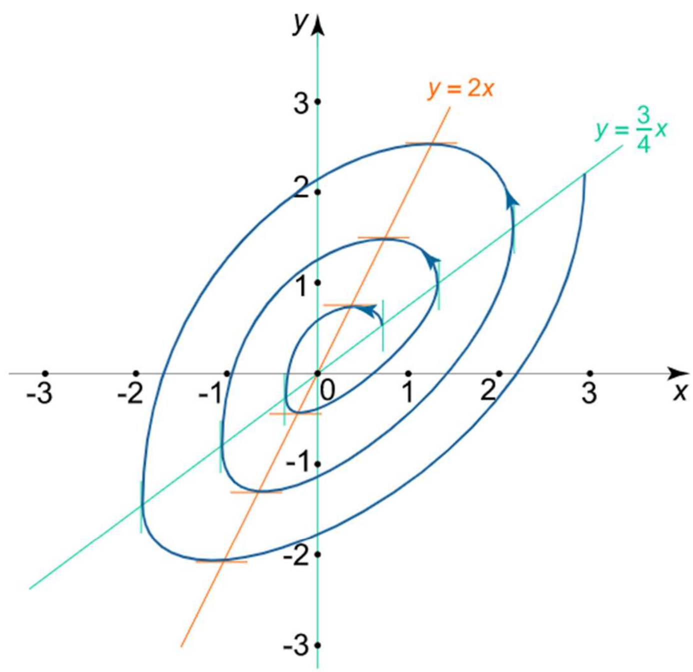


Figure 21.

Example 4.

Investigate the stability of the system depending on the parameter a :

$$\frac{dx}{dt} = ax + y, \quad \frac{dy}{dt} = x + ay.$$

Stability in the First Approximation or stability of nonlinear system

Example 1.

Investigate the stability of the zero equilibrium point of the system using the first approximation method.

$$\frac{dx}{dt} = y + 3x^2 + 2y^2, \quad \frac{dy}{dt} = -2x - y + xy.$$

Solution.

In this case, the functions f_1, f_2 have the form:

$$f_1(x,y) = y + 3x^2 + 2y^2, \quad f_2(x,y) = -2x - y + xy.$$

Obviously, they are continuous and infinitely differentiable in a neighborhood of the origin and equal to zero at $\mathbf{X} = \mathbf{0}$. Also, the order of the nonlinear terms in both functions is equal to or greater than 2. Thus, all requirements of the theorem on stability in the first approximation are satisfied. Compute the elements of the Jacobian matrix J at the equilibrium point $\mathbf{X} = \mathbf{0}$:

$$\frac{\partial f_1}{\partial x} = 6x, \quad \frac{\partial f_1}{\partial y} = 1 + 6y, \quad \frac{\partial f_2}{\partial x} = -2 + y, \quad \frac{\partial f_2}{\partial y} = -1 + x,$$

$$\Rightarrow \left.\frac{\partial f_1}{\partial x}\right|_{\substack{x=0\\y=0}} = 0, \quad \left.\frac{\partial f_1}{\partial y}\right|_{\substack{x=0\\y=0}} = 1, \quad \left.\frac{\partial f_2}{\partial x}\right|_{\substack{x=0\\y=0}} = -2, \quad \left.\frac{\partial f_2}{\partial y}\right|_{\substack{x=0\\y=0}} = -1, \quad \Rightarrow J = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}.$$

Find the eigenvalues:

$$\det(J - \lambda I) = 0, \quad \Rightarrow \begin{vmatrix} 0-\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix} = 0, \quad \Rightarrow \lambda(\lambda+1) + 2 = 0,$$

$$\Rightarrow \lambda^2 + \lambda + 2 = 0, \quad \Rightarrow D = -7, \quad \Rightarrow \lambda_{1,2} = \frac{-1 \pm \sqrt{-7}}{2} = -\frac{1}{2} \pm \frac{i\sqrt{7}}{2}.$$

The auxiliary equation has a pair of complex conjugate roots, the real part of which is negative:

$$\text{Re}[\lambda_1] < 0, \quad \text{Re}[\lambda_2] < 0.$$

Hence, the zero solution of the system is stable by the theorem on stability in the first approximation. This equilibrium point is a focus.

Example 2.

Find the equilibrium position of the system and investigate its stability in the first approximation.

$$\frac{dx}{dt} = x^2 - y, \ \ \frac{dy}{dt} = x - 1.$$

Solution.

We determine the equilibrium point from the system of algebraic equations:

$$\begin{cases} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} = 0 \end{cases}, \ \Rightarrow \begin{cases} x^2 - y = 0 \\ x - 1 = 0 \end{cases}, \ \Rightarrow \begin{cases} x = 1 \\ y = 1 \end{cases}.$$

As the equilibrium point $(1, 1)$ is not a zero solution, we make a coordinate transformation. We introduce the new variables u, v :

$$u = x - 1, \ \ v = y - 1.$$

Substituting into the original system, we obtain:

$$\begin{cases} \frac{d(u+1)}{dt} = (u+1)^2 - (v+1) \\ \frac{d(v+1)}{dt} = u + \cancel{1} - \cancel{1} \end{cases}, \ \Rightarrow \begin{cases} \frac{du}{dt} = u^2 + 2u + \cancel{1} - v - \cancel{1} \\ \frac{dv}{dt} = u \end{cases},$$

$$\Rightarrow \begin{cases} \frac{du}{dt} = u^2 + 2u - v \\ \frac{dv}{dt} = u \end{cases}$$

In order to determine the stability of the zero solution, we compute the Jacobian J. The right hand sides have the form:

$$f_1(u, v) = u^2 + 2u - v, \ \ f_2(u, v) = u.$$

Then

$$\frac{\partial f_1}{\partial u} = 2u + 2, \ \ \frac{\partial f_1}{\partial v} = -1, \ \ \frac{\partial f_2}{\partial u} = 1, \ \ \frac{\partial f_2}{\partial v} = 0.$$

Accordingly, the partial derivatives at the point $(u = 0, v = 0)$ are as follows:

$$\left.\frac{\partial f_1}{\partial u}\right|_{\substack{u=0\\v=0}} = 2, \quad \left.\frac{\partial f_1}{\partial v}\right|_{\substack{u=0\\v=0}} = -1, \quad \left.\frac{\partial f_2}{\partial u}\right|_{\substack{u=0\\v=0}} = 1, \quad \left.\frac{\partial f_2}{\partial v}\right|_{\substack{u=0\\v=0}} = 0.$$

Find the eigenvalues of the Jacobian J :

$$J = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \ \det(J - \lambda I) = 0, \quad \Rightarrow \begin{vmatrix} 2-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = 0,$$
$$\Rightarrow \lambda(\lambda - 2) + 1 = 0, \quad \Rightarrow \lambda^2 - 2\lambda + 1 = 0, \quad \Rightarrow (\lambda - 1)^2 = 0, \quad \Rightarrow \lambda_{1,2} = 1.$$

Thus, the matrix J has one eigenvalue $\lambda = 1$ with algebraic multiplicity 2.

Thus the zero solution of the linearized system and hence the equilibrium point (1, 1) is unstable node.

Example 3.

Determine the equilibrium positions of the system and explore their stability. Draw a schematic phase portrait of the corresponding linearized system.

$$\frac{dx}{dt} = e^{x+y} - 1, \ \frac{dy}{dt} = \ln(1 + x).$$

Solution.

We find the equilibrium points by solving the system of algebraic equations:

$$\begin{cases} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} = 0 \end{cases}, \quad \Rightarrow \begin{cases} e^{x+y} - 1 = 0 \\ \ln(1 + x) = 0 \end{cases}.$$

The second equation implies that $x = 0$. Substituting this into the first equation, we have:

$$e^y - 1 = 0, \quad \Rightarrow e^y = 1, \quad \Rightarrow y = 0.$$

Thus, the system has the unique equilibrium position $(x = 0, y = 0)$. To investigate the stability of the zero solution, we expand the right hand sides in the Maclaurin series:

$$f_1(x, y) = e^{x+y} - 1, \ f_2(x, y) = \ln(1 + x),$$

$$\frac{\partial f_1}{\partial x} = e^{x+y}, \ \frac{\partial f_1}{\partial y} = e^{x+y}, \ \frac{\partial f_2}{\partial x} = \frac{1}{1+x}, \ \frac{\partial f_2}{\partial y} = 0.$$

The partial derivatives at the point $(x=0, y=0)$ are equal to

$$\left.\frac{\partial f_1}{\partial x}\right|_{\substack{x=0\\y=0}} = 1, \ \left.\frac{\partial f_1}{\partial y}\right|_{\substack{x=0\\y=0}} = 1, \ \left.\frac{\partial f_2}{\partial x}\right|_{\substack{x=0\\y=0}} = 1, \ \left.\frac{\partial f_2}{\partial y}\right|_{\substack{x=0\\y=0}} = 0.$$

We find the eigenvalues of the Jacobian matrix J :

$$J = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \ \det(J - \lambda I) = 0, \ \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = 0, \ \Rightarrow \lambda(\lambda - 1) - 1 = 0,$$
$$\Rightarrow \lambda^2 - \lambda - 1 = 0, \ \Rightarrow D = 5, \ \Rightarrow \lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2} = \frac{1}{2} \pm \frac{\sqrt{5}}{2}.$$

As seen the eigenvalues are real numbers with different signs and approximately equal to

$$\lambda_1 = \frac{1}{2} + \frac{\sqrt{5}}{2} \approx 0.5 + 1.12 = 1.62, \ \ \lambda_2 = \frac{1}{2} - \frac{\sqrt{5}}{2} \approx 0.5 - 1.12 = -0.62$$

Therefore, the linearized system has the zero equilibrium point of the saddle type. A similar conclusion is true with respect to the original nonlinear system.

Construct a schematic phase portrait of the linearized system. Compute the eigenvectors $\mathbf{V}_1$, $\mathbf{V}_2$, associated with the eigenvalues λ_1 and λ_2. For the first eigenvalue λ_1, we obtain:

$$(J - \lambda_1 I)\,\mathbf{V}_1 = \mathbf{0}, \ \Rightarrow \begin{bmatrix} \frac{1}{2} - \frac{\sqrt{5}}{2} & 1 \\ 1 & -\frac{1}{2} - \frac{\sqrt{5}}{2} \end{bmatrix} \cdot \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \mathbf{0},$$
$$\Rightarrow \begin{cases} \left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right) V_{11} + V_{21} = 0 \\ V_{11} - \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) V_{21} = 0 \end{cases}, \ \Rightarrow \begin{cases} \left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right) V_{11} + V_{21} = 0 \\ \left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right) V_{11} + V_{21} = 0 \end{cases},$$
$$\Rightarrow \left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right) V_{11} + V_{21} = 0.$$

Let $V_{11} = t$. Then

$$V_{21} = -\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right) V_{11} = -\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right) t \approx 0.62t,$$
$$\Rightarrow \mathbf{V}_1 = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \begin{bmatrix} t \\ 0.62t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0.62 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0.62 \end{bmatrix}.$$

Similarly, we find the eigenvector $\mathbf{V}_2$ corresponding to the eigenvalue λ_2 :

$$(J-\lambda_2 I)\,\mathbf{V}_2=\mathbf{0},\quad\Rightarrow\begin{bmatrix}\frac{1}{2}+\frac{\sqrt{5}}{2} & 1\\ 1 & -\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)\end{bmatrix}\cdot\begin{bmatrix}V_{12}\\ V_{22}\end{bmatrix}=\mathbf{0},$$

$$\Rightarrow\begin{cases}\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)V_{12}+V_{22}=0\\ V_{12}-\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)V_{22}=0\end{cases},\quad\Rightarrow\begin{cases}\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)V_{12}+V_{22}=0\\ \left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)V_{12}+V_{22}=0\end{cases},$$

$$\Rightarrow\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)V_{12}+V_{22}=0.$$

Suppose that $V_{12}=t$. Hence,

$$V_{22}=-\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)V_{12}=-\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)t\approx-1.62t,$$

$$\Rightarrow\mathbf{V}_2=\begin{bmatrix}V_{12}\\ V_{22}\end{bmatrix}=\begin{bmatrix}t\\ -1.62t\end{bmatrix}=t\begin{bmatrix}1\\ -1.62\end{bmatrix}\sim\begin{bmatrix}1\\ -1.62\end{bmatrix}.$$

Draw the straight lines passing through the origin and directed along the vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ (Figure 2).

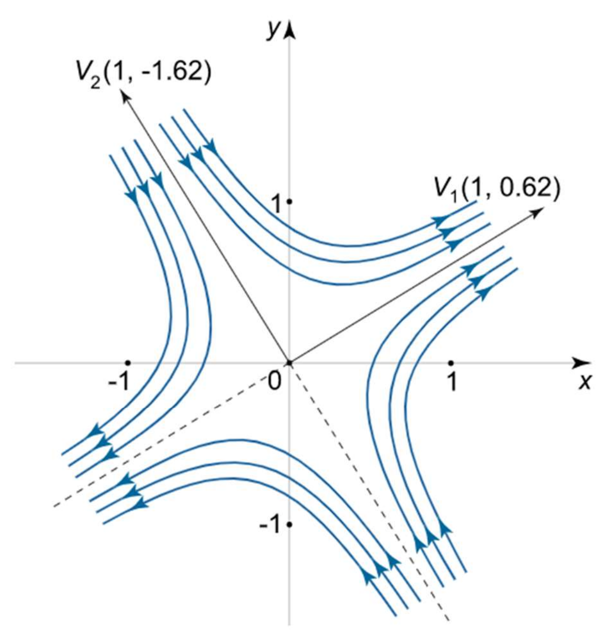


Figure 2.

These lines are separatrices of the saddle. Now we can depict the phase trajectories.

Now we specify the direction of the trajectories by selecting any point, for example, the point $(1, 0)$. The velocity of motion at this point is given by

$$\left.\frac{dx}{dt}\right|_{(1,0)} = e^{1+0} - 1 \approx 1.72, \quad \left.\frac{dy}{dt}\right|_{(1,0)} = \ln(1+1) = \ln 2 \approx 0.69$$

It is clear that the velocity vector is directed to the upper right side. On this basis, we can specify the direction of the phase trajectories.

This equilibrium point is rough. Therefore, the phase portrait of the original nonlinear system has the same shape in a neighborhood of the zero equilibrium point as shown in Figure 2 for the linearized system.

Example 4.

Using equations of the first approximation investigate the stability of the zero solution of the nonlinear system:

$$\frac{dx}{dt} = \tan(x+y) - y, \ \frac{dy}{dt} = 3\sin x + 2e^{y} - 2.$$

Solution.

We verify that the point $(x = 0, y = 0)$ is an equilibrium position for the given system:

$$f_1\left(0,0\right) = \tan 0 - 0 = 0, \ f_2\left(0,0\right) = 3\sin 0 + 2e^{0} - 2 = 0 + 2\cdot 1 - 2 = 0.$$

Expand the functions f_1, f_2 (which are continuously differentiable in a neighborhood of the zero point) in the Maclaurin series. The first order partial derivatives have the form

$$\frac{\partial f_1}{\partial x} = \frac{1}{\cos^2\left(x+y\right)}, \ \frac{\partial f_1}{\partial y} = \frac{1}{\cos^2\left(x+y\right)} - 1, \ \frac{\partial f_2}{\partial x} = 3\cos x, \ \frac{\partial f_2}{\partial y} = 2e^{y}.$$

The values of the derivatives at the point $(x = 0, y = 0)$ are

$$\left.\frac{\partial f_1}{\partial x}\right|_{(0,0)} = 1, \ \left.\frac{\partial f_1}{\partial y}\right|_{(0,0)} = 0, \ \left.\frac{\partial f_2}{\partial x}\right|_{(0,0)} = 3, \ \left.\frac{\partial f_2}{\partial y}\right|_{(0,0)} = 2.$$

We've got the Jacobian matrix J for the linearized system in the form:

$$J = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}.$$

Compute its eigenvalues:

$$\det(J - \lambda I) = 0, \quad \Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 3 & 2-\lambda \end{vmatrix} = 0, \quad \Rightarrow (\lambda - 1)(\lambda - 2) = 0,$$
$$\Rightarrow \lambda_1 = 1, \ \lambda_2 = 2.$$

Thus, according to the Lyapunov theorem on stability in the first approximation, the zero solution of the system is unstable. The zero equilibrium point is an unstable node.

We find the eigenvectors and draw a schematic phase portrait of the linearized system near zero. For the eigenvalue $\lambda_1 = 1$, the vector $\mathbf{V}_1$ has the following coordinates:

$$(J - \lambda_1 I)\,\mathbf{V}_1 = \mathbf{0}, \quad \Rightarrow \begin{bmatrix} 1-1 & 0 \\ 3 & 2-1 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \mathbf{0}, \quad \Rightarrow \begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \mathbf{0},$$
$$\Rightarrow 3V_{11} + V_{21} = 0.$$

Let $V_{11} = t$. Then

$$V_{21} = -3V_{11} = -3t, \quad \Rightarrow \mathbf{V}_1 = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \begin{bmatrix} t \\ -3t \end{bmatrix} = t \begin{bmatrix} 1 \\ -3 \end{bmatrix} \sim \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

Determine the eigenvector $\mathbf{V}_2$ for the eigenvalue $\lambda_2 = 2$:

$$(J - \lambda_2 I)\,\mathbf{V}_2 = \mathbf{0}, \quad \Rightarrow \begin{bmatrix} 1-2 & 0 \\ 3 & 2-2 \end{bmatrix} \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \mathbf{0}, \quad \Rightarrow \begin{bmatrix} -1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \mathbf{0},$$
$$\Rightarrow \begin{cases} -V_{12} + 0 \cdot V_{22} = 0 \\ 3V_{12} + 0 \cdot V_{22} = 0 \end{cases}.$$

Let $V_{12} = 0$, $V_{22} = 1$, that is

$$\mathbf{V}_2 = \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Now we can plot the asymptotic straight lines passing through the origin and parallel to the vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ (Figure 3).

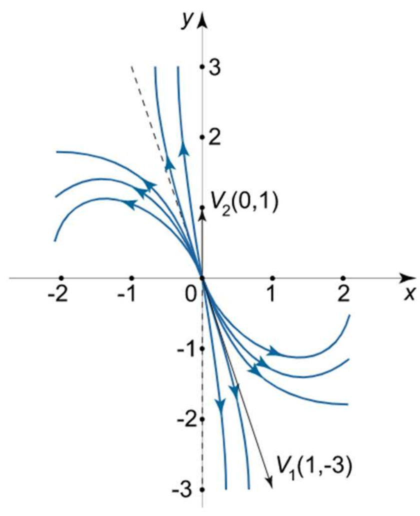


Figure 3.

We take into account that the phase trajectories asymptotically approach the line directed along the vector $\mathbf{V}_1$ with the smallest (in absolute value) eigenvalue $\lambda_1 = 1$. At a distance from the origin, the phase trajectories become parallel to the eigenvector $\mathbf{V}_2$, which is directed along the y-axis.

The original nonlinear system has the same phase portrait near the origin. This follows from the roughness (structural stability) of the zero equilibrium position.

Example 5.

Using the first approximation method, investigate the stability of the equilibrium point of the system

$$\frac{dx}{dt} = \ln(x+y), \;\; \frac{dy}{dt} = \arctan\frac{2x}{y}.$$

Example 6.

Using the first approximation method, investigate the stability of the zero solution of the system

$$\frac{dx}{dt} = \sin(x+y) - y, \;\; \frac{dy}{dt} = y^2 + 2x.$$

Method of Lyapunov Functions

Definition of the Lyapunov Function

A Lyapunov function is a scalar function defined on the phase space, which can be used to prove the stability of an equilibrium point. The Lyapunov function method is applied to study the stability of various differential equations and systems. Below, we restrict ourselves to the autonomous systems

$$\mathbf{X}' = \mathbf{f}\left(\mathbf{X}\right) \quad \text{or} \quad \frac{dx_i}{dt} = f_i\left(x_1, x_2, \ldots, x_n\right), \; i = 1, 2, \ldots, n,$$

with the zero equilibrium $\mathbf{X} \equiv \mathbf{0}$.

We suppose that we are given a continuously differentiable function

$$V\left(\mathbf{X}\right) = V\left(x_1, x_2, \ldots, x_n\right)$$

in a neighborhood U of the origin. Let $V\left(\mathbf{X}\right) > 0$ for all $\mathbf{X} \in U \backslash \left\{\mathbf{0}\right\}$, and $V\left(\mathbf{0}\right) = 0$ in the origin. For example, these are functions of the form

$$V\left(x_1, x_2\right) = ax_1^2 + bx_2^2, \; V\left(x_1, x_2\right) = ax_1^2 + bx_2^4, \; a, b > 0.$$

We find the total derivative of the function $V\left(\mathbf{X}\right)$ with respect to time t :

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial V}{\partial x_n}\frac{dx_n}{dt}.$$

This expression can be written as a scalar (dot) product of two vectors:

$$\frac{dV}{dt} = \left(\text{grad}\, V, \frac{d\mathbf{X}}{dt}\right), \; \text{where} \; \text{grad}\, V = \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \ldots, \frac{\partial V}{\partial x_n}\right),$$
$$\frac{d\mathbf{X}}{dt} = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt}\right).$$

Here, the first vector is the gradient of $V\left(\mathbf{X}\right)$, i.e. it's always directed toward the greatest increase in $V\left(\mathbf{X}\right)$. Typically, the function $V\left(\mathbf{X}\right)$ increases with the distance from the origin, i.e. provided $\left|\mathbf{X}\right| \to \infty$. The second vector in the scalar product is the velocity vector. At any point, it is tangent to the phase trajectory.

Consider the case when the derivative of $V(\mathbf{X})$ in a neighborhood U of the origin is negative:

$$\frac{dV}{dt} = \left(\operatorname{grad} V, \frac{d\mathbf{X}}{dt}\right) < 0.$$

This means that the angle φ between the gradient vector and the velocity vector is greater than 90°. For a function of two variables, it is shown schematically in Figures $1-2$.

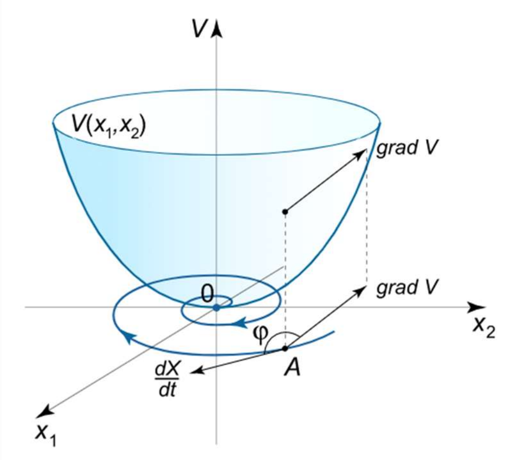


Figure 1.

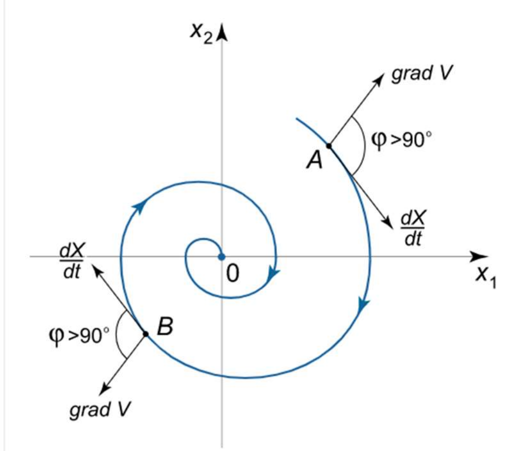


Figure 2.

Obviously, if the derivative $\frac{dV}{dt}$ along a phase trajectory is everywhere negative, then the trajectory tends to the origin, i.e. the system is stable. Otherwise, when the derivative $\frac{dV}{dt}$ is positive, the trajectory moves away from the origin, i.e. the system is unstable.

We now turn to the strict formulation.

Let a function $V(\mathbf{X})$ be continuously differentiable in a neighborhood U of the origin. The function $V(\mathbf{X})$ is called the Lyapunov function for an autonomous system

$$\mathbf{X}' = \mathbf{f}(\mathbf{X}),$$

if the following conditions are met:

1. $V(\mathbf{X}) > 0$ for all $\mathbf{X} \in U \setminus \{\mathbf{0}\}$;
2. $V(\mathbf{0}) = 0$;
3. $\frac{dV}{dt} \le 0$ for all $\mathbf{X} \in U$.

Stability Theorems

Theorem on stability in the sense of Lyapunov.

If in a neighborhood U of the zero solution $\mathbf{X} = \mathbf{0}$ of an autonomous system there is a Lyapunov function $V(\mathbf{X})$, then the equilibrium point $\mathbf{X} = \mathbf{0}$ of the system is Lyapunov stable.

Theorem on asymptotic stability.

If in a neighborhood U of the zero solution $\mathbf{X} = \mathbf{0}$ of an autonomous system there is a Lyapunov function $V(\mathbf{X})$ with a negative definite derivative $\frac{dV}{dt} < 0$ for all $\mathbf{X} \in U \backslash \{\mathbf{0}\}$, then the equilibrium point $\mathbf{X} = \mathbf{0}$ of the system is asymptotically stable.

As it can be seen, the total derivative $\frac{dV}{dt}$ must be strictly negative (negative definite) in a neighborhood of the origin for the asymptotic stability of the zero solution.

Instability Theorems

Lyapunov instability theorem.

Suppose that in a neighborhood U of the zero solution $\mathbf{X} = \mathbf{0}$ there is a continuously differentiable function $V(\mathbf{X})$ such that

1. $V(\mathbf{0}) = 0;$
2. $\frac{dV}{dt} > 0.$

If in the neighborhood U there are points at which $V(\mathbf{X}) > 0$, then the zero solution $\mathbf{X} = \mathbf{0}$ is unstable.

Chetaev instability theorem.

Suppose that in a neighborhood U of the zero solution $\mathbf{X} = \mathbf{0}$ of an autonomous system there exists a continuously differentiable function $V(\mathbf{X})$. Let the neighborhood U contain a subdomain U_1, including the origin (Figure 3) such that

1. $V(\mathbf{X}) > 0$ for all $\mathbf{X} \in U_1 \backslash \{\mathbf{0}\};$
2. $\frac{dV}{dt} > 0$ for all $\mathbf{X} \in U_1 \backslash \{\mathbf{0}\};$
3. $V(\mathbf{X}) = 0$ for all $\mathbf{X} \in \delta U_1$, where δU_1 denotes the boundary of the subdomain U_1.

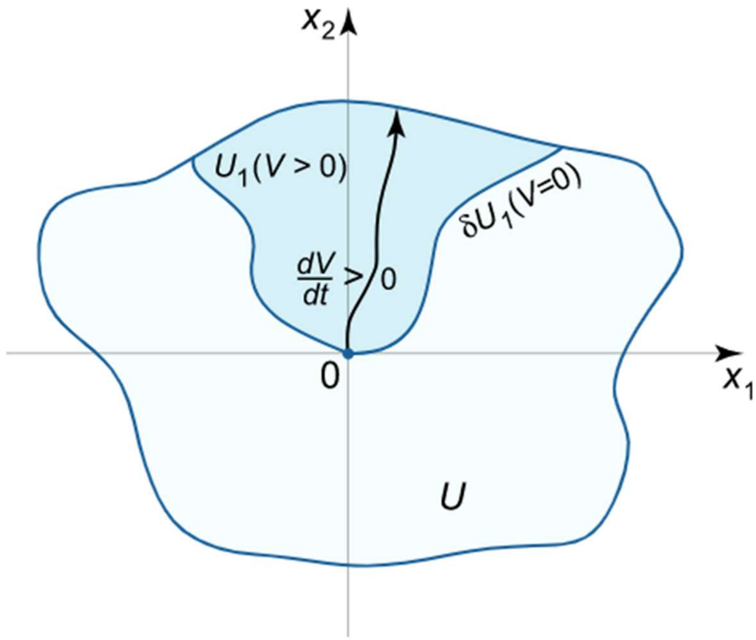


Figure 3.

Then the zero solution $\mathbf{X} = \mathbf{0}$ of the system is unstable. In this case, the phase trajectories in the subdomain U_1 will move away from the origin.

Thus, Lyapunov functions allow to determine the stability or instability of a system. The advantage of this method is that we do not need to know the actual solution $\mathbf{X}(t)$. In addition, this method allows to study the stability of equilibrium points of non-rough systems, for example, in the case when the equilibrium point is a center. The disadvantage is that there is no general method of constructing Lyapunov functions. In the particular case of homogeneous autonomous systems with constant coefficients, the Lyapunov function can be found as a quadratic form.

Example 1.

Investigate the stability of the zero solution of the system

$$\frac{dx}{dt} = -2x, \;\; \frac{dy}{dt} = x - y.$$

Solution.

This system is a linear homogeneous system with constant coefficients. We take as a Lyapunov function the quadratic form

$$V(\mathbf{X}) = V(x, y) = ax^2 + by^2,$$

where the coefficients a, b are to be determined.

Obviously, the function $V(x, y)$ is positive everywhere except at the origin, where it is zero. We calculate the total derivative of the function $V(x, y)$:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = 2ax(-2x) + 2by(x - y) = -4ax^2 + 2bxy - 2by^2$$
$$= -2b\left(\frac{4a}{2b}x^2 - xy + y^2\right) = -2b\left(\frac{2a}{b}x^2 - xy + y^2\right).$$

The expression in the brackets can be converted to a square of the difference if the following condition is satisfied:

$$\frac{2a}{b} = \frac{1}{4} \quad \text{or} \quad 8a = b.$$

We can take any suitable combination, for example, we set $a = 1, b = 8$. Then the derivative becomes

$$\frac{dV}{dt} = -16\left(\frac{x^2}{4} - xy + y^2\right) = -16\left(\frac{x}{2} - y\right)^2 < 0.$$

Thus, for the given system, there is a Lyapunov function, and its derivative is negative everywhere except at the origin. Hence, the zero solution of the system is asymptotically stable (stable node).

Example 2.

Investigate the stability of the zero solution of the system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x.$$

Solution.

Note that the first approximation method is not applicable for this system, since the zero solution is a "center" (that is the system is not rough):

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \det(A - \lambda I) = 0, \ \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0, \ \Rightarrow \lambda^2 + 1 = 0, \ \Rightarrow \lambda_{1,2} = \pm i.$$

We use the method of Lyapunov functions for the stability analysis. Let the function $V(\mathbf{X})$ have the form

$$V(\mathbf{X}) = V(x, y) = x^2 + y^2.$$

We calculate the derivative of the function $V(\mathbf{X})$ by virtue of the system:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = 2x \cdot y + 2y \cdot (-x) \equiv 0.$$

Thus, the derivative is identically zero. Hence, the function $V(\mathbf{X})$ is a Lyapunov function and the zero solution of the system is stable in the sense of Lyapunov. The condition of asymptotic stability is not satisfied (for this, the derivative $\frac{dV}{dt}$ must be negative).

Example 3.

Investigate the stability of the zero solution of the nonlinear system

$$\frac{dx}{dt} = -xy^2, \ \frac{dy}{dt} = 3yx^2.$$

Solution.

It is obvious that the Jacobian of the system at the point $(0, 0)$ is a zero matrix:

$$J = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}\right|_{\substack{x=0 \\ y=0}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalues of this matrix are zero: $\lambda_{1,2} = 0$. Therefore, the first approximation method is inapplicable.

Let's see what results can be obtained using a Lyapunov function. We choose as a Lyapunov function the quadratic form

$$V(\mathbf{X}) = V(x, y) = 3x^2 + y^2,$$

which is positive definite everywhere except at the origin. Calculate the total derivative:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = 6x\left(-xy^2\right) + 2y\left(3yx^2\right) = -6x^2y^2 + 6x^2y^2 \equiv 0.$$

Here again, as in the previous example, the derivative is identically zero. This means that the zero solution is stable (in the sense of Lyapunov).

Example 4.

Investigate the stability of the zero solution of the system using the method of Lyapunov functions.

$$\frac{dx}{dt} = y - 2x,\ \ \frac{dy}{dt} = 2x - y - x^3.$$

Solution.

As a Lyapunov candidate function we choose a function of the form

$$V(\mathbf{X}) = V(x, y) = (x + y)^2 + \frac{x^4}{2}.$$

Obviously, this function is positive definite everywhere except at the origin, where it is zero. We calculate its derivative (by virtue of the system):

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = \left(2x + 2y + 2x^3\right)(y - 2x) + (2x + 2y)\left(2x - y - x^3\right)$$
$$= \cancel{(2x + 2y)(y - 2x)} + 2x^3(y - 2x) - \cancel{(2x + 2y)(y - 2x)} - x^3(2x + 2y)$$
$$= \cancel{2x^3y} - 4x^4 - 2x^4 - \cancel{2x^3y} = -6x^4 \le 0.$$

As one can see, the derivative is negative definite everywhere except at $(0, 0)$. Then the zero solution is asymptotically stable.

Note that the first approximation method is inapplicable here because one of the eigenvalues is zero:

$$J = \left. \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \right|_{\substack{x=0 \\ y=0}} = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix};$$

$$\det(J - \lambda I) = 0, \quad \Rightarrow \begin{vmatrix} -2-\lambda & 1 \\ 2 & -1-\lambda \end{vmatrix} = 0, \quad \Rightarrow (-2-\lambda)(-1-\lambda) - 2 = 0,$$

$$\Rightarrow \cancel{2} + \lambda + 2\lambda + \lambda^2 - \cancel{2} = 0, \quad \Rightarrow \lambda^2 + 3\lambda = 0, \quad \Rightarrow \lambda(\lambda+3) = 0,$$

$$\Rightarrow \lambda_1 = 0, \ \lambda_2 = -3.$$

Example 5.

Using a Lyapunov function, investigate the stability of the zero solution of the system

$$\frac{dx}{dt} = x + 3y, \ \frac{dy}{dt} = 2x.$$

Example 6.

Investigate the stability of the zero solution of the system

$$\frac{dx}{dt} = x^3 + y, \ \frac{dy}{dt} = x + y^3.$$

Existence and Uniqueness Theorems

Let us return to the general initial value problem

$$y' = F(t, y), \quad y(t_0) = y_0, \tag{1}$$

The main theorems we have in mind put certain relatively mild conditions on F to insure that a solution exists, is unique, and/or both. We say that a solution ***exists*** if there is a function $y = y(t)$ defined on an interval containing t_0 as an interior point and satisfying (1). We say that a solution is ***unique*** if there is only one such function $y = y(t)$. Why are the concepts of existence and uniqueness important? Differential equations frequently model real-world systems as a function of time. Knowing that a solution exists means that the system has predictable future states. Further, when that solution is also unique, then for each future time there is only *one* possible state, leaving no room for ambiguity.

To illustrate these points and some of the theoretical aspects of the existence and uniqueness theorems that follow, consider the following example.

Example 1. Show that the following two functions satisfy the initial value problem

$$y' = 3y^{2/3}, \quad y(0) = 0. \tag{2}$$

1. $y(t) = 0$
2. $y(t) = t^3$

► **Solution.** Clearly, the constant function $y(t) = 0$ for $t \in \mathbb{R}$ is a solution. For the second function $y(t) = t^3$, observe that $y' = 3t^2$ while $3y^{2/3} = 3(t^3)^{2/3} = 3t^2$. Further, $y(0) = 0$. It follows that both of the functions $y(t) = 0$ and $y(t) = t^3$ are solutions of (2). ◄

If this differential equation modeled a real-world system, we would have problems accurately predicting future states. Should we use $y(t) = 0$ or $y(t) = t^3$? It is even worse than this, for further analysis of this differential equation reveals that there are many other solutions from which to choose. (See Example 9.) What is it about (2) that allows multiple solutions? More precisely, what conditions could we impose on (1) to guarantee that a solution exists and is unique? These questions are addressed in Picard's existence and uniqueness theorem, Theorem 5, stated below.

Thus far, our method for proving the existence of a solution to an ordinary differential equation has been to explicitly find one. This has been a reasonable approach for the categories of differential equations we have introduced thus far. However, there are many differential equations that do not fall into any of these categories, and knowing that a solution exists is a fundamental piece of information in the analysis of any given initial value problem.

Suppose $F(t, y)$ is a *continuous* function of (t, y) in the rectangle

$$\mathcal{R} := \{(t, y) : a \le t \le b\,,\, c \le y \le d\}$$

and (t_0, y_0) is an interior point of $\mathcal{R}$. The key to the proof of existence and uniqueness is the fact that a continuously differentiable function $y(t)$ is a solution of (1) if and only if it is a solution of the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} F(u, y(u))\,\mathrm{d}u. \tag{3}$$

To see the equivalence between (1) and (3), assume that $y(t)$ is a solution to (1), so that

$$y'(t) = F(t, y(t))$$

for all t in an interval containing t_0 as an interior point and $y(t_0) = y_0$. Replace t by u in this equation, integrate both sides from t_0 to t, and use the fundamental theorem of calculus to get

$$\int_{t_0}^{t} F(u, y(u))\,\mathrm{d}u = \int_{t_0}^{t} y'(u)\,\mathrm{d}u = y(t) - y(t_0) = y(t) - y_0,$$

which implies that $y(t)$ is a solution of (3). Conversely, if $y(t)$ is a continuously differentiable solution of (3), it follows that

$$g(t) := F(t, y(t))$$

is a continuous function of t since $F(t, y)$ is a continuous function of t and y. Apply the fundamental theorem of calculus to get

$$\begin{aligned} y'(t) &= \frac{\mathrm{d}}{\mathrm{d}t} y(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(y_0 + \int_{t_0}^{t} F(u, y(u))\,\mathrm{d}u\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(y_0 + \int_{t_0}^{t} g(u)\,\mathrm{d}u\right) = g(t) = F(t, y(t)), \end{aligned}$$

which is what it means for $y(t)$ to be a solution of (1). Since

$$y(t_0) = y_0 + \int_{t_0}^{t_0} F(u,\, y(u))\,\mathrm{d}u = y_0,$$

$y(t)$ also satisfies the initial value in (1).

We will refer to (3) as the *integral equation* corresponding to the initial value problem (1) and conversely, (1) is referred to as the *initial value problem* corresponding to the integral equation (3). What we have shown is that a solution to the initial value problem is a solution to the corresponding integral equation and vice versa.

Example 2. Find the integral equation corresponding to the initial value problem

$$y' = t + y, \quad y(0) = 1.$$

► **Solution.** In this case, $F(t,\, y) = t + y$, $t_0 = 0$ and $y_0 = 1$. Replace the independent variable t by u in $F(t,\, y(t))$ to get $F(u,\, y(u)) = u + y(u)$. Thus, the integral equation (3) corresponding to this initial value problem is

$$y(t) = 1 + \int_0^t (u + y(u))\,\mathrm{d}u.$$ ◄

For any continuous function y, define

$$\mathcal{T}y(t) = y_0 + \int_{t_0}^t F(u, y(u))\,\mathrm{d}u.$$

That is, $\mathcal{T}y$ is the right-hand side of (3) for any continuous function y. Given a function y, $\mathcal{T}$ produces a new function $\mathcal{T}y$. If we can find a function y so that $\mathcal{T}y = y$, we say y is a ***fixed point*** of $\mathcal{T}$. A fixed point y of $\mathcal{T}$ is precisely a solution to (3) since if $y = \mathcal{T}y$, then

$$y = \mathcal{T}y = y_0 + \int_{t_0}^t F(u, y(u)\,\mathrm{d}u,$$

which is what it means to be a solution to (3). To solve equations like the integral equation (3), mathematicians have developed a variety of so-called "fixed point theorems" for operators such as $\mathcal{T}$, each of which leads to an existence and/or uniqueness result for solutions to an integral equation. One of the oldest and most widely used of the existence and uniqueness theorems is due to Émile Picard (1856–1941). Assuming that the function $F(t, y)$ is sufficiently "nice," he first employed the ***method of successive approximations***. This method is an iterative procedure which begins with a crude approximation of a solution and improves it using a step-by-step procedure that brings us as close as we please to an exact and unique solution of (3). The process should remind students of Newton's method where successive approximations are used to find numerical solutions to $f(t) = c$ for some function f and constant c. The algorithmic procedure follows.

Algorithm 3. Perform the following sequence of steps to produce an *approximate solution* of (3), and hence to the initial value problem, (1).

Picard Approximations

1. A rough initial approximation to a solution of (3) is given by the constant function
$$y_0(t) := y_0.$$
2. Insert this initial approximation into the right-hand side of (3) and obtain the first approximation
$$y_1(t) := y_0 + \int_{t_0}^{t} F(u, y_0(u))\, \mathrm{d}u.$$
3. The next step is to generate the second approximation in the same way, that is,
$$y_2(t) := y_0 + \int_{t_0}^{t} F(u, y_1(u))\, \mathrm{d}u.$$
4. At the nth stage of the process, we have
$$y_n(t) := y_0 + \int_{t_0}^{t} F(u, y_{n-1}(u))\, \mathrm{d}u,$$
which is defined by substituting the previous approximation $y_{n-1}(t)$ into the right-hand side of (3).

In terms of the operator $\mathcal{T}$ introduced above, we can write

$$
\begin{aligned}
y_1 &= \mathcal{T} y_0 \\
y_2 &= \mathcal{T} y_1 = \mathcal{T}^2 y_0 \\
y_3 &= \mathcal{T} y_2 = \mathcal{T}^3 y_0 \\
&\vdots
\end{aligned}
$$

The result is a sequence of functions $y_0(t), y_1(t), y_2(t), \ldots$, defined on an interval containing t_0. We will refer to y_n as the ***nth Picard approximation*** and the sequence $y_0(t), y_1(t), \ldots, y_n(t)$ as the ***first n Picard approximations***. Note that the first n Picard approximations actually consist of $n + 1$ functions, since the starting approximation $y_0(t) = y_0$ is included.

Example 4. Find the first three Picard approximations for the initial value problem

$$y' = t + y, \quad y(0) = 1.$$

► **Solution.** The corresponding integral equation was computed in Example 2:

$$y(t) = 1 + \int_0^t (u + y(u))\, \mathrm{d}u.$$

We have

$$
\begin{aligned}
y_0(t) &= 1 \\
y_1(t) &= 1 + \int_0^t (u + y_0(u))\, \mathrm{d}u \\
&= 1 + \int_0^t (u + 1)\, \mathrm{d}u \\
&= 1 + \left(\frac{u^2}{2} + u\right)\Bigg|_0^t = 1 + \frac{t^2}{2} + t = 1 + t + \frac{t^2}{2}. \\
y_2(t) &= 1 + \int_0^t \left(u + 1 + u + \frac{u^2}{2}\right) \mathrm{d}u = 1 + \int_0^t \left(1 + 2u + \frac{u^2}{2}\right) \mathrm{d}u \\
&= 1 + \left(u + u^2 + \frac{u^3}{6}\right)\Bigg|_0^t = 1 + t + t^2 + \frac{t^3}{6}. \\
y_3(t) &= 1 + \int_0^t \left(u + 1 + u + u^2 + \frac{u^3}{6}\right) \mathrm{d}u = 1 + \int_0^t \left(1 + 2u + u^2 + \frac{u^3}{6}\right) \mathrm{d}u
\end{aligned}
$$

$$= 1 + \left(u + u^2 + \frac{u^3}{3} + \frac{u^4}{24}\right)\bigg|_0^t = 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{24}. \qquad \blacktriangleleft$$

It was one of Picard's great contributions to mathematics when he showed that the functions $y_n(t)$ converge to a unique, continuously differentiable solution $y(t)$ of the integral equation, (3), and thus of the initial value problem, (1), under the mild condition that the function $F(t, y)$ and its partial derivative $F_y(t, y) := \frac{\partial}{\partial y} F(t, y)$ are continuous functions of (t, y) on the rectangle $\mathcal{R}$.

Theorem 5 (Picard's Existence and Uniqueness Theorem).[13] *Let $F(t, y)$ and $F_y(t, y)$ be continuous functions of (t, y) on a rectangle*

$$\mathcal{R} = \{(t, y) : a \le t \le b\, ,\, c \le y \le d\}\, .$$

If (t_0, y_0) is an interior point of $\mathcal{R}$, then there exists a unique solution $y(t)$ of

$$y' = F(t, y)\, , \quad y(t_0) = y_0,$$

on some interval $[a', b']$ with $t_0 \in [a', b'] \subset [a, b]$. Moreover, the sequence of approximations $y_0(t) := y_0$

$$y_n(t) := y_0 + \int_{t_0}^{t} F(u, y_{n-1}(u))\, \mathrm{d}u,$$

computed by Algorithm 3 converges uniformly[14] *to $y(t)$ on the interval $[a', b']$.*

Example 6. Consider the initial value problem

$$y' = t + y \quad y(0) = 1.$$

For $n \ge 1$, find the nth Picard approximation and determine the limiting function $y = \lim_{n\to\infty} y_n$. Show that this function is a solution and, in fact, the only solution.

► **Solution.** In Example 4, we computed the first three Picard approximations:

$$y_1(t) = 1 + t + \frac{t^2}{2},$$
$$y_2(t) = 1 + t + t^2 + \frac{t^3}{3!},$$

$$y_3(t) = 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{4!}.$$

It is not hard to verify that

$$\begin{aligned} y_4(t) &= 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{12} + \frac{t^5}{5!} \\ &= 1 + t + 2\left(\frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}\right) + \frac{t^5}{5!} \end{aligned}$$

and inductively,

$$y_n(t) = 1 + t + 2\left(\frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!}\right) + \frac{t^{n+1}}{(n+1)!}.$$

Recall from calculus that $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$, so the part in parentheses in the expression for $y_n(t)$ is the first n terms of the expansion of e^t minus the first two:

$$\frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!} = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!}\right) - (1 + t).$$

Thus,

$$y(t) = \lim_{n\to\infty} y_n(t) = 1 + t + 2(e^t - (1 + t)) = -1 - t + 2e^t.$$

It is easy to verify by direct substitution that $y(t) = -1 - t + 2e^t$ is a solution to $y' = t + y$ with initial value $y(0) = 1$. Moreover, since the equation $y' = t + y$ is a first order linear differential equation, the techniques of Sect. 1.4 show that $y(t) = -1 - t + 2e^t$ is the *unique* solution since it is obtained by an explicit formula. Alternatively, Picard's theorem may be applied as follows. Consider any rectangle $\mathcal{R}$ about the point $(0, 1)$. Let $F(t, y) = t + y$. Then $F_y(t, y) = 1$. Both $F(t, y)$ and $F_y(t, y)$ are continuous functions on the whole (t, y)-plane and hence continuous on $\mathcal{R}$. Therefore, Picard's theorem implies that $y(t) = \lim_{n\to\infty} y_n(t)$ is the unique solution of the initial value problem

$$y' = t + y \quad y(0) = 1.$$

Hence, $y(t) = -1 - t + 2e^t$ is the only solution. ◀

Example 7. Consider the Riccati equation

$$y' = y^2 - t$$

with initial condition $y(0) = 0$. Determine whether Picard's theorem applies on a rectangle containing $(0, 0)$. What conclusions can be made? Determine the first three Picard approximations.

► **Solution.** Here, $F(t, y) = y^2 - t$ and $F_y(t, y) = 2y$ are continuous on all of $\mathbb{R}^2$ and hence on *any* rectangle that contains the origin. Thus, by Picard's Theorem, the initial value problem

$$y' = y^2 - t\,, \quad y(0) = 0$$

has a unique solution on some interval I containing 0. Picard's theorem does not tell us on what interval the solution is defined, only that there is *some* interval. The direction field for $y' = y^2 - t$ with the unique solution through the origin is given below and suggests that the maximal interval $I_{\max}$ on which the solution exists should be of the form $I_{\max} = (a, \infty)$ for some $-\infty \le a < -1$. However, without further analysis of the problem, we have no precise knowledge about the maximal domain of the solution.

Next we show how Picard's method of successive approximations works in this example. To use this method, we rewrite the initial value problem as an integral equation. In this example, $F(t, y) = y^2 - t$, $t_0 = 0$ and $y_0 = 0$. Thus, the corresponding integral equation is

$$y(t) = \int_0^t (y(u)^2 - u)\,\mathrm{d}u. \tag{4}$$

We start with our initial approximation $y_0(t) = 0$, plug it into (4), and obtain our first approximation

$$y_1(t) = \int_0^t (y_0(u)^2 - u)\,\mathrm{d}u = -\int_0^t u\,\mathrm{d}u = -\frac{1}{2}t^2.$$

The second iteration yields

$$y_2(t) = \int_0^t (y_1(u)^2 - u)\,\mathrm{d}u = \int_0^t \left(\frac{1}{4}u^4 - u\right)\,\mathrm{d}u = \frac{1}{4\cdot 5}t^5 - \frac{1}{2}t^2.$$

Since $y_2(0) = 0$ and

$$y_2(t)^2 - t = \frac{1}{4^2\cdot 5^2}t^{10} - \frac{1}{4\cdot 5}t^7 + \frac{1}{4}t^4 - t = \frac{1}{4^2\cdot 5^2}t^{10} - \frac{1}{4\cdot 5}t^7 + y_2'(t) \approx y_2'(t)$$

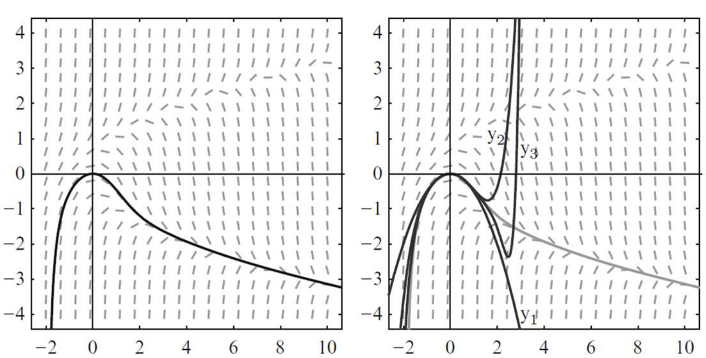
if t is close to 0, it follows that the second iterate $y_2(t)$ is already a "good" approximation of the exact solution for t close to 0. Since

$$y_2(t)^2 = \frac{1}{4^2\cdot 5^2}t^{10} - \frac{1}{4\cdot 5}t^7 + \frac{1}{4}t^4,$$

it follows that

$$\begin{aligned} y_3(t) &= \int_0^t \left(\frac{1}{4^2\cdot 5^2}u^{10} - \frac{1}{4\cdot 5}u^7 + \frac{1}{4}u^4 - u\right)\,\mathrm{d}u \\ &= \frac{1}{11\cdot 4^2\cdot 5^2}t^{11} - \frac{1}{4\cdot 5\cdot 8}t^8 + \frac{1}{4\cdot 5}t^5 - \frac{1}{2}t^2. \end{aligned}$$

According to Picard's theorem, the successive approximations $y_n(t)$ converge toward the exact solution $y(t)$, so we expect that $y_3(t)$ is an even better approximation of $y(t)$ for t close enough to 0. The graphs of y_1, y_2, and y_3 are given below.



Direction Field and Solution for $y' = y^2 - t$, $y(0) = 0$

Solution $y(t)$ and Picard Approximations y_1, y_2, and y_3

Although the Riccati equation looks rather simple in form, its solution cannot be obtained by methods developed in this chapter. In fact, the solution is not expressible in terms of elementary functions but requires special functions such as the Bessel functions. The calculation of the Picard approximations does not reveal a pattern by which we might guess what the nth term might be. This is rather typical. Only in special cases can we expect to find a such a general formula for y_n. ◄

If one only assumes that the function $F(t, y)$ is continuous on the rectangle $\mathcal{R}$, but makes no assumptions about $F_y(t, y)$, then Guiseppe Peano (1858–1932) showed that the initial value problem (1) still has a solution on some interval I with $t_0 \in I \subset [a, b]$. However, in this case, the solutions are not necessarily unique.

Theorem 8 (Peano's Existence Theorem[15]). *Let $F(t, y)$ be a continuous functions of (t, y) on a rectangle*

$$\mathcal{R} = \{(t, y) : a \le t \le b, c \le y \le d\}.$$

If (t_0, y_0) is an interior point of $\mathcal{R}$, then there exists a solution $y(t)$ of

$$y' = F(t, y), \quad y(t_0) = y_0,$$

on some interval $[a', b']$ with $t_0 \in [a', b'] \subset [a, b]$.

Let us reconsider the differential equation introduced in Example 1.

Example 9. Consider the initial value problem

$$y' = 3y^{2/3}, \quad y(t_0) = y_0. \tag{5}$$

Discuss the application of Picard's existence and uniqueness theorem and Peano's existence theorem.

► **Solution.** The function $F(t, y) = 3y^{2/3}$ is continuous for all (t, y), so Peano's existence theorem shows that the initial value problem (5) has a solution for all possible initial values $y(t_0) = y_0$. Moreover, $F_y(t, y) = \frac{2}{y^{1/3}}$ is continuous on any rectangle not containing a point of the form $(t, 0)$. Thus, Picard's existence and uniqueness theorem tells us that the solutions of (5) are unique *as long as the initial value y_0 is nonzero*. Assume that $y_0 \neq 0$. Since the differential equation $y' = 3y^{2/3}$ is separable, we can rewrite it in the differential form

$$\frac{1}{y^{2/3}}\,dy = 3dt,$$

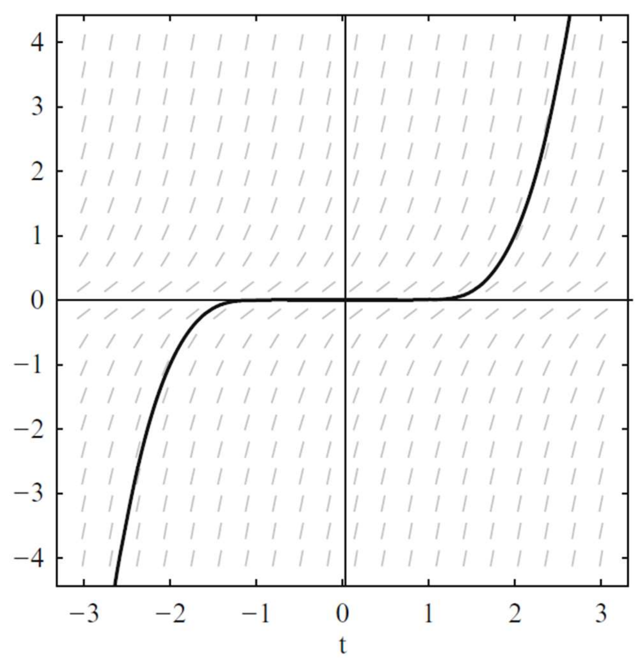
and integrate the differential form to get

$$3y^{1/3} = 3t + c.$$

Thus, the functions $y(t) = (t + c)^3$ for $t \in \mathbb{R}$ are solutions for $y' = 3y^{2/3}$. Clearly, the equilibrium solution $y(t) = 0$ does not satisfy the initial condition. The constant c is determined by $y(t_0) = y_0$. We get $c = y_0^{1/3} - t_0$, and thus $y(t) = (t + y_0^{1/3} - t_0)^3$ is the unique solution of *if* $y_0 \neq 0$. If $y_0 = 0$, then (5) admits more than one solution. Two of them are given in Example 1. However, there are many more. In fact, the following functions are all solutions:

$$y(t) = \begin{cases} (t-\alpha)^3 & \text{if } t < \alpha \\ 0 & \text{if } \alpha \leq t \leq \beta \\ (t-\beta)^3 & \text{if } t > \beta, \end{cases} \qquad (6)$$

where $t_0 \in [\alpha, \beta]$. The graph of one of these functions (where $\alpha = -1$, $\beta = 1$) is depicted below. What changes among the different functions is the length of the straight line segment joining α to β on the t-axis.



Graph of Equation (6)
$\alpha = -1$ and $\beta = 1$

Picard's theorem, Theorem 5, is called a local existence and uniqueness theorem because it guarantees the existence of a unique solution in some subinterval $I \subset [a, b]$. In contrast, the following important variant of Picard's theorem yields a unique solution on the whole interval $[a, b]$.

Theorem 10. *Let $F(t, y)$ be a continuous function of (t, y) that satisfies a Lipschitz condition on a strip $\mathcal{S} = \{(t, y) : a \le t \le b, -\infty < y < \infty\}$. That is, assume that*

$$|F(t, y_1) - F(t, y_2)| \le K|y_1 - y_2|$$

for some constant $K > 0$ and for all (t, y_1) and (t, y_2) in $\mathcal{S}$. If (t_0, y_0) is an interior point of $\mathcal{S}$, then there exists a unique solution of

$$y' = F(t, y), \quad y(t_0) = y_0,$$

on the interval $[a, b]$.

Example 11. Show that the following differential equations have unique solutions on all of $\mathbb{R}$:

1. $y' = e^{\sin ty}, \quad y(0) = 0$
2. $y' = |ty|, \quad y(0) = 0$

► **Solution.** For each differential equation, we will show that Theorem 10 applies on the strip $\mathcal{S} = \{(t, y) : -a \le t \le a, -\infty < y < \infty\}$ and thus guarantees a unique solution on the interval $[-a, a]$. Since a is arbitrary, the solution exists on $\mathbb{R}$.

1. Let $F(t, y) = e^{\sin ty}$. Here we will use the fact that the partial derivative of F with respect to y exists so we can apply the mean value theorem:

$$F(t, y_1) - F(t, y_2) = F_y(t, y_0)(y_1 - y_2), \tag{7}$$

where y_1 and y_2 are real numbers with y_0 between y_1 and y_2. Now focus on the partial derivative $F_y(t, y) = e^{\sin ty} t \cos(ty)$. Since the exponential function is increasing, the largest value of $e^{\sin ty}$ occurs when sin is at its maximum value of 1. Since $|\cos ty| \le 1$ and $|t| \le a$, we have $\left|F_y(t, y)\right| = \left|e^{\sin ty} t \cos ty\right| \le e^1 a = ea$. Now take the absolute value of (7) to get

$$|F(t, y_1) - F(t, y_2)| = \left|e^{\sin ty_1} - e^{\sin ty_2}\right| \le ea\,|y_1 - y_2|.$$

It follows that $F(t, y) = e^{\sin ty}$ satisfies the Lipschitz condition with $K = ea$. Theorem 10 now implies that $y' = e^{\sin ty}$, with $y(0) = 0$ has a unique solution on the interval $[-a, a]$. Since a is arbitrary, a solution exists and is unique on all of $\mathbb{R}$.

2. Let $F(t, y) = |ty|$. Here F does not have a partial derivative at $(0, 0)$. Nevertheless, it satisfies a Lipschitz condition for

$$|F(t, y_1) - F(t, y_2)| = ||ty_1| - |ty_2|| \leq |t| \, |y_1 - y_2| \leq a \, |y_1 - y_2| \, ,$$

since the maximum value of t on $[-a, a]$ is a. It follows that $F(t, y) = |ty|$

satisfies the Lipschitz condition with $K = a$. Theorem 10 now implies that

$$y' = |ty| \quad y(0) = 0,$$

has a unique solution on the interval $[-a, a]$. Since a is arbitrary, a solution exists and is unique on all of $\mathbb{R}$. ◄

Remark 12.

1. When Picard's theorem is applied to the initial value problem $y' = e^{\sin ty}$, $y(0) = 0$, we can only conclude that there is a unique solution in an interval about the origin. Theorem 10 thus tells us much more, namely, that the solution is in fact defined on the entire real line.
2. In the case of $y' = |ty|$, $y(0) = 0$, Picard's theorem does not apply at all since the absolute value function is not differentiable at 0. Nevertheless, Theorem 10 tells us that a unique solution exists on all of $\mathbb{R}$. Now that you know this, can you guess what that unique solution is?