قسم الرياضيات / مادة المنتهية/ المرحلة الاولى

الكورس الاول

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1. Systems of Linear Equations

Consider the following problem: At a carry-out pizza restaurant, an order of 3 slices of pizza, 4 breadsticks, and 2 soft drinks cost \$13.35. A second order of 5 slices of pizza, 2 breadsticks , and 3 soft drinks cost \$19.50. If four bread sticks and a can of soda cost \$0.30 more than a slice of pizza, what is the cost of each item?
Let x_1 be the cost of a slice of pizza, x_2 the cost of a breadsticks, and x_3 the cost of a soft drink. The assumptions of the problem yield the following three equations:

$$\begin{cases} 3x_1 + 4x_2 + 2x_3 = 13.35 \\ 5x_1 + 2x_2 + 3x_3 = 19.50 \\ \quad\quad\;\; 4x_2 + x_3 = 0.30 + x_1 \end{cases}$$

or equivalently

$$\begin{cases} 3x_1 + 4x_2 + 2x_3 = 13.35 \\ 5x_1 + 2x_2 + 3x_3 = 19.50 \\ -x_1 + 4x_2 + x_3 = 0.30. \end{cases}$$

Thus, the problem is to find the values of $x_1, x_2,$ and x_3. A system like the one above is called a linear system.
Many practical problems can be reduced to solving systems of linear equations. The main purpose of linear algebra is to find systematic methods for solving these systems. So it is natural to start our discussion of linear algebra by studying linear equations.

A linear equation in n variables is an equation of the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b \tag{1.1}$$

where $x_1, x_2, ..., x_n$ are the **unknowns** (i.e. quantities to be found) and $a_1, \cdots, a_n$ are the **coefficients** (i.e. given numbers). We assume that the a_i's are not all zero. Also given the number b known as the **constant term.** In the special case where $b = 0$, Equation (1.1) is called a **homogeneous linear equation.**
Observe that a linear equation does not involve any products, inverses, or roots of variables. All variables occur only to the first power and do not appear as arguments for trigonometric, logarithmic, or exponential functions.

Example 1.1

Determine whether the given equations are linear or not (i.e., non-linear).

(a) $3x_1 - 4x_2 + 5x_3 = 6$.
(b) $4x_1 - 5x_2 = x_1 x_2$.
(c) $x_2 = 2\sqrt{x_1} - 6$.
(d) $x_1 + \sin x_2 + x_3 = 1$.
(e) $x_1 - x_2 + x_3 = \sin 3$.

Solution

(a) The given equation is in the form given by (1.1) and therefore is linear.
(b) The equation is non-linear because the term on the right side of the equation involves a product of the variables x_1 and x_2.
(c) A non-linear equation because the term $2\sqrt{x_1}$ involves a square root of the variable x_1.
(d) Since x_2 is an argument of a trigonometric function, the given equation is non-linear.
(e) The equation is linear according to (1.1) ■

In the case of $n = 2$, sometimes we will drop the subscripts and use instead $x_1 = x$ and $x_2 = y$. For example, $ax + by = c$. Geometrically, this is a straight line in the $xy-$coordinate system. Likewise, for $n = 3$, we will use $x_1 = x, x_2 = y$, and $x_3 = z$ and write $ax + by + cz = d$ which is a plane in the $xyz-$coordinate system.

A **solution** of a linear equation (1.1) in n unknowns is a finite ordered collection of numbers $s_1, s_2, ..., s_n$ which make (1.1) a true equality when $x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$ are substituted in (1.1). The collection of all solutions of a linear equation is called **the solution set** or **the general solution.**

Example 1.2

Show that $(5 + 4s - 7t, s, t)$, where $s, t \in \mathbb{R}$, is a solution to the equation

$$x_1 - 4x_2 + 7x_3 = 5.$$

Solution

$x_1 = 5 + 4s - 7t, x_2 = s$, and $x_3 = t$ is a solution to the given equation because

$$x_1 - 4x_2 + 7x_3 = (5 + 4s - 7t) - 4s + 7t = 5$$ ■

Many problems in the sciences lead to solving more than one linear equation. The general situation can be described by a linear system.

A **system of linear equations** or simply a **linear system** is any finite collection of linear equations. A linear system of m equations in n variables has the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = b_m \end{cases}$$

Note that the coefficients a_{ij} consist of two subscripts. The subscript i indicates the equation in which the coefficient occurs, and the subscript j indicates which unknown it multiplies.

When a linear system has more equations than unknowns, we call the system **overdetermined.** When the system has more unknowns than equations then we call the system **underdetermined.**

A **solution** of a linear system in n unknowns is a finite ordered collection of numbers $s_1, s_2, \ldots, s_n$ for which the substitution

$$x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$$

makes each equation a true statement. In compact form, a solution is an ordered $n-$tuple of the form

$$(s_1, s_2, \cdots, s_n).$$

The collection of all solutions of a linear system is called the **solution set** or the **general solution.** To solve a linear system is to find its general solution.

A linear system can have infinitely many solutions (**dependent system**), exactly one solution (**independent system**) or no solutions at all. When a linear system has a solution we say that the system is **consistent**. Otherwise, the system is said to be **inconsistent**. Thus, for the case $n = 2$, a linear system is consistent if the two lines either intersect at one point (independent) or they coincide (dependent). In the case the two lines are parallel, the system is inconsistent. For the case, $n = 3$, replace a line by a plane.

Example 1.3

Find the general solution of the linear system

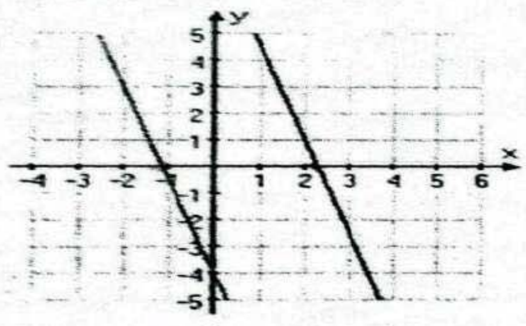
$$\begin{cases} x & + & y & = & 7 \\ 2x & + & 4y & = & 18. \end{cases}$$

... the first equation of the system by −2 and then add the resulting equation to the second equation to find $2y = 4$. Solving for y we find $y = 2$. Plugging this value in one of the equations of the given system and then solving for x one finds $x = 5$ ■

Example 1.4
Solve the system

$$\begin{cases} 7x + 2y = 16 \\ -21x - 6y = 24. \end{cases}$$

Solution.
Graphing the two lines we find

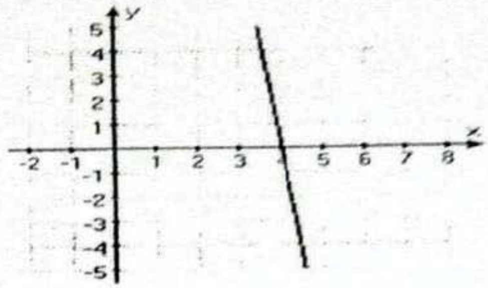


Thus, the system is inconsistent ■

Example 1.5
Solve the system

$$\begin{cases} 9x + y = 36 \\ 3x + \frac{1}{3}y = 12. \end{cases}$$

Solution.
Graphing the two lines we find



Thus, the system is consistent and dependent. Note that the two equations are basically the same: $9x + y = 36$. Letting $y = t$, where t is called a **parameter**, we can solve for x and find $x = \frac{36-t}{9}$. Thus, the general solution is defined by the **parametric equations**

$$x = \frac{36 - t}{9}, \quad y = t \blacksquare$$

Example 1.6
By letting $x_3 = t$, find the general solution of the linear system

$$\begin{cases} x_1 + x_2 + x_3 = 7 \\ 2x_1 + 4x_2 + x_3 = 18. \end{cases}$$

Solution.
By letting $x_3 = t$ the given system can be rewritten in the form

$$\begin{cases} x_1 + x_2 = 7 - t \\ 2x_1 + 4x_2 = 18 - t. \end{cases}$$

By multiplying the first equation by -2 and adding to the second equation one finds $x_2 = \frac{4+t}{2}$. Substituting this expression in one of the individual equations of the system and then solving for x_1 one finds $x_1 = \frac{10-3t}{2}$ $\blacksquare$

Practice Problems

Problem 1.1

Which of the following equations are not linear and why:
(a) $x_1^2 + 3x_2 - 2x_3 = 5$.
(b) $x_1 + x_1x_2 + 2x_3 = 1$.
(c) $x_1 + \frac{2}{x_2} + x_3 = 5$.

Problem 1.2

Show that $(2s + 12t + 13, s, -s - 3t - 3, t)$ is a solution to the system

$$\left\{\begin{array}{rcrcrcrcr} 2x_1 & + & 5x_2 & + & 9x_3 & + & 3x_4 & = & -1 \\ x_1 & + & 2x_2 & + & 4x_3 & & & = & 1 \end{array}\right.$$

Problem 1.3

Solve each of the following systems graphically:
(a)

$$\left\{\begin{array}{l} 4x_1 - 3x_2 = 0 \\ 2x_1 + 3x_2 = 18 \end{array}\right.$$

(b)

$$\left\{\begin{array}{l} 4x_1 - 6x_2 = 10 \\ 6x_1 - 9x_2 = 15 \end{array}\right.$$

(c)

$$\left\{\begin{array}{l} 2x_1 + x_2 = 3 \\ 2x_1 + x_2 = 1 \end{array}\right.$$

Which of the above systems is consistent and which is inconsistent?

Problem 1.4

Determine whether the system of equations is linear or non-linear.

(a)

$$\left\{\begin{array}{r} \ln x_1 + x_2 + x_3 = 3 \\ 2x_1 + x_2 - 5x_3 = 1 \\ -x_1 + 5x_2 + 3x_3 = -1. \end{array}\right.$$

(b)

$$\left\{\begin{array}{rcrcrcr} 3x_1 & + & 4x_2 & + & 2x_3 & = & 13.35 \\ 5x_1 & + & 2x_2 & + & 3x_3 & = & 19.50 \\ -x_1 & + & 4x_2 & + & x_3 & = & 0.30. \end{array}\right.$$

Problem 1.5
Find the parametric equations of the solution set to the equation $-x_1+5x_2+3x_3-2x_4=-1$.

Problem 1.6
Write a system of linear equations consisting of three equations in three unknowns with
(a) no solutions.
(b) exactly one solution.
(c) infinitely many solutions.

Problem 1.7
For what values of h and k the system below has (a) no solution, (b) a unique solution, and (c) many solutions.

$$\left\{\begin{array}{rcrcl} x_1 & + & 3x_2 & = & 2\\ 3x_1 & + & hx_2 & = & k.\end{array}\right.$$

Problem 1.8
True/False:
(a) A general solution of a linear system is an explicit description of all the solutions of the system.
(b) A linear system with either one solution or infinitely many solutions is said to be inconsistent.
(c) Finding a parametric description of the solution set of a linear system is the same as solving the system.
(d) A linear system with a unique solution is consistent and dependent.

Problem 1.9
Find a linear equation in the variables x and y that has the general solution $x=5+2t$ and $y=t$. +x − 2y = 5

Problem 1.10
Find a relationship between a, b, c so that the following system is consistent.

$$\left\{\begin{array}{rcrcrcl} x_1 & + & x_2 & + & 2x_3 & = & a\\ x_1 & & & + & x_3 & = & b\\ 2x_1 & + & x_2 & + & 3x_3 & = & c.\end{array}\right.$$

2. Equivalent Systems and Elementary Row Operations: The Elimination Method

Next, we shift our attention for solving linear systems of equations. In this section we introduce the concept of elementary row operations that will be vital for our algebraic method of solving linear systems.
First, we define what we mean by equivalent systems: Two linear systems are said to be **equivalent** if and only if they have the same set of solutions.

Example 2.1
Show that the system

$$\begin{cases} x_1 - 3x_2 = -7 \\ 2x_1 + x_2 = 7 \end{cases}$$

is equivalent to the system

$$\begin{cases} 8x_1 - 3x_2 = 7 \\ 3x_1 - 2x_2 = 0 \\ 10x_1 - 2x_2 = 14. \end{cases}$$

Solution.
Solving the first system one finds the solution $x_1 = 2, x_2 = 3$. Similarly, solving the second system one finds the solution $x_1 = 2$ and $x_2 = 3$. Hence, the two systems are equivalent ■

Example 2.2
Show that if $x_1 + kx_2 = c$ and $x_1 + \ell x_2 = d$ are equivalent then $k = \ell$ and $c = d$.

Solution.
For arbitrary t the ordered pair $(c - kt, t)$ is a solution to the second equation. That is $c - kt + \ell t = d$ for all $t \in \mathbb{R}$. In particular, if $t = 0$ we find $c = d$. Thus, $kt = \ell t$ for all $t \in \mathbb{R}$. Letting $t = 1$ we find $k = \ell$ ■

Our basic algebraic method for solving a linear system is known as the **method of elimination**. The method consists of reducing the original system to an equivalent system that is easier to solve. The reduced system has the shape of an upper (resp. lower) triangle. This new system can

be solved by a technique called **backward-substitution** (resp. **forward-substitution**): The unknowns are found starting from the bottom (resp. the top) of the system.
The three basic operations in the above method, known as the **elementary row operations**, are summarized as follows:
(I) Multiply an equation by a non-zero number.
(II) Replace an equation by the sum of this equation and another equation multiplied by a number.
(III) Interchange two equations.

To indicate which operation is being used in the process one can use the following shorthand notation. For example, $r_3 \leftarrow \frac{1}{2}r_3$ represents the row operation of type (I) where each entry of row 3 is being replaced by $\frac{1}{2}$ that entry. Similar interpretations for types (II) and (III) operations.
The following theorem asserts that the system obtained from the original system by means of elementary row operations has the same set of solutions as the original one.

Theorem 2.1
Suppose that an elementary row operation is performed on a linear system. Then the resulting system is equivalent to the original system.

Example 2.3
Use the elimination method described above to solve the system

$$\left\{\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & 3 \\ x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 2x_1 & - & 2x_2 & + & x_3 & = & 4. \end{array}\right.$$

Solution.
Step 1: We eliminate x_1 from the second and third equations by performing two operations $r_2 \leftarrow r_2 - r_1$ and $r_3 \leftarrow r_3 - 2r_1$ obtaining

$$\left\{\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & 3 \\ & - & 4x_2 & + & 3x_3 & = & -2 \\ & - & 4x_2 & + & 3x_3 & = & -2 \end{array}\right.$$

Step 2: The operation $r_3 \leftarrow r_3 - r_2$ leads to the system

$$\left\{\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & 3 \\ & - & 4x_2 & + & 3x_3 & = & -2 \end{array}\right.$$

By assigning x_3 an arbitrary value t we obtain the general solution $x_1 = \frac{t+10}{4}, x_2 = \frac{2+3t}{4}, x_3 = t$. This means that the linear system has infinitely many solutions (consistent and dependent). Every time we assign a value to t we obtain a different solution ■

Example 2.4
Determine if the following system is consistent or not

$$\left\{\begin{array}{rcrcrcr} 3x_1 & + & 4x_2 & + & x_3 & = & 1 \\ 2x_1 & + & 3x_2 & & & = & 0 \\ 4x_1 & + & 3x_2 & - & x_3 & = & -2. \end{array}\right.$$

Solution.
Step 1: To eliminate the variable x_1 from the second and third equations we perform the operations $r_2 \leftarrow 3r_2 - 2r_1$ and $r_3 \leftarrow 3r_3 - 4r_1$ obtaining the system

$$\left\{\begin{array}{rcrcrcr} 3x_1 & + & 4x_2 & + & x_3 & = & 1 \\ & & x_2 & - & 2x_3 & = & -2 \\ & - & 7x_2 & - & 7x_3 & = & -10. \end{array}\right.$$

Step 2: Now, to eliminate the variable x_2 from the third equation we apply the operation $r_3 \leftarrow r_3 + 7r_2$ to obtain

$$\left\{\begin{array}{rcrcrcr} 3x_1 & + & 4x_2 & + & x_3 & = & 1 \\ & & x_2 & - & 2x_3 & = & -2 \\ & & & - & 21x_3 & = & -24. \end{array}\right.$$

Solving the system by the method of backward substitution we find the unique solution $x_1 = -\frac{3}{7}, x_2 = \frac{2}{7}, x_3 = \frac{8}{7}$. Hence the system is consistent and independent ■

Example 2.5
Determine whether the following system is consistent:

$$\left\{\begin{array}{rcrcr} x_1 & - & 3x_2 & = & 4 \\ -3x_1 & + & 9x_2 & = & 8. \end{array}\right.$$

Solution.
Multiplying the first equation by 3 and adding the resulting equation to the second equation we find $0 = 20$ which is impossible. Hence, the given system is inconsistent ■

Practice Problems

Problem 2.1
Solve each of the following systems using the method of elimination:
(a)

$$\begin{cases} 4x_1 - 3x_2 = 0 \\ 2x_1 + 3x_2 = 18 \end{cases}$$

(b)

$$\begin{cases} 4x_1 - 6x_2 = 10 \\ 6x_1 - 9x_2 = 15 \end{cases}$$

(c)

$$\begin{cases} 2x_1 + x_2 = 3 \\ 2x_1 + x_2 = 1 \end{cases}$$

Which of the above systems is consistent and which is inconsistent?

Problem 2.2
Find the values of A,B,C in the following partial fraction

$$\frac{x^2 - x + 3}{(x^2+2)(2x-1)} = \frac{Ax+B}{x^2+2} + \frac{C}{2x-1}.$$

Problem 2.3
Find a quadratic equation of the form $y = ax^2 + bx + c$ that goes through the points $(-2, 20)$, $(1, 5)$, and $(3, 25)$.

Problem 2.4
Solve the following system using the method of elimination.

$$\begin{cases} 5x_1 - 5x_2 - 15x_3 = 40 \\ 4x_1 - 2x_2 - 6x_3 = 19 \\ 3x_1 - 6x_2 - 17x_3 = 41 \end{cases}$$

Problem 2.5
Solve the following system using elimination.

$$\begin{cases} 2x_1 + x_2 + x_3 = -1 \\ x_1 + 2x_2 + x_3 = 0 \\ 3x_1 \qquad - 2x_3 = 5 \end{cases}$$

Problem 2.6
Find the general solution of the linear system

$$\begin{cases} x_1 - 2x_2 + 3x_3 + x_4 = -3 \\ 2x_1 - x_2 + 3x_3 - x_4 = 0 \end{cases}$$

Problem 2.7
Find a, b, and c so that the system

$$\begin{cases} x_1 + ax_2 + cx_3 = 0 \\ bx_1 + cx_2 - 3x_3 = 1 \\ ax_1 + 2x_2 + bx_3 = 5 \end{cases}$$

has the solution $x_1 = 3, x_2 = -1, x_3 = 2$.

Problem 2.8
Show that the following systems are equivalent.

$$\begin{cases} 7x_1 + 2x_2 + 2x_3 = 21 \\ \quad\quad - 2x_2 + 3x_3 = 1 \\ \quad\quad\quad\quad 4x_3 = 12 \end{cases}$$

and

$$\begin{cases} 21x_1 + 6x_2 + 6x_3 = 63 \\ \quad\quad - 4x_2 + 6x_3 = 2 \\ \quad\quad\quad\quad x_3 = 3 \end{cases}$$

Problem 2.9
Solve the following system by elimination.

$$\begin{cases} 3x_1 + x_2 + 2x_3 = 13 \\ 2x_1 + 3x_2 + 4x_3 = 19 \\ x_1 + 4x_2 + 3x_3 = 15 \end{cases}$$

Problem 2.10
Solve the following system by elimination.

$$\begin{cases} x_1 - 2x_2 + 3x_3 = 7 \\ 2x_1 + x_2 + x_3 = 4 \\ -3x_1 + 2x_2 - 2x_3 = -10 \end{cases}$$

3. Solving Linear Systems Using Augmented Matrices

In this section we apply the elimination method described in the previous section to the rectangular array consisting of the coefficients of the unknowns and the right-hand side of a given system rather than to the individual equations. To elaborate, consider the linear system

$$\left\{\begin{array}{ll} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = b_m \end{array}\right.$$

We define the **augmented matrix** corresponding to the above system to be the rectangular array

$$\left[\begin{array}{ccccc} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

We then apply elementary row operations on the augmented matrix and reduces it to a triangular matrix. Then the corresponding system is triangular as well and is equivalent to the original system. Next, use either the backward-substitution or the forward-substitution technique to find the unknowns. We illustrate this technique in the following examples.

Example 3.1
Solve the following linear system using elementary row operations on the augmented matrix:

$$\left\{\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9. \end{array}\right.$$

Solution.
The augmented matrix for the system is

$$\left[\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

Step 1: The operations $r_2 \leftarrow \frac{1}{2}r_2$ and $r_3 \leftarrow r_3 + 4r_1$ give

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Step 2: The operation $r_3 \leftarrow r_3 + 3r_2$ gives

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \quad\quad x_2 - 4x_3 = 4 \\ \quad\quad\quad\quad x_3 = 3 \end{cases}$$

Using back-substitution we find the unique solution $x_1 = 29, x_2 = 16, x_3 = 3$ ■

Example 3.2
Solve the following linear system using the method described above.

$$\begin{cases} \quad\quad x_2 + 5x_3 = -4 \\ x_1 + 4x_2 + 3x_3 = -2 \\ 2x_1 + 7x_2 + x_3 = -1. \end{cases}$$

Solution.
The augmented matrix for the system is

$$\begin{bmatrix} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & 1 & -1 \end{bmatrix}$$

Step 1:The operation $r_2 \leftrightarrow r_1$ gives

$$\begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 2 & 7 & 1 & -1 \end{bmatrix}$$

Step 2: The operation $r_3 \leftarrow r_3 - 2r_1$ gives the system

$$\begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & -1 & -5 & 3 \end{bmatrix}$$

Step 3: The operation $r_3 \leftarrow r_3 + r_2$ gives

$$\begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The corresponding system of equations is

$$\left\{ \begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & 3x_3 & = & -2 \\ & & x_2 & + & 5x_3 & = & -4 \\ & & & & 0 & = & -1 \end{array} \right.$$

From the last equation we conclude that the system is inconsistent ■

Example 3.3
Determine if the following system is consistent.

$$\left\{ \begin{array}{rcrcrcr} & & x_2 & - & 4x_3 & = & 8 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & 1. \end{array} \right.$$

Solution.
The augmented matrix of the given system is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

Step 1: The operation $r_3 \leftarrow r_3 - 2r_2$ gives

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 1 & -2 & 3 & -1 \end{bmatrix}$$

Step 2: The operation $r_3 \leftrightarrow r_1$ leads to

$$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \end{bmatrix}$$

Step 3: Applying $r_2 \leftarrow r_2 - 2r_1$ to obtain

$$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -4 & 3 \\ 0 & 1 & -4 & 8 \end{bmatrix}$$

Step 4: Finally, the operation $r_3 \leftarrow r_3 - r_2$ gives

$$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Hence, the equivalent system is

$$\begin{cases} x_1 - 2x_2 + 3x_3 = -1 \\ \quad\quad x_2 - 4x_3 = 3 \\ \quad\quad\quad\quad\quad 0 = 5 \end{cases}$$

This last system has no solution (the last equation requires x_1, x_2, and x_3 to satisfy the equation $0x_1 + 0x_2 + 0x_3 = 5$ and no such x_1, x_2, and x_3 exist). Hence the original system is inconsistent ■

Pay close attention to the last row of the trinagular matrix of the previous exercise. This situation is typical of an inconsistent system.

Gaussian Elimination

The elimination method introduced in the previous section reduces the augmented matrix to a "nice" matrix (meaning the corresponding equations are easy to solve). Two of the "nice" matrices discussed in this section are matrices in either row-echelon form or reduced row-echelon form, concepts that we discuss next.

By a **leading entry** of a row in a matrix we mean the leftmost non-zero entry in the row.

A rectangular matrix is said to be in **row-echelon form** if it has the following three characterizations:

(1) All rows consisting entirely of zeros are at the bottom.
(2) The leading entry in each non-zero row is 1 and is located in a column to the right of the leading entry of the row above it.
(3) All entries in a column below a leading entry are zero.

The matrix is said to be in **reduced row-echelon form** if in addition to the above, the matrix has the following additional characterization:

(4) Each leading 1 is the only nonzero entry in its column.

Remark 4.1 From the definition above, note that a matrix in row-echelon form has zeros below each leading 1, whereas a matrix in reduced row-echelon form has zeros both above and below each leading 1.

Example 4.1

Determine which matrices are in row-echelon form (but not in reduced row-echelon form) and which are in reduced row-echelon form

(a)

$$\begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution.
(a)The given matrix is in row-echelon form but not in reduced row-echelon form since the $(1,2)-$entry is not zero.
(b) The given matrix satisfies the characterization of a reduced row-echelon form ■

The importance of the row-echelon matrices is indicated in the following theorem.

Theorem 4.1
Every nonzero matrix can be brought to (reduced) row-echelon form by a finite number of elementary row operations.

The process of reducing a matrix to a row-echelon form is known as **Gaussian elimination**. That of reducing a matrix to a reduced row-echelon form is known as **Gauss-Jordan elimination**.

Example 4.2
Use Gauss-Jordan elimination to transform the following matrix first into row-echelon form and then into reduced row-echelon form

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution.
The reduction of the given matrix to row-echelon form is as follows.

Step 1: $r_1 \leftrightarrow r_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Step 2: $r_2 \leftarrow r_2 + r_1$ and $r_3 \leftarrow r_3 + 2r_1$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}.$$

Step 3: $r_2 \leftarrow \frac{1}{2}r_2$ and $r_3 \leftarrow \frac{1}{5}r_3$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Step 4: $r_3 \leftarrow r_3 - r_2$ and $r_4 \leftarrow r_4 + 3r_2$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Step 5: $r_3 \leftrightarrow r_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 6: $r_5 \leftarrow -\frac{1}{5}r_5$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 7: $r_1 \leftarrow r_1 - 4r_2$

$$\begin{bmatrix} 1 & 0 & -3 & 3 & 5 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 8: $r_1 \leftarrow r_1 - 3r_3$ and $r_2 \leftarrow r_2 + 3r_3$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \blacksquare$$

Example 4.3

Use Gauss-Jordan elimination to transform the following matrix first into row-echelon form and then into reduced row-echelon form

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution.

By following the steps in the Gauss-Jordan algorithm we find

Step 1: $r_3 \leftarrow \frac{1}{3}r_3$

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 1 & -3 & 4 & -3 & 2 & 5 \end{bmatrix}$$

Step 2: $r_1 \leftrightarrow r_3$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 3: $r_2 \leftarrow r_2 - 3r_1$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 4: $r_2 \leftarrow \frac{1}{2}r_2$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 5: $r_3 \leftarrow r_3 - 3r_2$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Step 6: $r_1 \leftarrow r_1 + 3r_2$

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Step 7: $r_1 \leftarrow r_1 - 5r_3$ and $r_2 \leftarrow r_2 - r_3$

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \blacksquare$$

Remark 4.2
It can be shown that no matter how the elementary row operations are varied, one will always arrive at the same reduced row-echelon form; that is the reduced row echelon form is unique. On the contrary row-echelon form is not unique. However, the number of leading 1's of two different row-echelon forms is the same. That is, two row-echelon matrices have the same number of nonzero rows. This number is knwon as the **rank** of the matrix.

Example 4.4
Consider the system

$$\begin{cases} ax + by = k \\ cx + dy = l. \end{cases}$$

Show that if $ad - bc \neq 0$ then the reduced row-echelon form of the coefficient matrix is the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution.
The coefficient matrix is the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Assume first that $a \neq 0$. Using Gaussian elimination we reduce the above matrix into row-echelon form as follows:

Step 1: $r_2 \leftarrow ar_2 - cr_1$

$$\begin{bmatrix} a & b \\ 0 & ad - bc \end{bmatrix}$$

Step 2: $r_2 \leftarrow \frac{1}{ad-bc} r_2$

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

5. Echelon Forms and Solutions to Linear Systems

In this section we give a systematic procedure for solving systems of linear equations; it is based on the idea of reducing the augmented matrix to either the row-echelon form or the reduced row-echelon form. The new system is equivalent to the original system.

Unknowns corresponding to leading entries in the echelon augmented matrix are called **dependent or leading variables**. If an unknown is not dependent then it is called **free or independent** variable.

Example 5.1

Find the dependent and independent variables of the following system

$$\left\{\begin{array}{rcrcrcrcrcrcr} x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & & & = & 0 \\ 2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & + & 4x_5 & - & 3x_6 & = & -1 \\ & & & & 5x_3 & + & 10x_4 & & & + & 15x_6 & = & 5 \\ 2x_1 & + & 6x_2 & & & + & 8x_4 & + & 4x_5 & + & 18x_6 & = & 6 \end{array}\right.$$

Solution.

The augmented matrix for the system is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

Using the Gaussian algorithm we bring the augmented matrix to row-echelon form as follows:

Step 1: $r_2 \leftarrow r_2 - 2r_1$ and $r_4 \leftarrow r_4 - 2r_1$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

Step 2: $r_2 \leftarrow -r_2$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

Step 3: $r_3 \leftarrow r_3 - 5r_2$ and $r_4 \leftarrow r_4 - 4r_2$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

Step 4: $r_3 \leftrightarrow r_4$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 5: $r_3 \leftarrow \frac{1}{6}r_3$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The leading variables are x_1, x_3, and x_6. The free variables are x_2, x_4, and x_5 ■

One way to solve a linear system is to apply the elementary row operations to reduce the augmented matrix to a (reduced) row-echelon form. If the augmented matrix is in reduced row-echelon form then to obtain the general solution one just has to move all independent variables to the right side of the equations and consider them as parameters. The dependent variables are given in terms of these parameters.

Example 5.2
Solve the following linear system.

$$\begin{cases} x_1 + 2x_2 + x_4 = 6 \\ x_3 + 6x_4 = 7 \\ x_5 = 1. \end{cases}$$

Solution.
The augmented matrix is already in row-echelon form. The free variables are x_2 and x_4. So let $x_2 = s$ and $x_4 = t$. Solving the system starting from the bottom we find $x_1 = -2s - t + 6, x_3 = 7 - 6t$, and $x_5 = 1$ ■

If the augmented matrix does not have the reduced row-echelon form but the row-echelon form then the general solution also can be easily found by using the method of backward substitution.

Example 5.3
Solve the following linear system

$$\begin{cases} x_1 - 3x_2 + x_3 - x_4 = 2 \\ \quad\quad x_2 + 2x_3 - x_4 = 3 \\ \quad\quad\quad\quad x_3 + x_4 = 1. \end{cases}$$

Solution.
The augmented matrix is in row-echelon form. The free variable is $x_4 = t$. Solving for the leading variables we find, $x_1 = 11t + 4, x_2 = 3t + 1$, and $x_3 = 1 - t$ ■

The questions of existence and uniqueness of solutions are fundamental questions in linear algebra. The following theorem provides some relevant information.

Theorem 5.1
A system of m linear equations in n unknowns can have exactly one solution, infinitely many solutions, or no solutions at all.
(1) If the reduced augmented matrix has a row of the form $[0, 0, \cdots, 0, b]$ where b is a nonzero constant, then the system has no solutions.
(2) If the reduced augmented matrix has independent variables and no rows of the form $[0, 0, \cdots, 0, b]$ with $b \neq 0$ then the system has infinitely many solutions.
(3) If the reduced augmented matrix has no independent variables and no rows of the form $[0, 0, \cdots, 0, b]$ with $b \neq 0$, then the system has exactly one solution.

Example 5.4
Find the general solution of the system whose augmented matrix is given by

$$\begin{bmatrix} 1 & 2 & -7 \\ -1 & -1 & 1 \\ 2 & 1 & 5 \end{bmatrix}$$

Solution.
We first reduce the system to row-echelon form as follows.

Step 1: $r_2 \leftarrow r_2 + r_1$ and $r_3 \leftarrow r_3 - 2r_1$

$$\begin{bmatrix} 1 & 2 & -7 \\ 0 & 1 & -6 \\ 0 & -3 & 19 \end{bmatrix}$$

Step 2: $r_3 \leftarrow r_3 + 3r_2$

$$\begin{bmatrix} 1 & 2 & -7 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

The corresponding system is given by

$$\begin{cases} x_1 + 2x_2 = -7 \\ \quad\quad x_2 = -6 \\ \quad\quad\ 0 = 1 \end{cases}$$

Because of the last equation the system is inconsistent ■

Example 5.5
Find the general solution of the system whose augmented matrix is given by

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 7 & -3 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution.
By adding two times the second row to the first row we find the reduced row-echelon form of the augmented matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It follows that the free variables are $x_3 = s$ and $x_5 = t$. Solving for the leading variables we find $x_1 = -1 - t$, $x_2 = 1 + 3t$, and $x_4 = -4 - 5t$ ■

Example 5.6
Determine the value(s) of h such that the following matrix is the augmented matrix of a consistent linear system

$$\begin{bmatrix} 1 & 4 & 2 \\ -3 & h & -1 \end{bmatrix}$$

Solution.
By adding three times the first row to the second row we find

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 12+h & 5 \end{bmatrix}$$

The system is consistent if and only if $12 + h \neq 0$; that is, $h \neq -12$ ■

Example 5.7
Find (if possible) conditions on the numbers a, b, and c such that the following system is consistent

$$\left\{ \begin{array}{rcrcrcl} x_1 & + & 3x_2 & + & x_3 & = & a \\ -x_1 & - & 2x_2 & + & x_3 & = & b \\ 3x_1 & + & 7x_2 & - & x_3 & = & c \end{array} \right.$$

Solution.
The augmented matrix of the system is

$$\begin{bmatrix} 1 & 3 & 1 & a \\ -1 & -2 & 1 & b \\ 3 & 7 & -1 & c \end{bmatrix}$$

Now apply Gaussian elimination as follows.

Step 1: $r_2 \leftarrow r_2 + r_1$ and $r_3 \leftarrow r_3 - 3r_1$

$$\begin{bmatrix} 1 & 3 & 1 & a \\ 0 & 1 & 2 & b+a \\ 0 & -2 & -4 & c-3a \end{bmatrix}$$

Step 2: $r_3 \leftarrow r_3 + 2r_2$

$$\begin{bmatrix} 1 & 3 & 1 & a \\ 0 & 1 & 2 & b+a \\ 0 & 0 & 0 & c-a+2b \end{bmatrix}$$

The system has no solution if $c - a + 2b \neq 0$. The system has infinitely many solutions if $c - a + 2b = 0$. In this case, the solution is given by $x_1 = 5t - (2a + 3b), x_2 = (a + b) - 2t, x_3 = t$ ■

6. Homogeneous Systems of Linear Equations

A homogeneous linear system is any system of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \cdots\cdots\cdots\cdots &\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

Every homogeneous system is consistent, since $x_1 = 0, x_2 = 0, \cdots, x_n = 0$ is always a solution. This solution is called the **trivial solution**; any other solution is called **nontrivial.**

A homogeneous system has either a unique solution (the trivial solution) or infinitely many solutions. The following theorem provides a criterion where a homogeneous system is assured to have a nontrivial solution (and therefore infinitely many solutions).

Theorem 6.1

A homogeneous system in n unknowns and m equations has infinitely many solutions if either

(1) the rank of the coefficient matrix is less than n or

(2) the number of unknowns exceeds the number of equations, i.e. $m < n$. That is, the system is underdetermined.

Example 6.1

Solve the following homogeneous system using Gauss-Jordan elimination.

$$\begin{cases} 2x_1 + 2x_2 - x_3 + x_5 = 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0 \\ x_1 + x_2 - 2x_3 - x_5 = 0 \\ x_3 + x_4 + x_5 = 0. \end{cases}$$

Solution.

The reduction of the augmented matrix to reduced row-echelon form is outlined below.

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Step 1: $r_3 \leftarrow r_3 + r_2$

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Step 2: $r_3 \leftrightarrow r_4$ and $r_1 \leftrightarrow r_2$

$$\begin{bmatrix} -1 & -1 & 2 & -3 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{bmatrix}$$

Step 3: $r_2 \leftarrow r_2 + 2r_1$ and $r_4 \leftarrow -\frac{1}{3}r_4$

$$\begin{bmatrix} -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 3 & -6 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 4: $r_1 \leftarrow -r_1$ and $r_2 \leftarrow \frac{1}{3}r_2$

$$\begin{bmatrix} 1 & 1 & -2 & 3 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 5: $r_3 \leftarrow r_3 - r_2$

$$\begin{bmatrix} 1 & 1 & -2 & 3 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 6: $r_4 \leftarrow r_4 - \frac{1}{3}r_3$ and $r_3 \leftarrow \frac{1}{3}r_3$

$$\begin{bmatrix} 1 & 1 & -2 & 3 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Step 7: $r_1 \leftarrow r_1 - 3r_3$ and $r_2 \leftarrow r_2 + 2r_3$

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 8: $r_1 \leftarrow r_1 + 2r_2$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system is

$$\begin{cases} x_1 + x_2 + x_5 = 0 \\ x_3 + x_5 = 0 \\ x_4 = 0 \end{cases}$$

The free variables are $x_2 = s, x_5 = t$ and the general solution is given by the formula: $x_1 = -s - t, x_2 = s, x_3 = -t, x_4 = 0, x_5 = t$ ■

Example 6.2
Solve the following homogeneous system using Gaussian elimination.

$$\begin{cases} x_1 + 3x_2 + 5x_3 + x_4 = 0 \\ 4x_1 - 7x_2 - 3x_3 - x_4 = 0 \\ 3x_1 + 2x_2 + 7x_3 + 8x_4 = 0 \end{cases}$$

Solution.
The augmented matrix for the system is

$$\begin{bmatrix} 1 & 3 & 5 & 1 & 0 \\ 4 & -7 & -3 & -1 & 0 \\ 3 & 2 & 7 & 8 & 0 \end{bmatrix}$$

We reduce this matrix into a row-echelon form as follows.
Step 1: $r_2 \leftarrow r_2 - r_3$

$$\begin{bmatrix} 1 & 3 & 5 & 1 & 0 \\ 1 & -9 & -10 & -9 & 0 \\ 3 & 2 & 7 & 8 & 0 \end{bmatrix}$$

Step 2: $r_2 \leftarrow r_2 - r_1$ and $r_3 \leftarrow r_3 - 3r_1$

$$\begin{bmatrix} 1 & 3 & 5 & 1 & 0 \\ 0 & -12 & -15 & -10 & 0 \\ 0 & -7 & -8 & 5 & 0 \end{bmatrix}$$

Step 3: $r_2 \leftarrow -\frac{1}{12}r_2$

$$\begin{bmatrix} 1 & 3 & 5 & 1 & 0 \\ 0 & 1 & \frac{5}{4} & \frac{5}{6} & 0 \\ 0 & -7 & -8 & 5 & 0 \end{bmatrix}$$

Step 4: $r_3 \leftarrow r_3 + 7r_2$

$$\begin{bmatrix} 1 & 3 & 5 & 1 & 0 \\ 0 & 1 & \frac{5}{4} & \frac{5}{6} & 0 \\ 0 & 0 & \frac{3}{4} & \frac{65}{6} & 0 \end{bmatrix}$$

Step 5: $r_3 \leftarrow \frac{4}{3}r_3$

$$\begin{bmatrix} 1 & 3 & 5 & 1 & 0 \\ 0 & 1 & \frac{5}{4} & \frac{5}{6} & 0 \\ 0 & 0 & 1 & \frac{130}{9} & 0 \end{bmatrix}$$

We see that $x_4 = t$ is the only free variable. Solving for the leading variables using back substitution we find $x_1 = \frac{176}{9}t, x_2 = \frac{155}{9}t$, and $x_3 = -\frac{130}{9}t$ ■

Remark 6.1
Part (2) of Theorem 6.1 applies only to homogeneous linear systems. A non-homogeneous system (right-hand side has non-zero entries) with more unknowns than equations need not be consistent as shown in the next example.

Example 6.3
Show that the following system is inconsistent.

$$\left\{ \begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 0 \\ 2x_1 & + & 2x_2 & + & 2x_3 & = & 4. \end{array} \right.$$

Solution.
Multiplying the first equation by -2 and adding the resulting equation to the second we obtain $0 = 4$ which is impossible. So the system is inconsistent ■

Example 6.4
Show that if a homogeneous system of linear equations in n unknowns has a nontrivial solution then $rank(A) < n$, where A is the coefficient matrix.

Solution.
Since $rank(A) \leq n$, either $rank(A) = n$ or $rank(A) < n$. If $rank(A) < n$ then we are done. So suppose that $rank(A) = n$. Then there is a matrix B that is row equivalent to A and that has n nonzero rows. Moreover, B has the following form

$$\begin{bmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} & 0 \\ 0 & 1 & a_{23} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

The corresponding system is triangular and can be solved by back substitution to obtain the solution $x_1 = x_2 = \cdots = x_n = 0$ which is a contradiction. Thus we must have $rank(A) < n$ ■

8. Matrix Multiplication

In the previous section we discussed some basic properties associated with matrix addition and scalar multiplication. Here we introduce another important operation involving matrices-the product.
Let $A = (a_{ij})$ be a matrix of size $m \times n$ and $B = (b_{ij})$ be a matrix of size $n \times p$. Then the product matrix is a matrix of size $m \times p$ and entries

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj},$$

that is, c_{ij} is obtained by multiplying componentwise the entries of the i^{th} row of A by the entries of the j^{th} column of B. It is very important to keep in mind that the number of columns of the first matrix must be equal to the number of rows of the second matrix; otherwise the product is undefined.
An interesting question associated with matrix multiplication is the following: If A and B are square matrices then is it always true that $AB = BA$?
The answer to this question is negative. In general, matrix multiplication is not commutative, as the following example shows.

Example 8.1
Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

Show that $AB \neq BA$. Hence, matrix multiplication is not commutative.

Solution.
Using the definition of matrix multiplication we find

$$AB = \begin{bmatrix} -4 & 7 \\ 0 & 5 \end{bmatrix}, BA = \begin{bmatrix} -1 & 2 \\ 9 & 2 \end{bmatrix}$$

Hence, $AB \neq BA$ ■

Example 8.2
Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, C = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix}$$

(a) Compare $A(BC)$ and $(AB)C$.
(b) Compare $A(B + C)$ and $AB + AC$.
(c) Compute I_2A and AI_2, where I_2 is the 2×2 identity matrix.

Solution.

(a)

$$A(BC) = (AB)C = \begin{bmatrix} 70 & 14 \\ 235 & 56 \end{bmatrix}$$

(b)

$$A(B+C) = AB + AC = \begin{bmatrix} 16 & 7 \\ 59 & 33 \end{bmatrix}$$

(c) $AI_2 = I_2A = A$ ■

Example 8.3
Let A be a 3×2 and B be a 2×4 matrices.Show that if
(a) B has a column of zeros then the same is true for AB.
(b) A has a row of zeros then the same is true for AB.

Solution.
Write

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12} + b_{21} & a_{11}b_{12} + a_{12} + b_{22} & a_{11}b_{13} + a_{12} + b_{23} & a_{11}b_{14} + a_{12} + b_{24} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} & a_{21}b_{14} + a_{22}b_{24} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} & a_{31}b_{14} + a_{32}b_{24} \end{bmatrix}$$

(a) Suppose that $b_{11} = b_{21} = 0$. Then

$$AB = \begin{bmatrix} 0 & a_{11}b_{12} + a_{12} + b_{22} & a_{11}b_{13} + a_{12} + b_{23} & a_{11}b_{14} + a_{12} + b_{24} \\ 0 & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} & a_{21}b_{14} + a_{22}b_{24} \\ 0 & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} & a_{31}b_{14} + a_{32}b_{24} \end{bmatrix}$$

(b) Suppose that $a_{21} = a_{22} = 0$. Then

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12} + b_{21} & a_{11}b_{12} + a_{12} + b_{22} & a_{11}b_{13} + a_{12} + b_{23} & a_{11}b_{14} + a_{12} + b_{24} \\ 0 & 0 & 0 & 0 \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} & a_{31}b_{14} + a_{32}b_{24} \end{bmatrix}$$ ■

Next, consider a system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & = b_2 \\ \dots\dots\dots\dots\dots\dots & \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & = b_m \end{cases}$$

Then the matrix of the coefficients of the x_i's is called the **coefficient matrix:**

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The matrix of the coefficients of the x_i's and the right hand side coefficients is called the **augmented** matrix:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

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Now, if we let

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

then the above system can be represented in matrix notation as

$$Ax = b.$$

Example 8.4
Consider the linear system

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9. \end{cases}$$

(a) Find the coefficient and augmented matrices of the linear system.
(b) Find the matrix notation.

Solution.
(a) The coefficient matrix of this system is

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

and the augmented matrix is

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

(b) We can write the given system in matrix form as

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix} \blacksquare$$

As the reader has noticed so far, most of the basic rules of arithmetic of real numbers also hold for matrices but a few do not. In Example 8.1 we have seen that matrix multiplication is not commutative. The following exercise shows that the cancellation law of numbers does not hold for matrix product.

Example 8.5
(a) Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Compare AB and AC. Is it true that $B = C$?
(b) Find two square matrices A and B such that $AB = \mathbf{0}$ but $A \neq \mathbf{0}$ and $B \neq \mathbf{0}$.

Solution.
(a) Note that $B \neq C$ even though $AB = AC = 0$.
(b) The given matrices satisfy $AB = \mathbf{0}$ with $A \neq \mathbf{0}$ and $B \neq \mathbf{0}$ ■.

Matrix multiplication shares many properties of the product of real numbers which are listed in the following theorem

Theorem 8.1
Let A be a matrix of size $m \times n$. Then
(a) $A(BC) = (AB)C$, where B is of size $n \times p$, C of size $p \times q$.
(b) $A(B + C) = AB + AC$, where B and C are of size $n \times p$.
(c) $(B + C)A = BA + CA$, where B and C are of size $l \times m$.
(d) $c(AB) = (cA)B = A(cB)$, where c denotes a scalar.

The next theorem describes a property about the transpose of a matrix.

Theorem 8.2
Let $A = (a_{ij})$, $B = (b_{ij})$ be matrices of sizes $m \times n$ and $n \times m$ respectively. Then $(AB)^T = B^T A^T$.

Example 8.6
Let A be any matrix. Show that AA^T and A^TA are symmetric matrices.

Solution.
First note that for any matrix A the matrices AA^T and A^TA are well-defined. Since $(AA^T)^T = (A^T)^TA^T = AA^T$ then AA^T is symmetric. Similarly, $(A^TA)^T = A^T(A^T)^T = A^TA$ ■

Finally, we discuss the powers of a square matrix. Let A be a square matrix of size $n \times n$. Then the non-negative powers of A are defined as follows: $A^0 = I_n$, $A^1 = A$, and for $k \geq 2$, $A^k = (A^{k-1})A$.

Example 8.7
suppose that

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Compute A^3.

Solution.
Multiplying the matrix A by itself three times we obtain

$$A^3 = \begin{bmatrix} 37 & 54 \\ 81 & 118 \end{bmatrix} \blacksquare$$

Theorem 8.3
For any non-negative integers s, t we have
(a) $A^{s+t} = A^s A^t$
(b) $(A^s)^t = A^{st}$.

Example 8.8
Let A and B be two $n \times n$ matrices.
(a) Show that $tr(AB) = tr(BA)$.
(b) Show that $AB - BA = I_n$ is impossible.

Solution.
(a) Let $A = (a_{ij})$ and $B = (b_{ij})$. Then
$tr(AB) = \sum_{i=1}^{n}(\sum_{k=1}^{n} a_{ik}b_{ki}) = \sum_{i=1}^{n}(\sum_{k=1}^{n} b_{ik}a_{ki}) = tr(BA)$.
(b) If $AB - BA = I_n$ then $0 = tr(AB) - tr(BA) = tr(AB - BA) = tr(I_n) =$
$n \geq 1$, a contradiction $\blacksquare$

12. Determinants by Cofactor Expansion

The determinant of a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is the number

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

The determinant of a 3×3 matrix can be found using the determinants of 2×2 matrices using a cofactor expansion which we discuss next.
If A is a square matrix of order n then the **minor of entry** a_{ij}, denoted by M_{ij}, is the determinant of the submatrix obtained from A by deleting the i^{th} row and the j^{th} column. The **cofactor of entry** a_{ij} is the number $C_{ij} = (-1)^{i+j}M_{ij}$.

Example 12.1
Let

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

Find the minor and the cofactor of the entry $a_{32} = 4$.

Solution.
The minor of the entry a_{32} is

$$M_{32} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$$

and the cofactor is $C_{32} = (-1)^{3+2}M_{32} = -26$ ■

Example 12.2
Find the cofactors C_{11}, C_{12}, and C_{13} of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Solution.
We have

$$C_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{31}a_{23})$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22} \blacksquare$$

The determinant of a matrix A of order n can obtained by multiplying the entries of a row (or a column) by the corresponding cofactors and adding the resulting products. Any row or column chosen will result in the same answer. More precisely, we have the expansion along row i is

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

The expansion along column j is given by

$$|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

Any row or column chosen will result in the same answer.

Example 12.3
Find the determinant of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Solution.
Using the previous example, we can find the determinant using the cofactor along the first row to obtain

$$\begin{aligned} |A| =& a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ =& a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ =& a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \blacksquare \end{aligned}$$

Remark 12.1
In general, the best strategy for evaluating a determinant by cofactor expansion is to expand along a row or a column having the largest number of zeroes.

Example 12.4
Find the determinant of each of the following matrices.
(a)

$$A = \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Solution.
(a) Expanding along the first column we find

$$|A| = a_{31}C_{31} = -a_{31}a_{22}a_{13}.$$

(b) Again, by expanding along the first column we obtain

$$|A| = a_{41}C_{41} = a_{11}a_{32}a_{23}a_{44} \blacksquare$$

(c) Expanding along the last column we find

$$|A| = a_{44}C_{44} = a_{11}a_{22}a_{33}a_{44} \blacksquare$$

Example 12.5
Evaluate the determinant of the following matrix.

$$\begin{vmatrix} 2 & 7 & -3 & 8 & 3 \\ 0 & -3 & 7 & 5 & 1 \\ 0 & 0 & 6 & 7 & 6 \\ 0 & 0 & 0 & 9 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix}$$

Solution.
The given matrix is upper triangular so that the determinant is the product of entries on the main diagonal, i.e. equals to -1296 ■

Example 12.6
Use cofactor expansion along the first column to find $|A|$ where

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

Solution.
Expanding along the first column we find

$$\begin{aligned} |A| &= 3C_{11} + C_{21} + 2C_{31} + 3C_{41} \\ &= 3M_{11} - M_{21} + 2M_{31} - 3M_{41} \\ &= 3(-54) + 78 + 2(60) - 3(18) = -18 \end{aligned}$$ ■

14. Properties of the Determinant

In this section we shall exhibit some of the fundamental properties of the determinant. One of the immediate consequences of these properties will be an important determinant test for the invertibility of a square matrix.
The first result relates the invertibility of a square matrix to its determinant.

Theorem 14.1
If A is an $n \times n$ matrix then A is nonsingular if and only if $|A| \neq 0$.

Combining Theorem 11.1 with Theorem 14.1, we have

Theorem 14.2
The following statements are all equivalent:
(i) A is nonsingular.
(ii) $|A| \neq 0$.
(iii) A is row equivalent to I_n.
(iv) The homogeneous systen $Ax = \mathbf{0}$ has only the trivial solution.
(v) $rank(A) = n$.

Example 14.1
Prove that $|A| = 0$ if and only if $Ax = \mathbf{0}$ has a nontrivial solution.

Solution.
If $|A| = 0$ then according to Theorem 14.2 the homogeneous system $Ax = \mathbf{0}$ must have a nontrivial solution. Conversely, if the homogeneous system $Ax = \mathbf{0}$ has a nontrivial solution then A must be singular by Theorem 14.2. By Theorem 14.2 (a), $|A| = 0$ ■

Our next major result in this section concerns the determinant of a product of matrices.

Theorem 14.3
If A and B are $n \times n$ matrices then $|AB| = |A||B|$.

Example 14.2
Is it true that $|A + B| = |A| + |B|$?

Solution.
No. Consider the following matrices.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then $|A + B| = |\mathbf{0}| = 0$ and $|A| + |B| = -2$ ■

Example 14.3
Show that if A is invertible then $|A^{-1}| = \frac{1}{|A|}$.

Solution.
If A is invertible then $A^{-1}A = I_n$. Taking the determinant of both sides we find $|A^{-1}||A| = 1$. That is, $|A^{-1}| = \frac{1}{|A|}$. Note that since A is invertible then $|A| \neq 0$ ■

Example 14.4
Let A and B be two **similar** square matrices , i.e. there exists a nonsingular matrix P such that $A = P^{-1}BP$. Show that $|A| = |B|$.

Solution.
Using Theorem 14.3 and Example 14.3 we have, $|A| = |P^{-1}BP| = |P^{-1}||B||P| = \frac{1}{|P|}|B||P| = |B|$. Note that since P is nonsingular then $|P| \neq 0$ ■

Step 7: $r_1 \leftarrow r_1 - 5r_3$ and $r_2 \leftarrow r_2 - r_3$

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \blacksquare$$

Remark 4.2
It can be shown that no matter how the elementary row operations are varied, one will always arrive at the same reduced row-echelon form; that is the reduced row echelon form is unique. On the contrary row-echelon form is **not** unique. However, the number of leading 1's of two different row-echelon forms is the same. That is, two row-echelon matrices have the same number of non-ero rows. This number is knwon as the rank of the matrix.

Example 4.4
Consider the system

$$\begin{cases} ax + by = k \\ cx + dy = l. \end{cases}$$

Show that if $ad - bc \neq 0$ then the reduced row-echelon form of the coefficient matrix is the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution.
The coefficient matrix is the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Assume first that $a \neq 0$. Using Gaussian elimination we reduce the above matrix into row-echelon form as follows:

Step 1: $r_2 \leftarrow ar_2 - cr_1$

$$\begin{bmatrix} a & b \\ 0 & ad - bc \end{bmatrix}$$

Step 2: $r_2 \leftarrow \frac{1}{ad-bc} r_2$

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

Step 3: $r_1 \leftarrow r_1 - br_2$

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

Step 4: $r_1 \leftarrow \frac{1}{a} r_1$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Next, assume that $a = 0$. Then $c \neq 0$ and $b \neq 0$. Following the steps of Gauss-Jordan elimination algorithm we find

Step 1: $r_1 \leftrightarrow r_2$

$$\begin{bmatrix} c & d \\ 0 & b \end{bmatrix}$$

Step 2: $r_1 \leftarrow \frac{1}{c} r_1$ and $r_2 \leftarrow \frac{1}{b} r_2$

$$\begin{bmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{bmatrix}$$

Step 3: $r_1 \leftarrow r_1 - \frac{d}{c} r_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \blacksquare$$

Example 4.5
Find the rank of each of the following matrices
(a)

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}$$

(b)

$$B = \begin{bmatrix} 3 & 1 & 0 & 1 & -9 \\ 0 & -2 & 12 & -8 & -6 \\ 2 & -3 & 22 & -14 & -17 \end{bmatrix}$$

Solution.
(a) We use Gaussian elimination to reduce the given matrix into row-echelon form as follows:

Step 1: $r_2 \leftarrow r_2 - r_1$

$$\begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

Step 2: $r_1 \leftrightarrow r_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

Step 3: $r_2 \leftarrow r_2 - 2r_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$$

Step 4: $r_3 \leftarrow r_3 - r_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

Thus, $rank(A) = 3$.

(b) As in (a), we reduce the matrix into row-echelon form as follows:

Step 1: $r_1 \leftarrow r_1 - r_3$

$$\begin{bmatrix} 1 & 4 & -22 & 15 & 8 \\ 0 & -2 & 12 & -8 & -6 \\ 2 & -3 & 22 & -14 & -17 \end{bmatrix}$$

Step 2: $r_3 \leftarrow r_3 - 2r_1$

$$\begin{bmatrix} 1 & 4 & -22 & 15 & 25 \\ 0 & -2 & 12 & -8 & -6 \\ 0 & -11 & -22 & -44 & -33 \end{bmatrix}$$

Step 3: $r_2 \leftarrow -\frac{1}{2}r_2$

$$\begin{bmatrix} 1 & 4 & -22 & 15 & 8 \\ 0 & 1 & -6 & 4 & 3 \\ 0 & -11 & -22 & -44 & -33 \end{bmatrix}$$

Step 4: $r_3 \leftarrow r_3 + 11r_2$

$$\begin{bmatrix} 1 & 4 & -22 & 15 & 8 \\ 0 & 1 & -6 & 4 & 3 \\ 0 & 0 & -88 & 0 & 0 \end{bmatrix}$$

Step 5: $r_3 \leftarrow \frac{1}{8} r_3$

$r_3 \leftarrow -\frac{1}{88} r_3$

$$\begin{bmatrix} 1 & 4 & -22 & 15 & 8 \\ 0 & 1 & -6 & 4 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Hence, $rank(B) = 3$ ■

rank(B) = 2

9. The Inverse of a Square Matrix

Most problems in practice reduces to a system with matrix notation $Ax = b$. Thus, in order to get x we must somehow be able to eliminate the coefficient matrix A. One is tempted to try to divide by A. Unfortunately such an operation has not been defined for matrices. In this section we introduce a special type of square matrices and formulate the matrix analogue of numerical division. Recall that the $n \times n$ identity square matrix is the matrix I_n whose main diagonal entries are 1 and off diagonal entries are 0.

A square matrix A of size n is called **invertible** or **non-singular** if there exists a square matrix B of the same size such that $AB = BA = I_n$. In this case B is called the **inverse of** A. A square matrix that is not invertible is called **singular**.

Example 9.1

Show that the matrix

$$B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

is the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution.

Using matrix multiplication one checks that $AB = BA = I_2$ ■

Example 9.2

Show that the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is singular.

Solution.

Let $B = (b_{ij})$ be a 2×2 matrix. If $BA = I_2$ then the $(2,2)$-th entry of BA is zero while the $(2,2)-$entry of I_2 is 1, which is impossible. Thus, A is singular ■

It is important to keep in mind that the concept of invertibility is defined only for square matrices. In other words, it is possible to have a matrix A of size $m \times n$ and a matrix B of size $n \times m$ such that $AB = I_m$. It would be wrong to conclude that A is invertible and B is its inverse.

Example 9.3
Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Show that $AB = I_2$.

Solution.
Simple matrix multiplication shows that $AB = I_2$. However, this does not imply that B is the inverse of A since BA is undefnied so that the condition $BA = I_2$ fails ■

Example 9.4
Show that the identity matrix is invertible but the zero matrix is not.

Solution.
Since $I_n I_n = I_n$, I_n is nonsingular and its inverse is I_n. Now, for any $n \times n$ matrix B we have $B\mathbf{0} = \mathbf{0} \neq I_n$ so that the zero matrix is not invertible ■

Now if A is a nonsingular matrix then how many different inverses does it possess? The answer to this question is provided by the following theorem.

Theorem 9.1
The inverse of a matrix is unique.

Proof.
Suppose A has two inverses B and C. We will show that $B = C$. Indeed, $B = BI_n = B(AC) = (BA)C = I_nC = C$ ■

Since an invertible matrix A has a unique inverse, we will denote it from now on by A^{-1}.
For an invertible matrix A one can now define the negative power of a square matrix as follows: For any positive integer $n \geq 1$, we define $A^{-n} = (A^{-1})^n$.
The next theorem lists some of the useful facts about inverse matrices.

Theorem 9.2
Let A and B be two square matrices of the same size $n \times n$.
(a) If A and B are invertible matrices then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
(b) If A is invertible then A^{-1} is invertible and $(A^{-1})^{-1} = A$.
(c) If A is invertible then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Proof.

(a) If A and B are invertible then $AA^{-1} = A^{-1}A = I_n$ and $BB^{-1} = B^{-1}B = I_n$. In This case, $(AB)(B^{-1}A^{-1}) = A[B(B^{-1}A^{-1})] = A[(BB^{-1})A^{-1}] = A(I_nA^{-1}) = AA^{-1} = I_n$. Similarly, $(B^{-1}A^{-1})(AB) = I_n$. It follows that $B^{-1}A^{-1}$ is the inverse of AB.

(b) Since $A^{-1}A = AA^{-1} = I_n$, A is the inverse of A^{-1}, i.e. $(A^{-1})^{-1} = A$.

(c) Since $AA^{-1} = A^{-1}A = I_n$, by taking the transpose of both sides we get $(A^{-1})^TA^T = A^T(A^{-1})^T = I_n$. This shows that A^T is invertible with inverse $(A^{-1})^T$ ■

Example 9.5

(a) Under what conditions a diagonal matrix is invertible?

(b) Is the sum of two invertible matrices necessarily invertible?

Solution.

(a) Let $D = (d_{ii})$ be a diagonal $n \times n$ matrix. Let $B = (b_{ij})$ be an $n \times n$ matrix such that $DB = I_n$ and let $DB = (c_{ij})$. Then using matrix multiplication we find $c_{ij} = \sum_{k=1}^{n} d_{ik}b_{kj}$. If $i \neq j$ then $c_{ij} = d_{ii}b_{ij} = 0$ and $c_{ii} = d_{ii}b_{ii} = 1$. If $d_{ii} \neq 0$ for all $1 \leq i \leq n$ then $b_{ij} = 0$ for $i \neq j$ and $b_{ii} = \frac{1}{d_{ii}}$. Thus, if $d_{11}d_{22} \cdots d_{nn} \neq 0$ then D is invertible and its inverse is the diagonal matrix $D^{-1} = (\frac{1}{d_{ii}})$.

(b) The following two matrices are invertible but their sum , which is the zero matrix, is not.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} ■$$

Example 9.6

Consider the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Ex 9-5

invertible

Show that if $ad - bc \neq 0$ then A^{-1} exists and is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Solution.
Let

$$B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

be a matrix such that $BA = I_2$. Then using matrix multiplication we find

$$\begin{bmatrix} ax + cy & bx + dy \\ az + cw & bz + dw \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating corresponding entries we obtain the following systems of linear equations in the unknowns x, y, z and w.

$$\begin{cases} ax + cy = 1 \\ bx + dy = 0 \end{cases}$$

and

$$\begin{cases} az + cw = 0 \\ bz + dw = 1 \end{cases}$$

In the first system, using elimination we find $(ad - bc)y = -b$ and $(ad - bc)x = d$. Similarly, using the second system we find $(ad - bc)z = -c$ and $(ad - bc)w = a$. If $ad - bc \neq 0$ then one can solve for x, y, z, and w and in this case $B = A^{-1}$ as given in the statement of the problem ■

Finally, we mention here that matrix inverses can be used to solve systems of linear equations as suggested by the following theorem.

Theorem 9.3
If A is an $n \times n$ invertible matrix and b is a column matrix then the equation $Ax = b$ has a unique solution $x = A^{-1}b$.

Proof.
Since $A(A^{-1}b) = (AA^{-1})b = I_n b = b$, we find that $A^{-1}b$ is a solution to the equation $Ax = b$. Now, if y is another solution then $y = I_n y = (A^{-1}A)y = A^{-1}(Ay) = A^{-1}b$ ■

Example 9.7
If A is invertible and $k \neq 0$ show that $(kA)^{-1} = \frac{1}{k}A^{-1}$.

Solution.
Suppose that A is invertible and $k \neq 0$. Then $(kA)A^{-1} = k(AA^{-1}) = kI_n$. This implies $(kA)(\frac{1}{k}A^{-1}) = I_n$. Thus, kA is invertible with inverse equals to $\frac{1}{k}A^{-1}$ ■

16. Application of Determinants to Systems: Cramer's Rule

Cramer's rule is another method for solving a linear system of n equations in n unknowns. This method is reasonable for inverting, for example, a 3×3 matrix by hand; however, the inversion method discussed before is more efficient for larger matrices.

Theorem 16.1
Let $Ax = b$ be a matrix equation with $A = (a_{ij})$, $x = (x_i)$, $b = (b_i)$. Then we have the following matrix equation

$$\begin{bmatrix} |A|x_1 \\ |A|x_2 \\ \vdots \\ |A|x_n \end{bmatrix} = \begin{bmatrix} |A_1| \\ |A_2| \\ \vdots \\ |A_n| \end{bmatrix}$$

where A_i is the matrix obtained from A by replacing its i^{th} column by b. It follows that

(1) If $|A| \neq 0$ then the above system has a unique solution given by

$$x_i = \frac{|A_i|}{|A|},$$

where $1 \leq i \leq n$.
(2) If $|A| = 0$ and $|A_i| \neq 0$ for some i then the system has no solution.
(3) If $|A| = |A_1| = \cdots = |A_n| = 0$ then the system has an infinite number of solutions.

Proof.
We have the following chain of equalities

$$\begin{aligned} |A|x &= |A|(I_n x) \\ &= (|A|I_n)x \\ &= adj(A)Ax \\ &= adj(A)b \end{aligned}$$

since $A^{-1} = \frac{adj(A)}{|A|}$

Example 16.2
Use Cramer's rule to solve

$$\left\{\begin{array}{rcrcrcr} 5x_1 & - & 3x_2 & - & 10x_3 & = & -9 \\ 2x_1 & + & 2x_2 & - & 3x_3 & = & 4 \\ -3x_1 & - & x_2 & + & 5x_3 & = & 1. \end{array}\right.$$

Solution.
By Cramer's rule we have

$$A = \left[\begin{array}{rrr} 5 & -3 & -10 \\ 2 & 2 & -3 \\ -3 & -1 & 5 \end{array}\right], |A| = -2.$$

$$A_1 = \left[\begin{array}{rrr} -9 & -3 & -10 \\ 4 & 2 & -3 \\ 1 & -1 & 5 \end{array}\right], |A_1| = 66.$$

$$A_2 = \left[\begin{array}{rrr} 5 & -9 & -10 \\ 2 & 4 & -3 \\ -3 & 1 & 5 \end{array}\right], |A_2| = -16.$$

$$A_3 = \left[\begin{array}{rrr} 5 & -3 & -9 \\ 2 & 2 & 4 \\ -3 & -1 & 1 \end{array}\right], |A_3| = 36.$$

Thus, $x_1 = \frac{|A_1|}{|A|} = -33, x_2 = \frac{|A_2|}{|A|} = 8, x_3 = \frac{|A_3|}{|A|} = -18$ ■