

Discrete structures

2020-2021

الدراسات الاولى الفصل الاول

أستاذ المادة

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Discrete Structures

Lec.1

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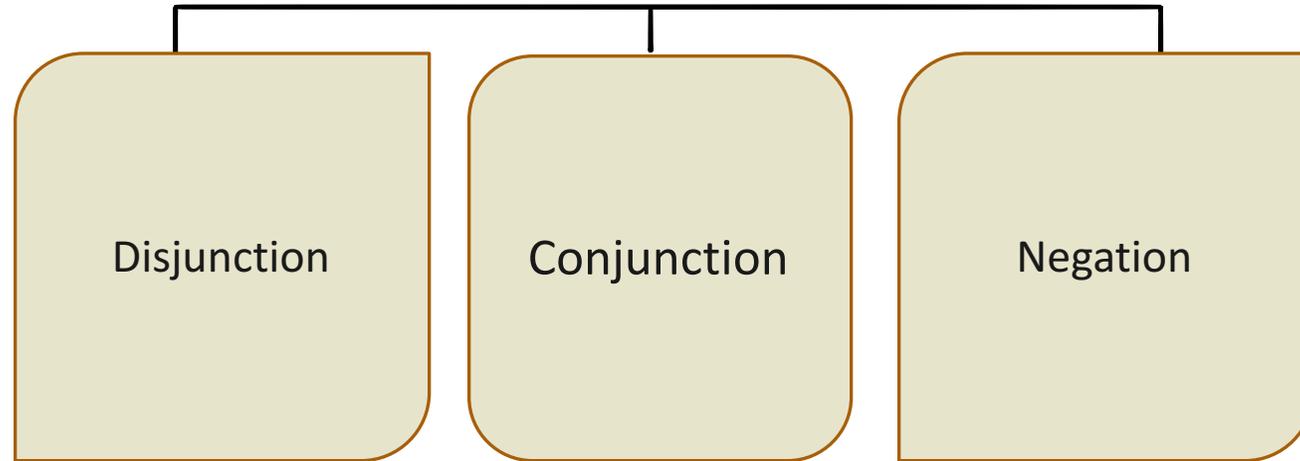
Mathematical logical

1. Statement (proposition)

A statement is a declarative sentence which is either true or false but not both. The statement is also known as proposition.

Proposition	Not proposition
Sun rises in the east.	X is a dog.
Two is less than five.	

2. Logical connectives



Truth Table (Disjunction)

P	Q	$(P \vee Q)$
T	T	T
T	F	T
F	T	T
F	F	F

Truth Table (Conjunction)

P	Q	$(P \wedge Q)$
T	T	T
T	F	F
F	T	F
F	F	F

Truth Table (Negation)

P	$\neg P$
T	F
F	T

Logical connectives

Negation

Negation = **not** = $\neg p$, $\sim p$, p'

Consider the example of a statement.

P: London is the capital of Iraq.

$\neg p$: London is not the capital of Iraq.

As we all know that Baghdad is the capital of Iraq , the truth value for the statements P is false (F) and $\neg p$ is true (T).

Rule: If P is true, then $\neg p$ is false and if P is false, then $\neg p$ is true.

Truth Table (Negation)

P	$\neg P$
T	F
F	T

Logical connectives

Conjunction

The conjunction of two statements P and Q is also a statement denoted by $(P \wedge Q)$. We use the connective **And** for conjunction.

Consider the example where P and Q are two statements.

P : $2+3=5$

Q : 5 is a prime number.

$(P \wedge Q)$: $2 + 3 = 5$ and 5 is a prime number.

Rule: $(P \wedge Q)$ is true if both P and Q are true, otherwise false.

Truth Table (Conjunction)

P	Q	$(P \wedge Q)$
T	T	T
T	F	F
F	T	F
F	F	F

Logical connectives

Disjunction

The disjunction of two statements P and Q is also a statement denoted by $(P \vee Q)$. We use the connective **Or** for disjunction.

Consider the example where P and Q are two statements

P : $2+3$ is not equal to 5

Q : 5 is a prime number

$(P \vee Q)$: $2 + 3$ is not equal to 5 or 5 is a prime number.

Rule: $(P \vee Q)$ is true, if either P or Q is true and it is false when both P and Q are false.

Truth Table (Disjunction)

P	Q	$(P \vee Q)$
T	T	T
T	F	T
F	T	T
F	F	F

Thank
you!

Discrete mathematics

Lec.2

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3.conditional

Let P and Q be any two statements. Then the statement $P \rightarrow Q$ is called a conditional statement. This can be put in any one of the following forms.

(a) If P , then Q (b) P only if Q

(C) P implies Q (d) Q if P

Ex) “I will take you boating on Sunday, if it is not raining”.

Rule: An implication (conditional) $P \rightarrow Q$ is false only when the hypothesis (P) is true and conclusion (Q) is false, otherwise true.

Truth Table (Conditional)

P	Q	$(P \rightarrow Q)$
T	T	T
T	F	F
F	T	T
F	F	T

4. Bi-conditional

Let P and Q be any two statements. Then the statement $P \leftrightarrow Q$ is called a bi-conditional statement. This $P \leftrightarrow Q$ can be put in any one of the following forms.

(a) P if and only if Q

(b) P is necessary and sufficient of Q

(c) P is necessary and sufficient for Q

(d) P implies and implied by Q

Rule: $(P \leftrightarrow Q)$ is true only when both P and Q have identical truth values, otherwise false.

Truth Table (Bi-Conditional)

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$(P \leftrightarrow Q)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

5.converse

Let P and Q be any two statements. The converse statement of the conditional $P \rightarrow Q$ is given as $Q \rightarrow P$.

6.Inverse

Let P and Q be any two statements. The inverse statement of the conditional $(P \rightarrow Q)$ is given as $(\neg p \rightarrow \neg Q)$.

7. Contra positive

Let P and Q be any two statements. The contra positive statement of the conditional $(P \rightarrow Q)$ is given As $(\neg Q \rightarrow \neg P)$.

Truth Table (Contra Positive)

P	Q	$P \rightarrow Q$	$\neg Q$	$\neg P$	$(\neg Q \rightarrow \neg P)$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Rule: From the truth table it is observed that both conditional $(P \rightarrow Q)$ and contra positive $(\neg Q \rightarrow \neg P)$ have same truth values.

8. Tautology

If the truth values of a composite statement are always true.

So, $(P \wedge (P \rightarrow Q)) \rightarrow Q$ is a tautology.

Truth Table

P	Q	$(P \rightarrow Q)$	$P \wedge (P \rightarrow Q)$	$(P \wedge (P \rightarrow Q)) \rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

9. Contradiction

If the truth values of a composite statement are always false.

So, $\neg R \equiv \neg (P \rightarrow (Q \rightarrow (P \wedge Q)))$ is a contradiction.

Truth Table

P	Q	$(P \wedge Q)$	$Q \rightarrow (P \wedge Q)$	$(P \rightarrow (Q \rightarrow (P \wedge Q)))$	$\neg R$
T	T	T	T	T	F
T	F	F	T	T	F
F	T	F	F	T	F
F	F	F	T	T	F

10. Algebra of propositions

Commutative Laws: $P \wedge Q \equiv Q \wedge P$ and
 $P \vee Q \equiv Q \vee P$

Associative Laws: $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$ and
 $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$

Distributive Laws: $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ and
 $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

Idempotent Laws: $P \wedge P \equiv P$ and
 $P \vee P \equiv P$

Absorption Laws: $P \vee (P \wedge Q) \equiv P$ and
 $P \wedge (P \vee Q) \equiv P$

11. Duality law

Two formulae P and P^* are said to be duals of each other if either one can be obtained from the other by interchanging \wedge by \vee and \vee by \wedge . The two connectives \wedge and \vee are called dual to each other.

Consider the formulae

$$P \equiv (P \vee Q) \wedge R \text{ and } P^* \equiv (P \wedge Q) \vee R$$

Which are dual to each other.

12. de morgan's laws

If P and Q be two statements, then

$$(i) \neg(P \wedge Q) \Leftrightarrow (\neg P) \vee (\neg Q) \text{ and}$$

$$(ii) \neg(P \vee Q) \Leftrightarrow (\neg P) \wedge (\neg Q)$$

Example Construct the truth table for $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$.

Solution: The given compound statement is $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$ where P and Q are two atomic statements.

P	Q	$\neg P$	$P \rightarrow Q$	$\neg P \vee Q$	$(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Example *Construct the truth table for $P \rightarrow (Q \leftrightarrow P \wedge Q)$.*

Solution: The given compound statement is $P \rightarrow (Q \leftrightarrow P \wedge Q)$, where P and Q are two atomic statements.

P	Q	$P \wedge Q$	$Q \leftrightarrow P \wedge Q$	$P \rightarrow (Q \leftrightarrow P \wedge Q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	F	F	T	T

Thank
you!

Discrete mathematics

LEC.3

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SETS

Collection of well defined objects is called a set. Well defined means distinct and distinguishable. The objects are called as elements of the set.

Ex: $A = \{a, b, c, d\}$ and $B = \{b, a, d, c\}$ are equal sets.

The symbol \in stands for 'belongs to'. $X \in A$ means x is an element of the set A .

Sets

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graph TD; Sets --> Set_builder_method[Set builder method]; Sets --> Tabular_method[Tabular method];
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Set builder
method

Tabular
method

1. Tabular Method

Expressing the elements of a set within a parenthesis where the elements are separated by commas is known as tabular method, roster method or method of extension.

Consider the example:

$$A = \{1, 3, 5, 7, 9, 11, 13, 15\}$$

2. Set Builder Method

$$S = \{x \mid P(x)\}$$

where $P(x)$ is the property that describes the elements of the set.

$$A = \{x \mid x = 2n + 1; 0 \leq n \leq 7; n \in \mathbb{I}\}$$

$$= \{1, 3, 5, 7, 9, 11, 13, 15\}$$

Types of sets

- Finite set
- Infinite set
- Singleton set
- Pair set
- Empty set
- Set of sets
- Universal set

- 1. Finite set:** A set which contains finite number of elements is known as finite set. Consider the example of finite set as. $A = \{a, b, c, d, e\}$.
- 2. Infinite set:** A set which contains infinite number of elements is known as infinite set. Consider the example of infinite set as
 $N = \{1, 2, 3, 4, \dots\}$
 $I = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- 3. Singleton set:** A set which contains only one element is known as a singleton set. Consider the example. $S = \{9\}$
- 4. Pair set:** A set which contains only two elements is known as a pair set. Consider the examples
 $S = \{e, f\}$
 $S = \{\{a\}, \{1, 3, 5\}\}$

5. Empty set : A set which contains no element is known as empty set. The empty set is also known as void set or null set. denoted by \emptyset .

Consider the examples

(i) $f = \{x : x \neq x\}$

(ii) $f = \{x : x \text{ is a month of the year containing 368 days}\}$

6. Set of sets: A set which contains sets is known as set of sets.

Consider the example $A = \{\{a, b\}, \{1\}, \{1, 2, 3, 4\}, \{u, v\}, \{\text{Book, Pen}\}\}$

7. Universal Set: A set which is superset of all the sets under consideration or particular discussion is known as universal set. denoted by U

lets

$$A = \{a, b, c\}$$

$$B = \{a, e, i, o, u\}$$

$$C = \{p, q, r, s\}$$

So, we can take the universal set U as $\{a, b, c, \dots, z\}$

$$U = \{a, b, c, d, e, \dots, z\}$$

Thank you!

Discrete Mathematics

LEC.4

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CARDINALITY OF A SET

If S be a set, then the number of elements present in the set S is known as cardinality of S and is denoted by $|S|$.

$$A = \{2, 4, 8, 16, 32, 64, 128, 256\}$$

$$|A| = 8$$

CARDINALITY OF A SET

Equivalent Sets

Equivalent Sets: Two sets A and B are said to be equivalent if they contains equal number of elements.

In other words A and B are said to be equivalent if they have same cardinality, i.e. $|A| = |B|$.

denoted as $A \approx B$.

SUBSET AND SUPERSSET

Set A is said to be a subset of B or set B is said to be the superset of A if each element of A is also an element of the set B.

We write $A \subseteq B$.

i.e., $A \subseteq B \leftrightarrow \{x \in A \rightarrow x \in B; \forall x \in A\}$

Example

- (i) Let $A = \{1, 2, 3, 4, 5, 6\}$
 $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
So $A \subseteq B$.

(ii) Let $A = \{a, b, c\}$
 $B = \{b, c, a\}$
so, $A \subseteq B$ and $B \subseteq A$.

(iii) Let $A = \{\}$ and $B = \{1, 2, 3\}$
So, $A \subseteq B$.

SUBSET AND SUPERSET

1. Equal Sets

2. Proper Subset

Equal Sets: Two sets A and B are said to be equal if and only if every element of A is in B and every element of B is in A

i.e. $A \subseteq B$ and $B \subseteq A$. Mathematically

$$A = B \leftrightarrow \{ A \subseteq B \text{ and } B \subseteq A \}$$

i.e.,

$$A = B \leftrightarrow \{ x \in A \leftrightarrow x \in B \}$$

Consider the example: Let $A = \{x, y, z, p, q, r\}$

$$B = \{p, q, r, x, y, z\}$$

So, $B \subseteq A$ and $A \subseteq B$. Thus $A = B$.

SUBSET AND SUPERSET

1. Equal Sets

2. Proper Subset

Proper Subset: Set A is said to be a proper subset of B if each element of A is also an element of B and set B has at least one element which is not an element of set A.

We write $A \subset B$

$$A \subset B \leftrightarrow \{x \in A \rightarrow x \in B \text{ and for at least one } y \in B \rightarrow y \notin A\}.$$

Consider an example

Let

$$A = \{a, b, c, d\}$$

$$B = \{a, b, c, d, e, f, g\}$$

Here for $x \in A$ we have $x \in B$ and $y = e \in B$ such that $y = e \notin A$. Thus $A \subset B$.

Rule: 1. Every set is a subset of itself, *i.e.* $A \subseteq A$.

2. Empty set is a subset of every set, *i.e.* $\phi \subseteq A$.

Thank you!



Discrete mathematics

LEC.5

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POWER SET

If A be a set, then the set of all subsets of A is known as power set of A and is denoted as $P(A)$

Mathematically, $P(A) = \{X : X \subseteq A\}$

Consider the example:

Let

$$A = \{a\}$$

⇒

$$P(A) = \{\Phi, \{a\}\}$$

Let

$$A = \{a, b\}$$

⇒

$$P(A) = \{\{a\}, \{b\}, \{a, b\}, \Phi\}$$

Let

$$A = \{a, b, c\}$$

⇒

$$P(A) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}, \Phi\}$$

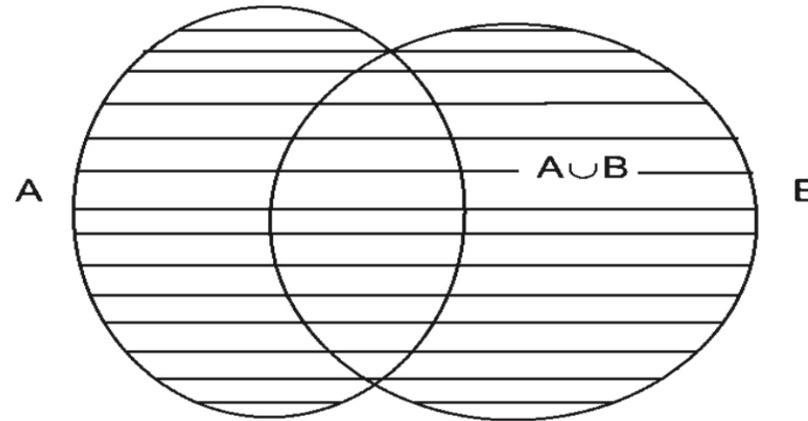
$$|A| = n \Rightarrow |P(A)| = 2^n.$$

Power set

1. Union
2. Intersection
3. Difference
4. Symmetric Difference
5. Complement of a set
6. Theorem

1. Union: If A and B be two sets, then the union ($A \cup B$) is defined as a set of all those elements which are either in A or in B or in both. $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Venn diagram



Consider the example:

Let

$$A = \{a, b, c, d, e\}$$

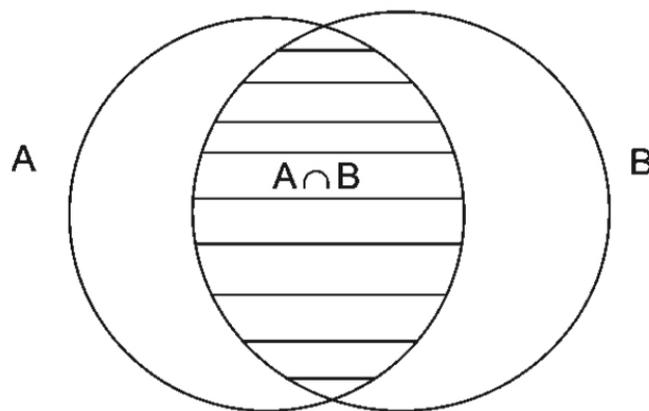
$$B = \{a, e, i, o, u\}$$

Therefore,

$$(A \cup B) = \{a, b, c, d, e, i, o, u\}$$

2. Intersection: If A and B be two sets, then the intersection ($A \cap B$) is defined as a set of all those elements which are common to both the sets. $(A \cap B) = \{x : x \in A \text{ and } x \in B\}$

Venn diagram



Consider the example:

Let

$$A = \{a, b, c, d, e\}$$

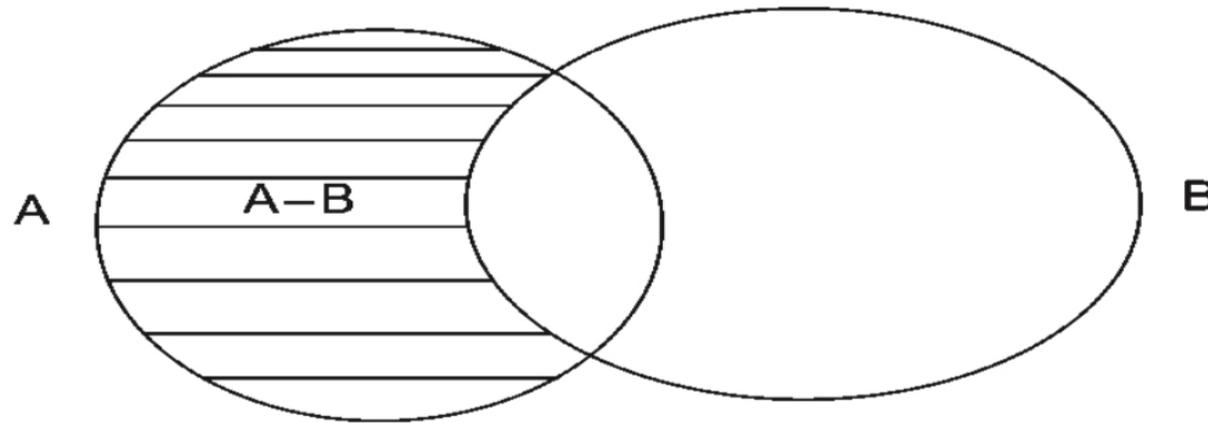
$$B = \{a, e, i, o, u\}$$

Therefore,

$$(A \cap B) = \{a, e\}$$

3.Difference: If A and B be two sets, then the difference (A - B) is defined as a set of all those elements of A which are not in B.

$$(A - B) = \{x \mid x \in A \text{ and } x \notin B\}$$



Consider the example:

Let $A = \{a, b, c, d, e, f\}$

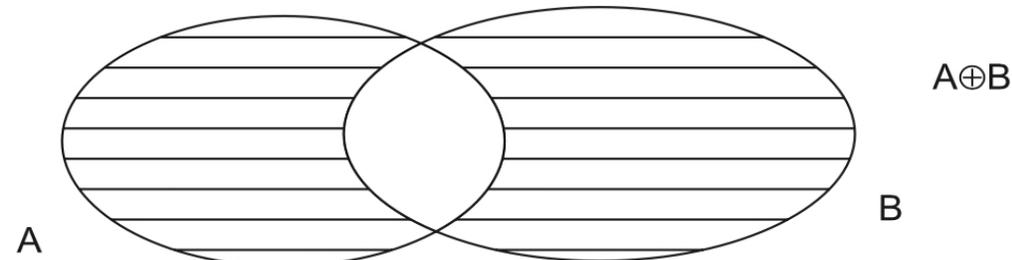
$B = \{a, c, i, o, u, k\}$

Therefore $(A - B) = \{b, d, e, f\}$

Symmetric Difference: If A and B be two sets, then the symmetric difference $(A \Delta B)$ or $(A \oplus B)$ is defined as a set of all those elements which are either in A or in B but not in both.

$$(A \oplus B) = (A - B) \cup (B - A)$$

Venn diagram



Consider the example:

Let

$$A = \{a, b, c, k, p, q, r, s\}$$

$$B = \{b, k, q, m, n, o, t\}$$

So,

$$(A - B) = \{a, c, p, r, s\}$$

and

$$(B - A) = \{m, n, o, t\}$$

Therefore,

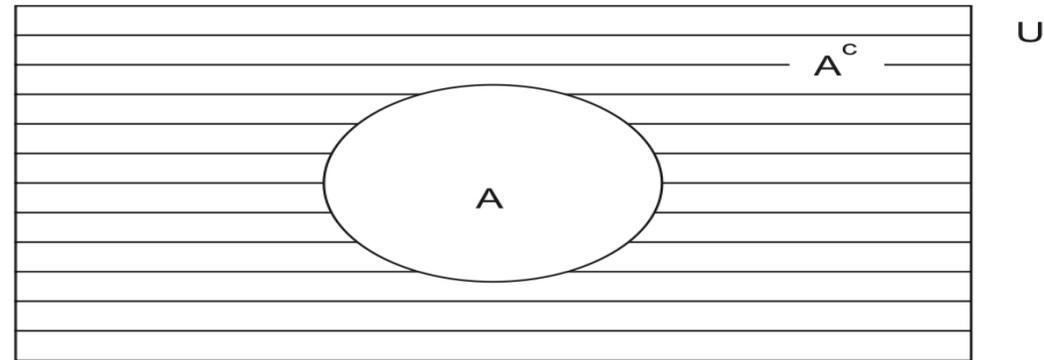
$$(A \oplus B) = (A - B) \cup (B - A)$$

$$= \{a, c, p, r, s, m, n, o, t\}$$

5. Complement of a Set: If A be a set , then the complement of A is given as A^c , A' or \bar{A} and is defined as a set of all those elements of the universal set U which are not in A .

$$A^c = \{x \mid x \in U \text{ and } x \notin A\}$$

Venn diagram



Consider the example:

Let

$$A = \{b, c, k, d, i, p, q, r, s, t\}$$

So, we can take the universal set $U = \{a, b, c, \dots, x, y, z\}$.

Therefore,

$$\begin{aligned} A^c &= U - A \\ &= \{a, e, f, g, h, j, l, m, n, o, u, v, w, x, y, z\} \end{aligned}$$

6.Theorem: Let A, B and C be subsets of the universal set U.
Then the following important laws hold.

(a) Commutative laws:

$$(A \cup B) = (B \cup A) \quad ; \quad (A \cap B) = (B \cap A)$$

(b) Associative laws:

$$A \cup (B \cup C) = (A \cup B) \cup C \quad ; \quad A \cap (B \cap C) = (A \cap B) \cap C$$

(c) Idempotent laws:

$$(A \cup A) = A \quad ; \quad (A \cap A) = A$$

(d) Identity laws:

$$(A \cup \phi) = A \quad ; \quad (A \cap U) = A$$

(e) Bound laws:

$$(A \cup U) = U \quad ; \quad (A \cap \phi) = \phi$$

(f) Absorption laws:

$$A \cup (A \cap B) = A \quad ; \quad A \cap (A \cup B) = A$$

(g) Complement laws:

$$(A \cup A^c) = U \quad ; \quad (A \cap A^c) = \phi$$

(h) Involution law:

$$(A^c)^c = A$$

(i) Distributive laws :

$$(i) \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$(ii) \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Thank you!



Discrete mathematics

LEC.6

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Binary Relation

Let A and B be two sets. Then any subset R of the Cartesian product $(A \times B)$ is a relation (binary relation) from the set A to the set B . Symbolically $R \subseteq (A \times B)$.

$$R = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

If $(x, y) \in R$, then we write $x R y$ and say that x is related to y . If $(x, y) \notin R$, then we write $x \not R y$ and say that x is not related to y . If $A = B$, then R is a relation (binary relation) on A .

Consider the example $A = \{1, 2, 3, 4, 5\}$ and $B = \{5, 6, 7, 8, 9\}$ and let the relation R from the set A to the set B as

$$R = \{(x, y) \mid x \in A \text{ and } y = 2x + 3 \in B\}$$

i.e.,

$$R = \{(1, 5), (2, 7), (3, 9), (4, 11), (5, 13)\}$$

i.e.,

$$R \subseteq A \times B$$

1.Domain of a Relation: Let R be a relation from the set A to the set B. Then the set of all first constituents of the ordered pairs present in the relation R is known as domain of R . Denoted by dom. R or D(R). Mathematically,

$$D(R) = \{x \mid (x, y) \in R, \text{ for } x \in A\}$$

$$D(R) \subseteq A$$

- **Range of a Relation:** Let R be a relation from the set A to the set B. Then the set of all second constituents of the ordered pairs present in the relation R is known as range of R. Denoted by rng.R or R(R).Mathematically

Mathematically,

$$R(R) = \{y \mid (x, y) \in R, \text{ for } y \in B\}$$

i.e.,

$$R(R) \subseteq B$$

Consider the example: Let $A = \{a, b, c, d\}$ and $B = \{5, 6, 7\}$. Let us define a relation R from the set A to the set B as below.

$$R = \{(a, 5), (a, 6), (c, 6), (d, 6)\}$$

So,

$$D(R) = \{a, c, d\} \text{ and } R(R) = \{5, 6\}$$

2. INVERSE RELATION: Let R be a relation from the set A to the set B . Then the inverse of the relation R is a relation from the set B to the set A . It is denoted by R^{-1} and is defined as

$$R^{-1} = \{(y, x) \mid (x, y) \in R\}$$

Consider the example: Let $A = \{1, 2, 3, 4, 5\}$

and

$$B = \{4, 9, 16, 17, 25\}$$

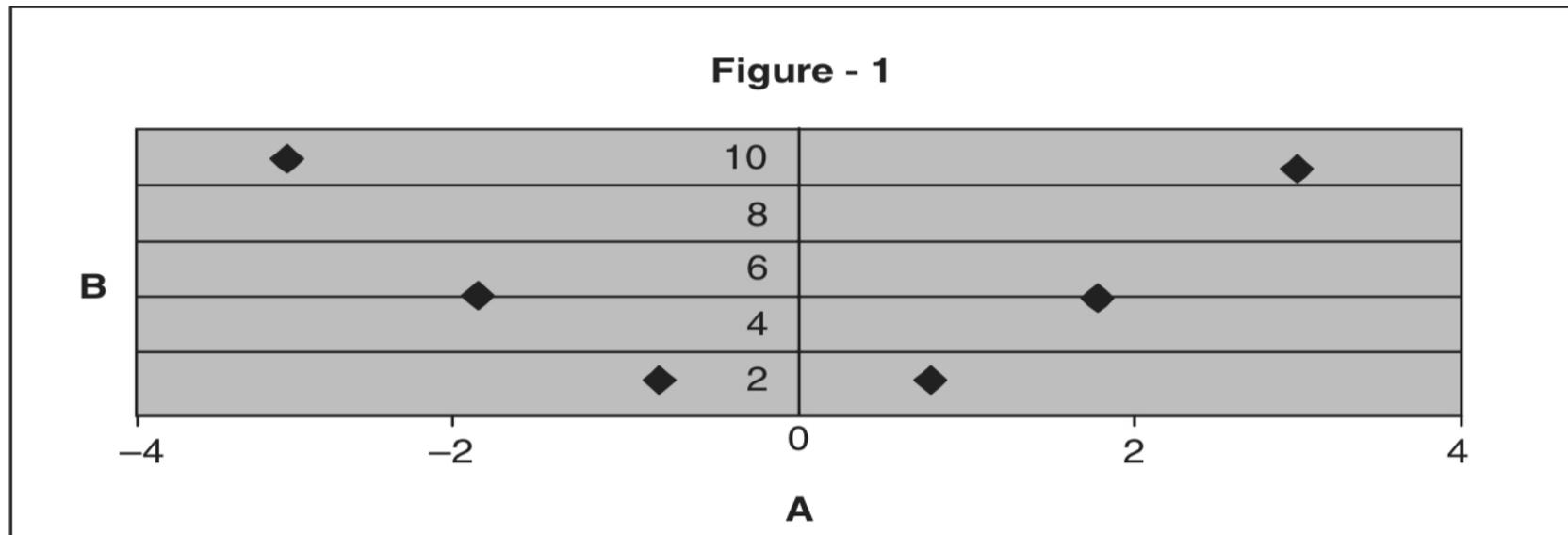
Let us consider the relation R from the set A to the set B as $R = \{(2, 4), (3, 9), (4, 16), (3, 17)\}$

Therefore, $R^{-1} = \{(4, 2), (9, 3), (16, 4), (17, 3)\}$.

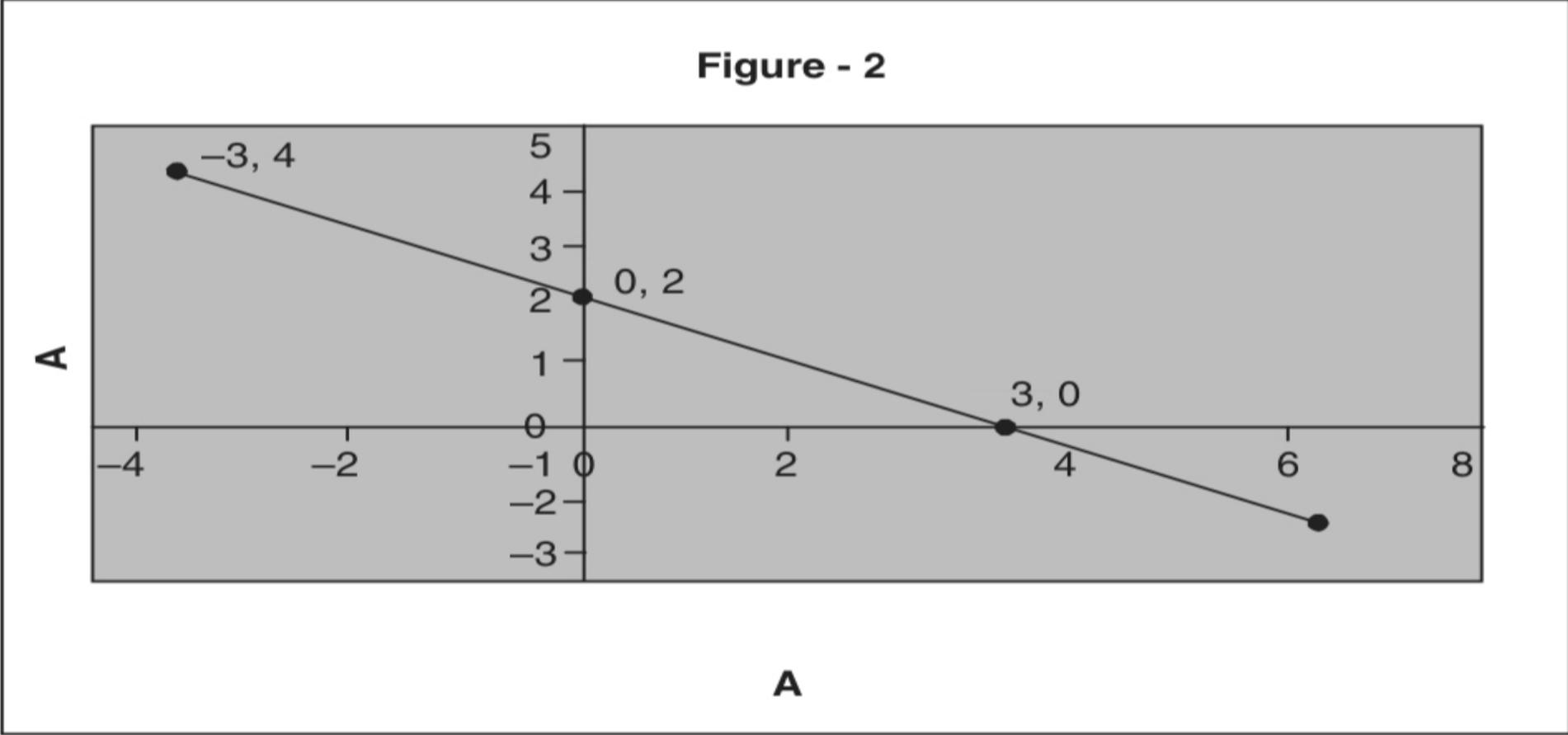
3. GRAPH OF RELATION: Let R be a relation from the set A to the set B ; that is R is a subset of $(A \times B)$. Since $(A \times B)$ can be represented by the set of points on the coordinate diagram of $(A \times B)$,

Let $A = \{-3, -2, -1, 1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $x R y$ such that $y = x^2$. Thus we have

$$R = \{(-1, 1), (1, 1), (-2, 4), (2, 4), (-3, 9), (3, 9)\}$$



Consider another example: Let $A = \{x \mid x \text{ is a real number}\}$ and $x R y$ such that $2x + 3y \leq 6$. Thus, we have $R = \{(x, y) \mid 2x + 3y \leq 6 \text{ and } x, y \in A\}$.



4.KINDS OF RELATION

- ❖ One-One
- ❖ One-Many
- ❖ Many-One
- ❖ Many-Many

5. ARROW DIAGRAM: We use arrow diagrams to represent relations.

Consider the example: Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8\}$. Let us define the relations from the set A to the set B as

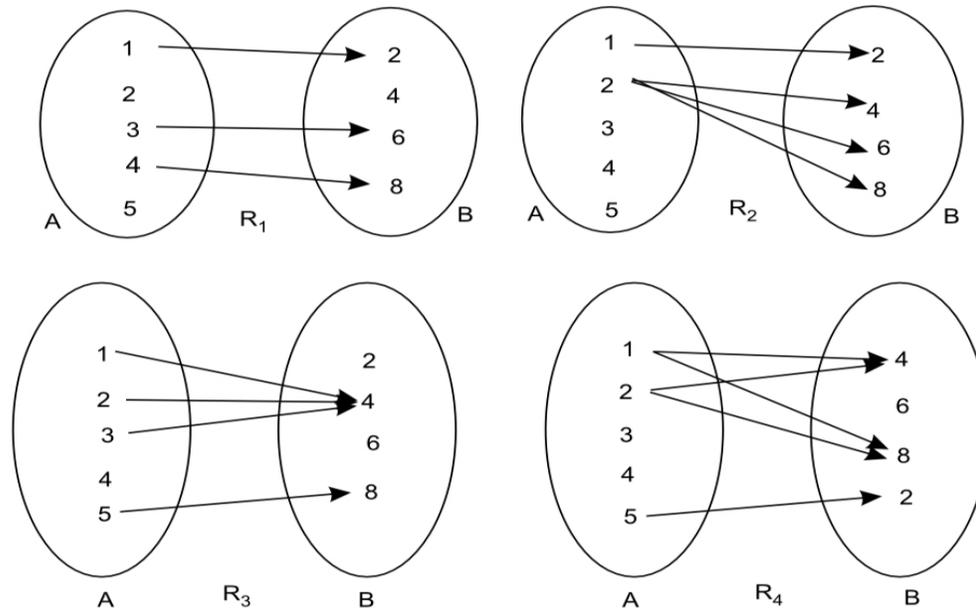
$$R_1 = \{(1, 2), (3, 6), (4, 8)\}$$

$$R_2 = \{(2, 4), (2, 6), (2, 8), (1, 2)\}$$

$$R_3 = \{(1, 4), (2, 4), (3, 4), (5, 8)\}$$

and

$$R_4 = \{(1, 4), (2, 4), (1, 8), (2, 8), (5, 2)\}$$



The arrow diagrams for the above relations are given above. From the above diagrams it is clear that R_1, R_2, R_3 and R_4 are One-One, One-Many, Many-One and Many-Many relations respectively.

Thank
you

Discrete mathematics

LEC.7

أ.م. عهد سعيد الحسني

6. VOID RELATION

Let R be a relation on a set A ; that is R is a subset of $(A \times A)$. Then the relation R is said to be an identity relation if $(x, x) \in R$. Generally denoted by I_A . Mathematically,

$$I_A = \{(x, x) \mid x \in A\}$$

7.IDENTITY RELATION

A relation R from a set A to a set B is said to be a void relation or empty relation if $R = \phi$.

Consider the example: Let $A = \{3, 5, 7\}$; $B = \{2, 4, 8\}$; $R \subseteq A \times B$ and $x R y \mid x$ divides y ; $x \in A$, $y \in B$. Hence, we observe that $R = \phi \subseteq A \times B$ is a void relation from the set A to the set B .

Consider the example: Let $A = \{a, b, c\}$ and I_A be a relation on A such that $I_A = \{(a, a), (b, b), (c, c)\}$. This is an identity relation on A .

8. UNIVERSAL RELATION: relation R from a set A to a set B is said to be an universal relation if R is equal to $(A \times B)$. That is $R = (A \times B)$.

Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Therefore the universal relation R from the set A to the set B is given as

$$R = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}.$$

9.RELATION MATRIX (MATRIX OF THE RELATION)

Let $A = \{a_1, a_2, a_3, \dots, a_i, \dots, a_k\}$
and $B = \{b_1, b_2, b_3, \dots, b_j, \dots, b_l\}$
be two finite sets and R be a relation from the set A to the set B . Then the matrix of the relation R , *i.e.*, $M(R)$ is defined as

$$M(R) = [m_{ij}] \text{ of order } (k \times l)$$

where

$$m_{ij} = \begin{cases} 1; & \text{if } a_i R b_j \\ 0; & \text{if } a_i \not R b_j \end{cases}$$

$$M(R) = \begin{matrix} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

10 COMPOSITION OF RELATIONS

Let R_1 be a relation from the set A to the set B and R_2 be a relation from the set B to the set C . That is R_1 is a subset of $(A \times B)$ and R_2 is a subset of $(B \times C)$. Then the composition of R_1 and R_2 is given by R_1R_2 and is defined by

$$R_1R_2 = \{(x, z) \in (A \times C) \mid \text{for some } y \in B, (x, y) \in R_1 \text{ and } (y, z) \in R_2\}$$

Consider the example: Let $A = \{1, 2, 4, 5, 7\}$;

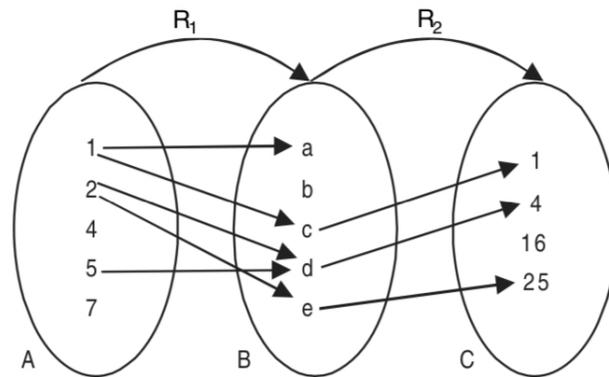
$$B = \{a, b, c, d, e\}$$

and

$$C = \{1, 4, 16, 25\}$$

Consider the relations $R_1: A \rightarrow B$ and $R_2: B \rightarrow C$ as

$R_1 = \{(1, a), (1, c), (2, d), (2, e), (5, d)\}$ and $R_2 = \{(c, 1), (d, 4), (e, 25)\}$. The arrow diagram is given as



So,

$$R_1R_2 = \{(1, 1), (2, 4), (2, 25), (5, 4)\}$$

11. TYPES OF RELATIONS

This section discusses a number of different important types of relations on a set A that are important for the study of finite state systems.

1. Reflexive Relations

A relation R defined on a set A is said to be reflexive if $(x, x) \in R$ for every element $x \in A$.

i.e., $x R x \quad \forall x \in A$

Consider the following relations on the set $A = \{1, 3, 5, 7\}$

$$R_1 = \{(1, 1), (1, 3), (1, 5), (5, 5), (5, 7)\}$$

$$R_2 = \{(1, 3), (1, 5), (5, 7), (3, 7)\}$$

$$R_3 = \{(1, 1), (1, 3), (3, 3), (5, 5), (5, 7), (1, 7), (7, 7)\}$$

From the above relations it is clear that R_3 is a reflexive relation. R_1 is not a reflexive relation as $(3, 3) \notin R_1$ and $(7, 7) \notin R_1$. Similarly, R_2 is also not reflexive.

2. Symmetric Relations

A relation R defined on a set A is said to be symmetric if $(x, y) \in R$ then $(y, x) \in R$.

i.e., $x R y \Rightarrow y R x$.

Consider the following relations on the set $A = \{1, 3, 5, 7\}$

$$R_1 = \{(1, 1), (1, 3), (3, 5), (3, 1), (5, 3), (5, 5)\}$$

$$R_2 = \{(1, 1), (1, 3), (3, 1), (3, 5), (5, 3), (5, 7), (7, 7)\}$$

From the above relations it is clear that R_1 is a symmetric relation, but R_2 is not a symmetric relation as $(5, 7) \in R_2 \Rightarrow (7, 5) \notin R_2$.

3. Transitive Relations:

A relation R defined on a set A is said to be transitive if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

i.e., $x R y$ and $y R z \Rightarrow x R z$

Consider the following relations on the set $A = \{1, 3, 5, 7\}$.

$$R_1 = \{(1, 1), (1, 3), (1, 5), (1, 7), (3, 3), (3, 5), (3, 7), (5, 3), (5, 5), (5, 7)\}$$

$$R_2 = \{(1, 1), (1, 3), (3, 5), (5, 5), (7, 7)\}$$

From the above relations it is clear that R_1 is a transitive relation. The relation R_2 is not transitive as $(1, 3) \in R_2, (3, 5) \in R_2 \Rightarrow (1, 5) \notin R_2$.

12. EQUIVALENCE RELATION: A relation R defined on a set A is said to be an equivalence relation in A if and only if R is reflexive, symmetric and transitive.

- **Theorem:** If R be an equivalence relation defined in a set A , then R^{-1} is also an equivalence relation in the set A .

Example Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ defined by the rule $(x, y) \in R$ if $x + y \leq 6$. Find the followings.

(a) List the elements of R

(b) List the elements of R^{-1}

(c) Domain of R

(d) Range of R

(e) Range of R^{-1}

(f) Domain of R^{-1}

Check that domain of R is equal to range of R^{-1} and range of R is equal to domain of R^{-1} .

Solution: Let $A = \{1, 2, 3, 4, 5\}$ and $R = \{(x, y) \in R \mid x + y \leq 6 ; x, y \in A\}$

(a) $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (5, 1)\}$

(b) $R^{-1} = \{(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (1, 2), (2, 2), (3, 2), (4, 2), (1, 3), (2, 3), (3, 3), (1, 4), (2, 4), (1, 5)\}$

(c) Domain of R *i.e.*, $D(R) = \{1, 2, 3, 4, 5\}$

(d) Range of R *i.e.*, $R(R) = \{1, 2, 3, 4, 5\}$

(e) Range of R^{-1} *i.e.*, $R(R^{-1}) = \{1, 2, 3, 4, 5\}$

(f) Domain of R^{-1} *i.e.*, $D(R^{-1}) = \{1, 2, 3, 4, 5\}$

From this it is clear that $D(R) = R(R^{-1})$ and $R(R) = D(R^{-1})$.

Thank
you

Discrete mathematics

LEC.8

أ.م. عهد سعيد الحسني

Function

Let A and B be two non-empty sets. A relation f from the set A to the set B is said to be a function if it satisfies the following two conditions.

(i) $D(f) = A$ and

(ii) if $(x_1, y_1) \in f$ and $(x_2, y_2) \in f$, then $y_1 = y_2$.

In other words a relation f from the set A to the set B is said to be a function if for each element x in A there exists unique element y in B . A function from A to B is sometimes denoted as $f: A \rightarrow B$.

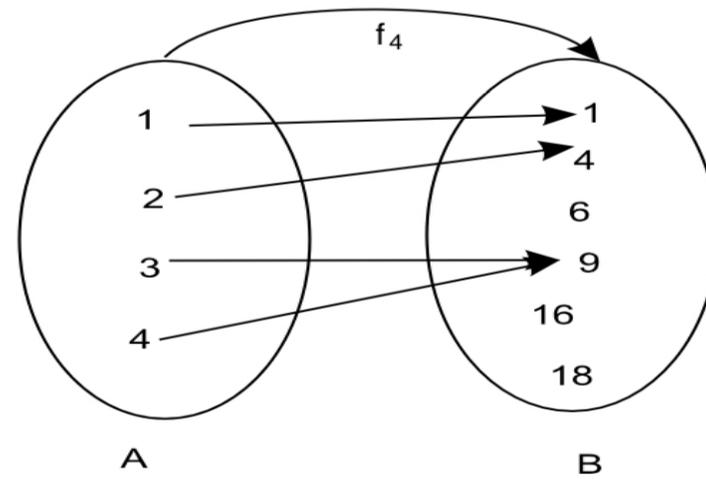
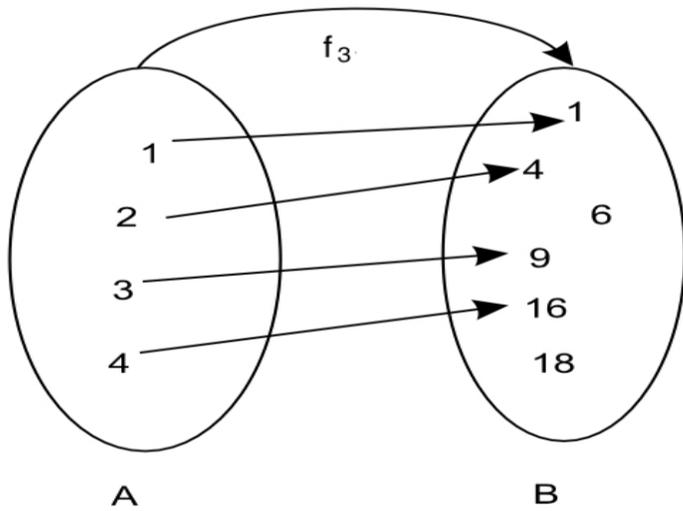
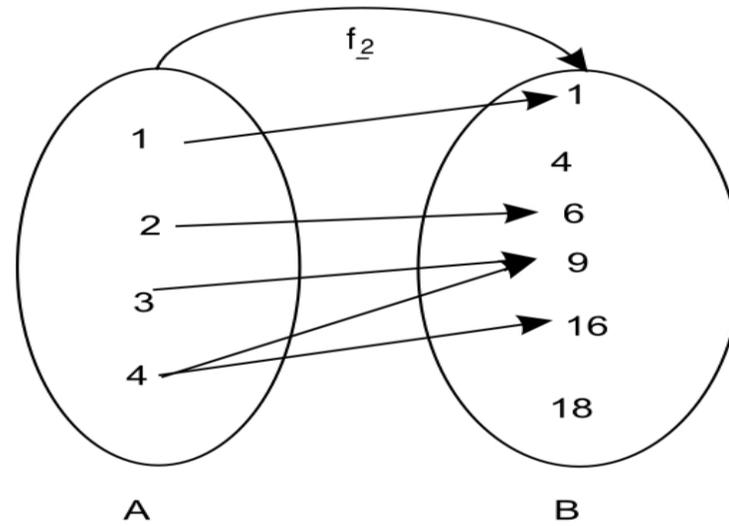
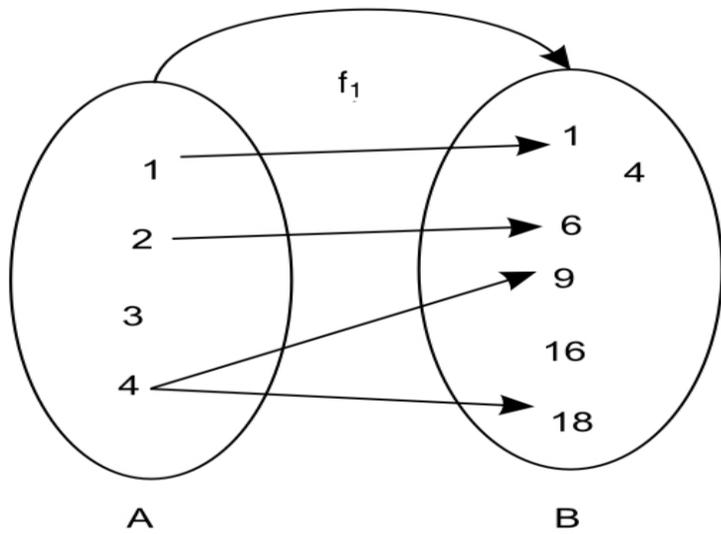
Consider the following relations from the set $A = \{1, 2, 3, 4\}$ to the set $B = \{1, 4, 6, 9, 16, 18\}$.

$$f_1 = \{(1, 1), (2, 6), (4, 9), (4, 18)\}$$

$$f_2 = \{(1, 1), (2, 6), (3, 9), (4, 9), (4, 16)\}$$

$$f_3 = \{(1, 1), (2, 4), (3, 9), (4, 16)\}$$

$$f_4 = \{(1, 1), (2, 4), (3, 9), (4, 9)\}$$



1. a) Domain and Co-domain of a Function

Suppose that f be a function from the set A to the set B . The set A is called the domain of the function f where as the set B is called the co-domain of the function f .

Consider the function f from the set $A = \{a, b, c, d\}$ to the set $B = \{1, 2, 3, 4\}$ as

$$f = \{(a, 1), (b, 2), (c, 2), (d, 4)\}$$

Therefore, domain of $f = \{a, b, c, d\}$ and co-domain of $f = \{1, 2, 3, 4\}$. *i.e.*, $D(f) = \{a, b, c, d\}$ and Co-domain $f = \{1, 2, 3, 4\}$.

b) Range of a Function

Let f be a function from the set A to the set B . The element $y \in B$ which the function f associates to an element $x \in A$ is called the image of x or the value of the function f for x . From the definition of function it is clear that each element of A has a unique image on B . Therefore the range of a function $f: A \rightarrow B$ is defined as the image of its domain A . Mathematically,

$$R(f) \text{ or } \text{rng}(f) = \{y = f(x) : x \in A\}$$

It is clear that $R(f) \subseteq B$.

Consider the function f from $A = \{a, b, c\}$ to $B = \{1, 3, 5, 7, 9\}$ as $f = \{(a, 3), (b, 5), (c, 5)\}$. Therefore, $R(f) = \{3, 5\}$.

2.EQUALITY OF FUNCTIONS

If f and g are functions from A to B , then they are said to be equal *i.e.*, $f = g$ if the following conditions hold.

(a) $D(f) = D(g)$

(b) $R(f) = R(g)$

(c) $f(x) = g(x) \quad \forall x \in A$.

Consider $f(x) = 3x^2 + 6: \mathbb{R} \rightarrow \mathbb{R}$ and $g(x) = 3x^2 + 6: \mathbb{C} \rightarrow \mathbb{C}$, where \mathbb{R} and \mathbb{C} are the set of real numbers and complex numbers respectively. Now it is clear that $D(f) \neq D(g)$. Therefore $f(x) \neq g(x)$.

Let us consider $A = \{1, 2, 3, 4\}$; $B = \{1, 2, 7, 8, 17, 18, 31, 32\}$ and the function $f: A \rightarrow B$ defined by $f = \{(1, 2), (2, 8), (3, 18), (4, 32)\}$. Consider another function $g: A \rightarrow \mathbb{N}$ defined by $g(x) = 2x^2$. Now it is clear that $D(f) = \{1, 2, 3, 4\}$ with $f(1) = 2, f(2) = 8, f(3) = 18, f(4) = 32$.

Similarly $D(g) = A = \{1, 2, 3, 4\}$ with $g(1) = 2, g(2) = 8, g(3) = 18, g(4) = 32$. Therefore, we get

(a) $D(f) = \{1, 2, 3, 4\} = D(g)$

(b) $R(f) = \{2, 8, 18, 32\} = R(g)$ and

(c) $f(x) = g(x) \quad \forall x \in \{1, 2, 3, 4\}$.

This implies f and g are equal. *i.e.*, $f = g$.

3.TYPES OF FUNCTION

1. One-One Function:

A function $f: A \rightarrow B$ is said to be an one-one function or injective if $f(x_1) = f(x_2)$, then $x_1 = x_2$ for $x_1, x_2 \in A$. *i.e.*, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

Consider a function $f: \mathbb{Q} \rightarrow \mathbb{Q}$ defined by $f(x) = 4x + 3; x \in \mathbb{Q}$.

Suppose that $f(x_1) = f(x_2)$ for $x_1, x_2 \in \mathbb{Q}$.

$$\Rightarrow 4x_1 + 3 = 4x_2 + 3$$

$$\Rightarrow 4x_1 = 4x_2$$

$$\Rightarrow x_1 = x_2$$

i.e., $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. So, $f(x) = (4x + 3): \mathbb{Q} \rightarrow \mathbb{Q}$ is One-One.

Consider another function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2; x \in \mathbb{R}$. Suppose that $f(x_1) = f(x_2)$

$$\Rightarrow x_1^2 = x_2^2$$

$$\Rightarrow x_1 = \pm x_2$$

$$\Rightarrow x_1 \neq x_2$$

i.e., $f(x_1) = f(x_2) \Rightarrow x_1 \neq x_2$. It is also clear that $f(1) = 1 = f(-1)$; but $1 \neq -1$. Therefore $f(x) = x^2: \mathbb{R} \rightarrow \mathbb{R}; x \in \mathbb{R}$ is not One-One.

3.TYPES OF FUNCTION

2 Onto Function

A function $f: A \rightarrow B$ is said to be an onto function or surjective if for every $y \in B$ there exists at least one element $x \in A$ such that $f(x) = y$.

In other words a function $f: A \rightarrow B$ is said to be an Onto function if $f(A) = B$. *i.e.*, range of f is equal to co-domain of f .

Consider a function $f: \mathbb{Q} \rightarrow \mathbb{Q}$ defined by $f(x) = 4x + 3, x \in \mathbb{Q}$. Then for every $y \in$ co-domain set \mathbb{Q} there exists $x = \frac{y-3}{4}$ belongs to domain set \mathbb{Q} . Therefore, $f(x) = 4x + 3$ is an Onto function.

3 One-One Onto Function

A function $f: A \rightarrow B$ is said to be an one-one onto function or bijective if f is both one-one and onto function.

Consider a function $f: \mathbb{Q} \rightarrow \mathbb{Q}$ defined by $f(x) = 4x + 3, x \in \mathbb{Q}$. From the above discussions it is clear that $f(x) = 4x + 3, x \in \mathbb{Q}$ is an one-one onto function.

4 Into Function

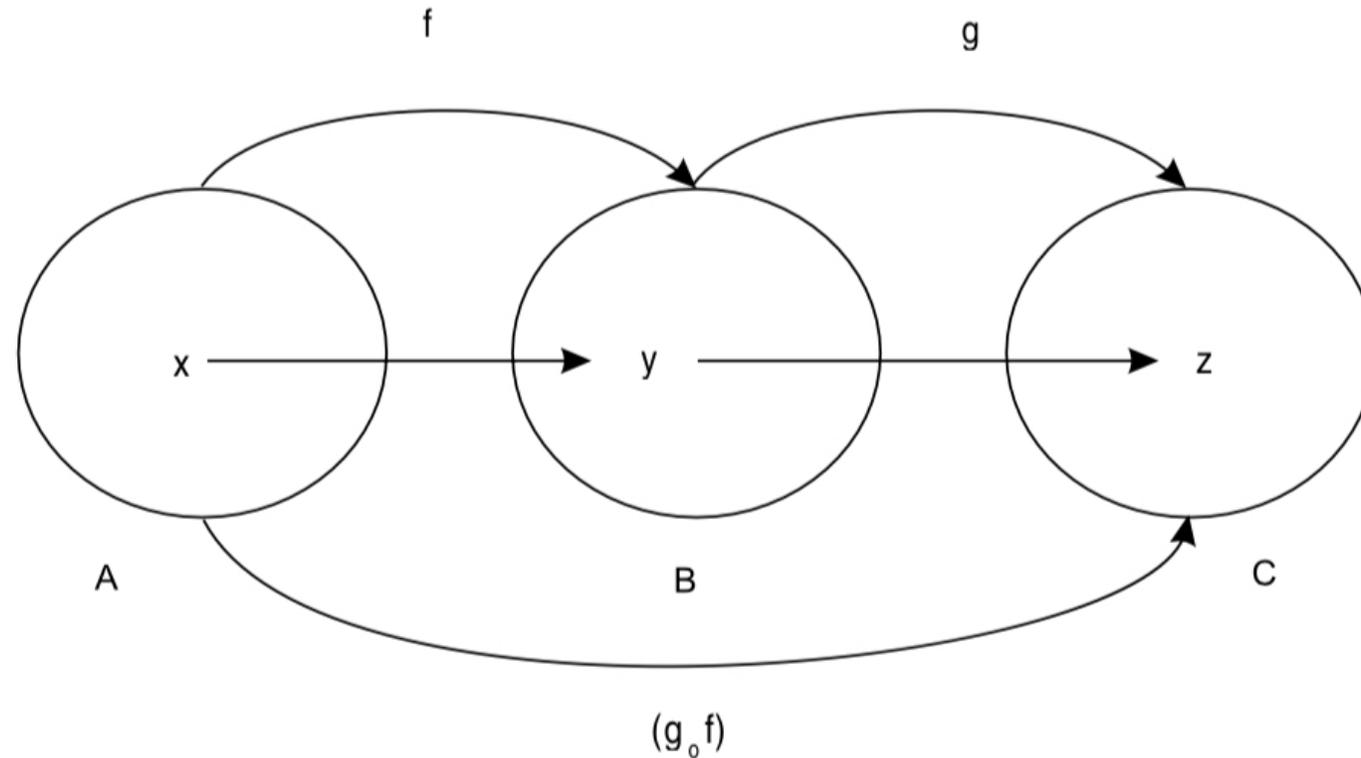
A function $f: A \rightarrow B$ is said to be an into function if for at least one $y \in B$ there exists no element $x \in A$ such that $f(x) = y$. In other words

A function $f: A \rightarrow B$ is said to be an into function if $f(A) \subset B$, *i.e.*, range of f is a proper subset of co-domain of f .

Consider a function $f: \mathbb{Q} \rightarrow \mathbb{R}$ defined by $f(x) = x + 4, x \in \mathbb{Q}$. Hence, it is clear that for $y = \sqrt{3} \in \mathbb{R}$ there exists no element $x = \sqrt{3} - 4 \in \mathbb{Q}$. Therefore, $f(x) = x + 4 : \mathbb{Q} \rightarrow \mathbb{R}$ is an into function.

4 COMPOSITION OF FUNCTIONS

Let f be a function from the set A to the set B and g be a function from the set B to the set C . Then the composition of the functions f and g is given as $(g \circ f)$ or gf . This is a function from the set A to the set C . It may also be noted that domain of g is equal to co-domain of f .



As f is a function from the set A to the set B , then for every $x \in A$ there exists unique $y \in B$ such that $y = f(x)$. Similarly g is a function from the set B to the set C , then for every $y \in B$ there exists unique $z \in C$ such that $z = g(y)$. Again $(g \circ f)$ is a function from the set A to the set C , so we get

$$(g \circ f)(x) = z \text{ for all } x \in A.$$

i.e.,
$$(g \circ f)(x) = g(y)$$

i.e.,
$$(g \circ f)(x) = g(f(x))$$

Consider two functions $f(x) = 2x + 5$ and $g(x) = 3x$.

Therefore $(g \circ f)(x) = g(f(x))$

$$= g(2x + 5)$$

$$= 3(2x + 5)$$

i.e.,
$$(g \circ f)(x) = 6x + 15$$

Similarly,
$$(f \circ g)(x) = f(g(x))$$

$$= f(3x)$$

$$= 2(3x) + 5$$

i.e.,
$$(f \circ g)(x) = 6x + 5$$

4.composition of function

➤ Theorem

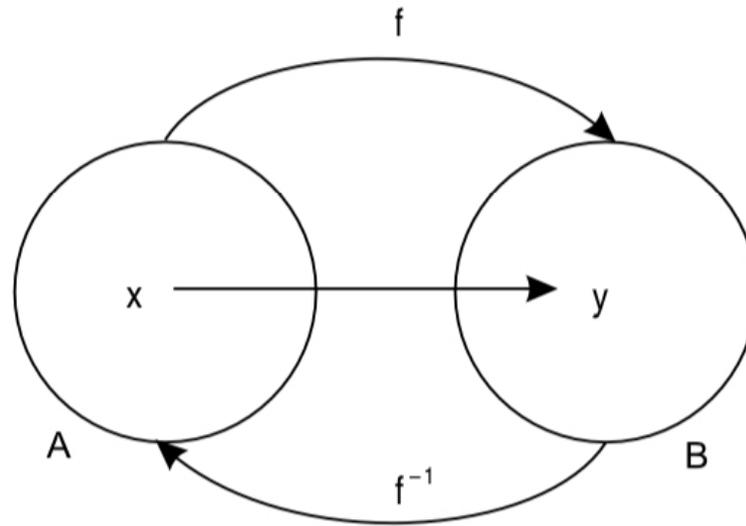
Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two functions. Then $(g \circ f)$ is one-one if both f and g are one-one and $(g \circ f)$ is onto if both f and g are onto.

➤ Theorem

If $f: A \rightarrow B$; $g: B \rightarrow C$ and $h: C \rightarrow D$, then $h \circ (g \circ f) = (h \circ g) \circ f$, i.e., composition of functions holds the associative law.

5. INVERSE FUNCTION

Let $f: A \rightarrow B$ be a bijective function. Then the inverse of f , *i.e.* f^{-1} be a function from B to A . Since f is a function from A to B , for every $x \in A$, there exists unique $y \in B$ such that $f(x) = y$.

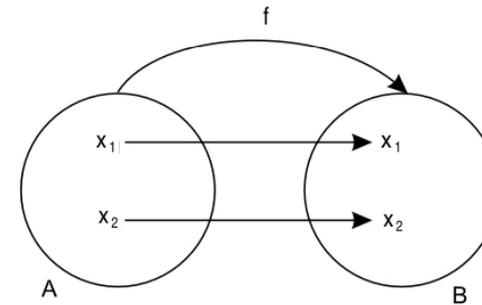


Since $f^{-1}: B \rightarrow A$ for every $y \in B$ there exists unique $x \in A$ such that $f^{-1}(y) = x$, *i.e.*,
 $f^{-1}(f(x)) = x$.

SOME IMPORTANT FUNCTIONS

1 Identity Function

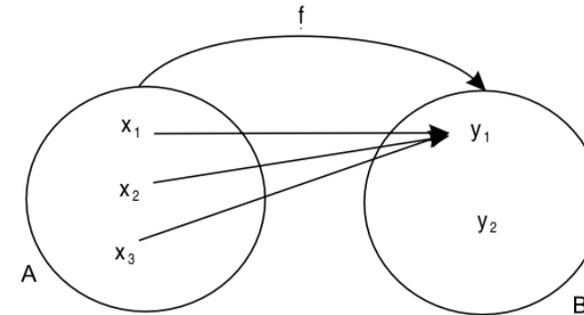
Let A be a set. The function $f: A \rightarrow A$ is said to be an identity function if for every $x \in A$, $f(x) = x$. Mathematically $f(x) = x \forall x \in A$.



2 Constant Function

The function $f: A \rightarrow B$ is said to be a constant function if for every $x \in A$ there exists unique $y \in B$ such that $f(x) = y$. Mathematically,

$$f(x) = y \quad \forall x \in A$$



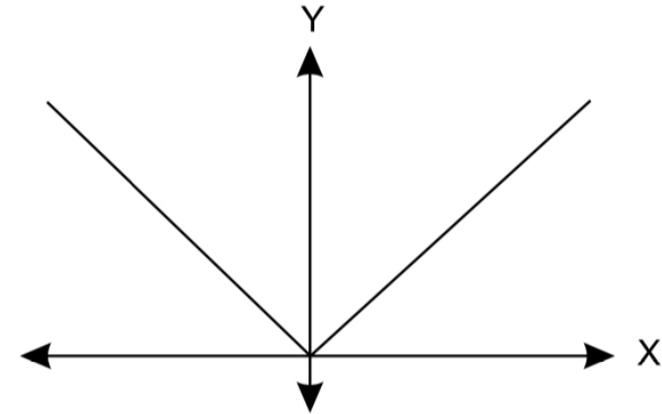
SOME IMPORTANT FUNCTIONS

3 Absolute Function

The absolute function or absolute value function $f(x) = |x|$ is defined as

$$|x| = \begin{cases} x; & \text{if } x \geq 0 \\ -x; & \text{if } x < 0 \end{cases}$$

The graph of $f = \{(x, |x|) : x \in \mathbb{R}\}$ is shown in the following figure.



Thank
you

جامعة بغداد / كلية علوم الحاسوب

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Discrete mathematics

Lec 9

Matrix I

Square matrices

A matrix with the same number of rows as columns is called square matrix . A square matrix with n rows and n columns is said to be order n ,and is called an n -square matrix.

Def: Main diagonal

The main diagonal or simply diagonal ,of asquare matrix $A=[a_{ij}]$ consists of the numbers: $a_{11},a_{22},a_{33},\dots,a_{nn}$.

Ex:The matrix

Is square matrix of order 3 .The numbers along the main diagonal are 1,-4,2.

Def:(Trace of the matrix)

If $A=[a_{ij}]$ is an $n \times n$ matrix , then the trace of A which is denoted by $\text{tr}(A)$ is defined as the sum of all elements on the main diagonal of A .

$\text{Tr}(A)=$

Ex :The trace of the following matrix is 4

Trace of the matrix has the following properties:-

1) $\text{Tr}(A+B)=\text{Tr}(A)+\text{Tr}(B)$

2) $\text{Tr}(c A)=c \text{Tr}(A)$

3) $\text{Tr}(AB)=\text{Tr}(BA)$

4) $\text{Tr}(A^T)=\text{Tr}(A)$

Def:Identity matrix

The n -square matrix with 1's along the main diagonal and 0's else where , is called the unit matrix and will be denoted and defined by

$$I_n =$$

Note : The unit matrix (I_n) plays the same role in matrix multiplication as the number 1 does in the usual multiplication of number ,specifically

$$A I_n = I_n A = A$$

For any square matrix A .

Def:Diagonal matrix

Asquare matrix $A=[a_{ij}]$ for which every term off the main diagonal is zero ,that is , $a_{ij}=0$ for $i \neq j$,is called diagonal matrix.

Ex: $G=$, $H=$

Are diagonal matrices.

Def:Scalar matrix

A diagonal matrix $A=[a_{ij}]$ for which all terms on the main diagonal are equal

,that is , $a_{ij}=c$ for $i=j$ and $a_{ij}=0$ for $i \neq j$, is called scalar matrix.

Ex: The following are scalar matrices

$J=$,

Def:Upper triangular matrix

A matrix $A=[a_{ij}]$ is called upper triangular matrix if $a_{ij}=0$ for $i > j$, and defined by

Ex:Suppose that

Is an upper triangular matrix of order 3.

Def: Lower triangular matrix :

An n- square matrix $A=[a_{ij}]$ is called lower triangular matrix if $a_{ij}=0$ for $i < j$,and defined by

Ex:Suppose that $A=$

Is a lower triangular matrix of order 4.

Def:(Periodic matrix)

If A is an n-square matrix .Such that $A^{k+1}=A$ where k is positive integer , then A is called periodic matrix .

Ex:Suppose that $A=$,is a periodic matrix.

Def:(Idempotent matrix)

If A is an n-square matrix such that $A^2=A$,then A is called idempotent matrix.

Ex:Suppose that ,is idempotent matrix.

Def:(Nilpotent matrix)

If A is an n-square matrix such that $A^p=0$, where P is a positive integer ,then A

is called nilpotent matrix.

Ex: Suppose that is nilpotent matrix of order 3..

Transpose of matrix :

If $A = [a_{ij}]$ is an $m \times n$ matrix, then the $n \times m$ matrix $A^T = [a_{ji}^T]$ where $a_{ij}^T = a_{ji}$,

(1) is called the transpose of A , thus the transpose of A is obtained by interchanging the rows and columns of A .

=

The transpose operation on matrices satisfies the following properties :-

Theorem:1) $(A+B)^T = A^T + B^T$

2) $(KA)^T = K A^T$, for K scalar.

3) $(AB)^T = B^T \cdot A^T$

4) $(A^T)^T = A$

Def:(Symmetric matrix)

A matrix $A = [a_{ij}]$ is called symmetric if $A^T = A$.

That is, A is symmetric if it is a square matrix for which $a_{ij} = a_{ji}$

If A is symmetric, then the elements of A are symmetric with respect to the main diagonal of A .

Ex: The matrices $A = I_3 =$

are symmetric.

Def:Skew symmetric matrix:

A matrix $A = [a_{ij}]$ is called skew symmetric matrix if $A^T = -A$ that is, A is skew symmetric matrix if and only if $a_{ij} = -a_{ji}$, for all i, j

Ex: $A =$, A is skew symmetric matrix .

Def:(Orthogonal matrix)

If A is an n -square matrix such that $A \cdot A^T = A^T \cdot A = I_n$, then A is called

orthogonal matrix

Ex: A is an orthogonal matrix.

Notes: 1) we can form powers of square matrix A by defining

$$A^2 = A \cdot A, A^3 = A^2 \cdot A, \dots, \text{ and } A^0 = I$$

In general, if p and q be non negative integers and A be square matrix, then

$$1) A^p \cdot A^q = A^{p+q}$$

$$2) (A^p)^q = A^{p \cdot q}$$

$$3) \text{ If } AB = BA, \text{ then } (AB)^p = A^p B^p$$

$$4) (cA)^p = c^p A^p \text{ where } c \text{ is scalar.}$$

2) We can form polynomials in A . That is, for any polynomials

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

we define $f(A)$ to be the matrix

$$f(A) = a_0 I + a_1 A + a_2 A^2 + \dots + a_n A^n$$

In the case that $f(A)$ is the zero matrix, then A is said zero or root of the polynomial $f(x)$.

Ex: Let $A =$, then $A^2 =$

$$\text{If } f(x) = x^2 + 3x - 10, \text{ then}$$

$$f(x) = +3 + (-10) =$$

Thus A is zero of the polynomial $f(x)$.

Invertible matrices :

Square matrix A is said to be invertible if there exist a matrix B with the property that $AB = BA = I$ (The identity matrix), such a matrix B is unique, it is called the inverse of A is denoted by A^{-1} . Observe that B is the inverse of A if and only if A is the inverse of B . For example

Suppose that $A =$ and $B =$

Then $AB =$ and $BA =$

Thus A and B are inverses.

Note: It is known that $AB = I_n$ if and only if $BA = I_n$, hence it is necessary to

test only one product to determine whether two given matrices are inverse as in the following example .

Ex : A= and B= A.B=

Thus the two matrices are invertible and are inverses of each other .

Note :-If A and B are two square matrices with the same size then

$$(A B)^{-1} = B^{-1} . A^{-1}$$

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Discrete mathematics

Lec 10

Matrix II

Determinants

To each n -square matrix $A = [a_{ij}]$ we assign a specific real number called the determinant of A , denoted by $\det(A)$ or $|A|$ where

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

The determinants of order one, two and three are defined as follows:

1) $|a_{11}| = a_{11}$

2) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

3) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix}$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Ex:-1) $|6| = 6$

2) $\begin{vmatrix} 2 & 1 \\ -4 & 6 \end{vmatrix} = 12 + 4 = 16$ 3) $\begin{vmatrix} 2 & 1 & 3 \\ 4 & 6 & -1 \\ 5 & 1 & 0 \end{vmatrix} = -81$

Properties of determinants:-

1) The determinants of matrix and its transpose are equal

That is $|A| = |A^T|$

Ex: Suppose that $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$, then $|A| = |A^T| = 6$

2) If matrix B result from matrix A by interchanging two rows (columns) of A ,then
 $| B | = - | A |$

Ex: $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, $| A | = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 4 + 3 = 7$

Suppose that $B = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}$, then $| B | = - | \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} | = -7$

3) If two rows (columns) of A are equal , then $| A | = 0$

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 7 \\ 1 & 2 & 3 \end{bmatrix}$, then $| A | = 0$

4) If a row (column) of A consists entirely of zero , then $| A | = 0$

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$ $| A | = 0$

5) If B is obtained from A by multiplying a row (column) of A by a real number c , then $| B | = c | A |$

Ex: Suppose that $B = \begin{bmatrix} 2 & -1 \\ 3 & 9 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$$3. | A | = 3 \cdot 7 = 21$$

$| B | = 21$, Observe that $| B | = 3 \cdot | A | = 21$.

6) If a matrix $A = [a_{ij}]$ is upper (lower) triangular matrix then

$$| A | = a_{11} a_{22} \dots a_{nn}$$

That is , the determinant of a triangular matrix is the product of the elements on the main diagonal.

Ex: $A = \begin{bmatrix} 7 & 5 & 12 \\ 0 & 1 & 18 \\ 0 & 0 & 6 \end{bmatrix}$, then $| A | = 7 \cdot 1 \cdot 6 = 42$

7)The determinant of aproduct of two matrices is the product of their determinants ,that is, $| A B | = | A | \cdot | B |$

Ex:let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, then $| A | = -2$, $| B | = 5$

$| A | | B | = -2 \cdot 5 = -10$,also

$A \cdot B = \begin{bmatrix} 4 & 3 \\ 10 & 5 \end{bmatrix}$, $| A B | = - \begin{vmatrix} 4 & 3 \\ 10 & 5 \end{vmatrix} = 20 - 30 = -10$

Observe that, $| A B | = | A | \cdot | B | = -10$