UNIVERSITY OF BAGHDAD - COLLEGE OF SCIENCE - DEPT. OF MATHEMATICS

FUNCTIONAL ANALYSIS

FOURTH CLASS

SECOND CORES

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Chapter 1

Basic Elements of Metric Spaces

1.1 Vector Spaces

Definition 1.1.1. A vector space $\mathcal{V}$ is a collection of objects with a (vector) addition and scalar multiplication defined that closed under both operations and which in addition satisfies the following axioms:

1. $(\alpha + \beta)x = \alpha x + \beta x$ for all $x \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{F}$.
2. $\alpha(\beta x) = (\alpha\beta)x$.
3. $x + y = y + x$ for all $x, y \in \mathcal{V}$.
4. $x + (y + z) = (x + y) + z$ for all $x, y, z \in \mathcal{V}$.
5. $\alpha(x + y) = \alpha x + \alpha y$.
6. There is $0 \in \mathcal{V}$ such that $0 + x = x$; 0 is usually called the origin.
7. $0x = 0$.
8. $ex = x$ where e is the multiplicative unit in $\mathbb{F}$.

Examples 1.1.2.

1. $\mathbb{R}^n = \{(a_1, a_2, \cdots, a_n) | a_1, a_2, \cdots, a_n \in \mathbb{R}\}$ n-dimensional space, for all $n \geq 1$.

2. $\mathbb{C}^2$ and $\mathbb{C}^n$ respectively to $\mathbb{R}^2$ and $\mathbb{R}^n$ where the underlying field is $\mathbb{C}$, the complex numbers.

3. $P_n = \left\{\sum_{j=0}^{n} a_j x^j | a_0, a_1, \cdots, a_n \in \mathbb{R}\right\}$ is called **the polynomial space of all polynomials of degree "n"**. Note this includes not just the polynomials of exactly degree "n" but also those of lesser degree.

4. $\ell_p = \{(a_i, \cdots) | a_i \in \mathbb{R}, \sum |a_i|^p < \infty\}$. This space is comprised of vectors in the form of infinite-tuples of numbers. Properly we would write

$$\ell_p(\mathbb{R}) \text{ or } \ell_p(\mathbb{C})$$

to designate the field.

Definition 1.1.3. Let $\mathcal{V}$ be a vector space and $\mathcal{U} \subset \mathcal{V}$. We will call $\mathcal{U}$ **a subspace of $\mathcal{V}$** if $\mathcal{U}$ is closed under vector addition, scalar multiplication and satisfies all of the vector space axioms.

Example 1.1.4. let $\mathcal{V} = \mathbb{R}^3 = \{(a, b, c) | a, b, c \in \mathbb{R}\}$
$\mathcal{U} = \{(a, b, 0) | a, b \in \mathbb{R}\}$.
Clearly $U \subset \mathcal{V}$ and also $\mathcal{U}$ is a subspace of $\mathcal{V}$.

Definition 1.1.5. let $S \subset \mathcal{V}$, a vector space, have the form

$$S = \{v_1, v_2, \cdots, v_k\}.$$

The span of S is the set

$$\mathcal{U} = \left\{\sum_{j=1}^{k} a_j v_j | a_1, \cdots, a_k \in \mathbb{R}\right\}.$$

Remark 1.1.6. span of the set S is a subspace.

Definition 1.1.7. Let X be a vector space. A set of "n" vectors $\{x_1, \cdots, x_n\} \subset X$ is called **linear independent**, if the following equation gives that

$$\sum_{j=1}^{n} \alpha_j x_j = 0 \Rightarrow \alpha_1 = \cdots = \alpha_n = 0$$

is the only solution. If there is just one $\alpha_i \neq 0$ then the system $\{x_1, \cdots, x_n\}$ is called **linear dependent**.

Definition 1.1.8. The set $B = \{x_1, \cdots, x_n\}$ is called a **basis** of X if:

1. the elements of B are linear independent.
2. and $span\,\{x_1, \cdots, x_n\} = X$.

Remark 1.1.9. If every $x \in X$ can be expressed as a unique linear combination of the elements out of the set $\{x_1, \cdots, x_n\}$ then that set is called a **basis** of X.

Definition 1.1.10. The number of elements, needed to describe a vector space X, is called the **dimension** of X, abbreviated by $\dim X$.

Remark 1.1.11. Let X be a vector space.

1. If $X = \{0\}$, then $\dim X = 0$.
2. if X has a basis $\{x_1, \cdots, x_n\}$, then $\dim X = n$.
3. If $X \neq \{0\}$ has no finite basis, then $\dim X = \infty$.

Definition 1.1.12. Let V, W be two vector spaces. A function $T : V \to W$ is called a **linear transformation** from V to W if the following hold for all vectors u, v in V and for all scalars k.

1. $T(u+v) = T(u) + T(v)$
2. $T(ku) = kT(u)$

Definition 1.1.13. Let $T : V \to W$ is a linear transformation.

1. The set of all vectors v in V for which $T(v) = \vec{0}$ is called the *kernel* of T.

   We denote the kernel of T by $\ker(T)$.

   i.e., $\ker(T) := \{v \in V : T(v) = 0\}$.

2. The set of all images $T(v)$ of vectors in V via the transformation T is called the *range* of T. We denote the range of T by $R(T)$.

   i.e., $R(T) := \{T(v) : v \in V\}$.

1.2 Metric spaces

Definition 1.2.1. A **metric space** is a non-empty set X with a function

$$d(.,.) : X \times X \to \mathbb{R}$$

satisfying, for x, y, and z in X,

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ if and only if $x = y$,
3. $d(x, y) = d(y, x)$,
4. $d(x, y) + d(y, z) \geq d(x, z)$ (the triangle inequality).

Examples 1.2.2.

1. Let $X = \mathbb{C}$, with $d(z, w) = |z - w|$.

2. Suppose X is a non-empty set and that $d : X \times X \to \mathbb{R}$. defined as
$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ \\ 0 & \text{if } x = y \end{cases}$$
is a metric which is called **the discrete metric on X.**

3. For any integer $n \geq 1$, the function $d_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by
$$d_1(x, y) = \sum_{j=1}^{n} |x_j - y_j|$$
is a metric on the set $\mathbb{R}^n$.

4. For any integer $n \geq 1$, the function $d_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by
$$d_2(x, y) = \left(\sum_{j=1}^{n} |x_j - y_j|^2 \right)^{1/2}$$
is a metric on the set $\mathbb{R}^n$. This metric will be called **the standard metric on $\mathbb{R}^n$.**

5. For any integer $n \geq 1$, the function $d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by
$$d_p(x, y) = \left(\sum_{j=1}^{n} |x_j - y_j|^p \right)^{1/p}$$
is a metric on the set $\mathbb{R}^n$.

6. For any integer $n \geq 1$, the function $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by
$$d_\infty(x, y) = \max_{1 \leq j \leq n} |x_j - y_j|$$
is a metric on the set $\mathbb{R}^n$.

1.3 Normed vector spaces

Definition 1.3.1. A **normed linear space** $(V, \|.\|)$ is a (real or complex) linear space V together with a function

$$\|.\| : V \to \mathbb{R}.$$

called a **norm** satisfying four conditions:

1. $\|v\| \geq 0$ for all $v \in V$.
2. $\|v\| = 0$ if and only if $v = 0$.
3. $\|\lambda v\| = |\lambda| \, \|v\|$ for all $v \in V$ and $\lambda \in \mathbb{R}$.
4. $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$ (triangle inequality).

Examples 1.3.2.

1. $V = \mathbb{R}$. with $\|x\| = |x|$.
2. $V = \mathbb{C}$. with $\|z\| = |z|$.
3. $V = \mathbb{R}^n$ with

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2};$$

for $x = (x_1, x_2, \cdots, x_n)$. This is the **usual, standard, or Euclidean norm** on $\mathbb{R}^n$. It is usually denoted by $\|.\|_2$.

Proof. For $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$

(a) $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \geq 0$.

(b) $\|x\|^2 = 0$ iff $\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = 0$ iff $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$ iff $x_1 = \cdots = x_n = 0$.

(c)

$$\begin{aligned} \|\lambda x\| &= \sqrt{(\lambda x_1)^2 + (\lambda x_2)^2 + \cdots + (\lambda x_n)^2} \\ &= \sqrt{\lambda^2 x_1^2 + \lambda^2 x_2^2 + \cdots + \lambda^2 x_n^2} \\ &= \sqrt{\lambda^2 (x_1^2 + x_2^2 + \cdots + x_n^2)} \\ &= |\lambda| \, \|x\| \end{aligned}$$

(d)

$$\begin{aligned} \|x + y\| &= \sqrt{(x_1 + y_1)^2 + (x_2 + y_2)^2 + \cdots + (x_n + y_n)^2} \\ &\leq \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} + \sqrt{y_1^2 + y_2^2 + \cdots + y_n^2} \\ &= \|x\| + \|y\| \end{aligned}$$

□

4. Let $C_{\mathbb{F}}[a, b]$ be the vector space of continuous functions, where $\mathbb{F}$ is a field either real or complex, on $[a, b]$, under pointwise addition and scalar multiplication.

$$C_{\mathbb{F}}[a, b] = \{f : [a, b] \to \mathbb{F} : f \text{ is continuous function}\}$$

Define a norm on $C[a, b]$ as

$$\|f\| = \sup\{|f(x)| : x \in [a, b]\}.$$

Then $(C_{\mathbb{F}}[a, b], \|.\|)$ is a normed space.

Proof. Let $f, g \in C_{\mathbb{F}}[a,b]$ and $\lambda \in_{\mathbb{F}}$

(a) $\|f\| = \sup\{|f(x)| : x \in [a,b]\} \geq 0$.

(b) $\|f\| = 0$ iff $\sup\{|f(x)| : x \in [a,b]\} = 0$ iff $f(x) = 0$ for all $x \in [a,b]$.

(c)

$$\begin{aligned} \|\lambda f\| &= \sup\{|\lambda f(x)| : x \in [a,b]\} \\ &= |\lambda| \sup\{|f(x)| : x \in [a,b]\} \\ &= |\lambda| \, \|f\| \end{aligned}$$

(d)

$$\begin{aligned} \|f + g\| &= \sup\{|f(x) + g(x)| : x \in [a,b]\} \\ &\leq \sup\{|f(x)| : x \in [a,b]\} + \sup\{|g(x)| : x \in [a,b]\} \\ &= \|f\| + \|g\| \end{aligned}$$

□

Remark 1.3.3. A normed vector space $(X, \|.\|)$ is a metric space with the metric

$$d(x,y) = \|x - y\|.$$

Homework 1. For $1 \leq p < \infty$, define

$$\ell^p(\mathbb{N}) := \left\{ x = \{x_n\}_{n=1}^{\infty} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

with the p-norm

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$$

And for $p = \infty$ define

$$\|x\|_\infty = \sup\{|x_n| : n \geq 1\}$$

Prove that $\left(\ell^p(\mathbb{N}), \|x\|_p\right)$ is normed space for all $1 \leq p < \infty$ and $p = \infty$.

1.4 Inner Product spaces

Definition 1.4.1. Let $\mathcal{X}$ be a vector space over a field $\mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. An **inner product** is a map $\langle . \rangle : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{F}$ satisfying, for x, y, and z in $\mathcal{X}$ and scalars $\alpha \in \mathbb{F}$,

1. $\langle x, x \rangle \geq 0$ with $\langle x, x \rangle = 0$ (if and) only if $x = 0$,
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all x, y in $\mathcal{X}$,
3. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
4. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

Remark 1.4.2. The inner product is tool to fined:

1. The length of a vector x; $\langle x, x \rangle = \|x\|^2$.
2. The angle between two vectors x, y. Note that:
   - $\langle x, y \rangle = 0$, if and only if $x \perp y$.
   - If $\langle x, y \rangle > 0$, then $0 < \theta < \pi/2$.
   - If $\langle x, y \rangle < 0$, then $\pi/2 < \theta < \pi$.
3. The scalar projection of vector u in the direction of vector v which is $|\langle u, v \rangle|$.

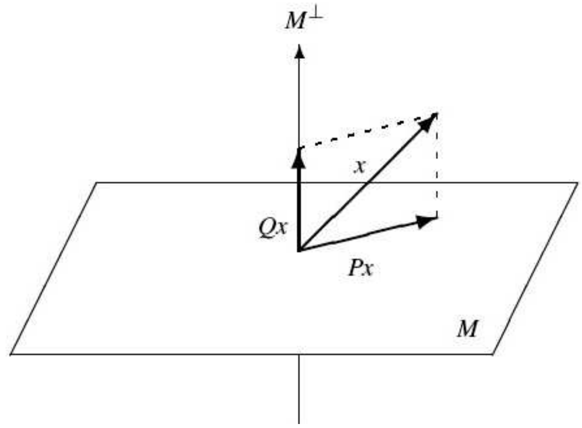

Figure 1.1: The projective vector

Example 1.4.3.

1. The function $\langle .,. \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$, is an inner product on $\mathbb{R}^n$. This inner product will be called **the standard inner product on $\mathbb{R}^n$**.

2. The function $\langle .,. \rangle : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$ defined by $\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$, is an inner product on $\mathbb{C}^n$. This inner product will be called **the standard inner product on $\mathbb{C}^n$**.

3. If $a = \{a_i\}$, $b = \{b_i\} \in \ell^2$ then the sequence $\left\{a_i \overline{b_i}\right\} \in \ell^1$ and the function $\langle .,. \rangle : \mathbb{F}^n \times \mathbb{F}^n \longrightarrow \mathbb{F}$ defined by $\langle a, b \rangle = \sum_{i=1}^{n} a_i \overline{b_i}$ is an inner product on ℓ^2. This inner product will be called **the standard inner product on ℓ^2**.

Proposition 1.4.4 (Cauchy-Schwarz inequality)**.** *If $\langle .,. \rangle$ is an inner product on a vector space $\mathcal{X}$, then for all x and y in $\mathcal{X}$ we have*

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle .$$

In particular, The equality in Cauchy - Schwarz inequality holds if and only if x and y are dependent

Proof. If one of the two vectors is zero then both sides are zero so we may assume that both x, y are non-zero. Let $t \in \mathbb{C}$. Then

$$\begin{aligned} 0 \leq \|x+ty\|^2 &= \langle x+ty, x+ty \rangle \\ &= \langle x,x \rangle + \langle x,ty \rangle + \langle yt,x \rangle + \langle ty,ty \rangle \\ &= \langle x,x \rangle + \bar{t}\langle x,y \rangle + t\overline{\langle x,y \rangle} + |t|^2 \langle y,y \rangle \\ &= \langle x,x \rangle + 2Re(t\overline{\langle x,y \rangle}) + |t|^2 \langle y,y \rangle \end{aligned}$$

Now choose $t := -\frac{\langle x,y \rangle}{\langle y,y \rangle}$. Then we get

$$0 \leq \langle x,x \rangle + 2Re(-\frac{|\langle x,y \rangle|^2}{\langle y,y \rangle}) + \frac{|\langle x,y \rangle|^2}{\langle y,y \rangle} = \langle x,x \rangle - \frac{|\langle x,y \rangle|^2}{\langle y,y \rangle}$$

And hence $|\langle x,y \rangle| \leq \|x\|\|y\|$ Note that if $y = \lambda x$ for $\lambda \in \mathbb{C}$ then equality holds:

$$|\lambda|^2 |\langle x,x \rangle| = |\lambda|^2 \|x\|\|x\|$$

Hence

$$|\langle x,x \rangle| = \|x\|^2$$

□

Proposition 1.4.5. *If $\langle .,. \rangle$ is an inner product on a vector space $\mathcal{X}$, then for all x and y in $\mathcal{X}$ we have*

$$\|x\| = \langle x,x \rangle^{\frac{1}{2}}$$

is a norm on $\mathcal{X}$.

Remark 1.4.6. The norm $\|x\| = \langle x,x \rangle^{\frac{1}{2}}$ defined in above proposition (1.4.5) on the inner product space $\mathcal{X}$ is said to be **induced by the inner product** $\langle .,. \rangle$.

Theorem 1.4.7 (The Parallelogram Rule). *Let $\mathcal{X}$ be an inner product space with inner product $\langle .,. \rangle$ and induced norm $\|.\|$. Then for all $x,\ y \in \mathcal{X}$:*

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

Its name comes from picturing the relationship for vectors in, say, $\mathbb{R}^2$; see Figure 3.2

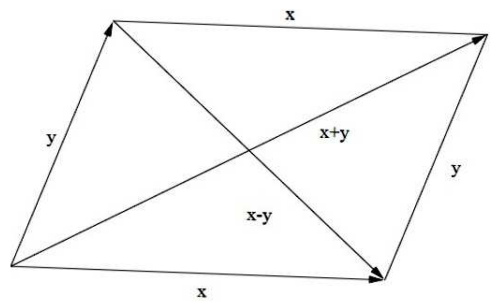

Figure 1.2: The parallelogram equality

Remark 1.4.8. One way to show that a given norm on a vector space is not induced by an inner product is to show that it does not satisfy the parallelogram rule.

Example 1.4.9. The standard norm on the space $C[0,1]$ is not induced by an inner product.

Solution. Consider the functions $f, g \in C[0,1]$ defined by $f(x) = 1,\ g(x) = x$, $x \in [0,1]$. From the definition of the standard norm on $C[0,1]$ we have

$$\begin{aligned} \|f+g\|^2 + \|f-g\|^2 &= 4+1=5 \\ 2\left(\|f\|^2 + \|g\|^2\right) &= 2(1+1)=4 \end{aligned}$$

Thus the parallelogram rule does not hold and so the norm cannot be induced by an inner product.

Chapter 2

Complete Metric Spaces

2.1 Convergence Sequences

Definition 2.1.1. A sequence $\{x_n\}$ in a metric space $(\mathcal{X}, d)$ **converges to** $x \in \mathcal{X}$ (or the sequence $\{x_n\}$ is **convergent**) if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$, for all $n > N$.

Definition 2.1.2. A sequence $\{x_n\}$ in a metric space $(\mathcal{X}, d)$ is called **bounded** if there exists $x \in X$ and $M \in \mathbb{N}$ such that $d(x_n, x) < M$ for all $n \in \mathbb{N}$.

Proposition 2.1.3. *Every convergence sequence in a metric space is bounded.*

Proof. Let $\{x_n\}$ be a convergent sequence in a metric space X to a point x.

$\therefore \forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > N$

In particular, this is true when $\varepsilon = 1$

$\therefore \exists M \in \mathbb{N}$, such that $d(x_n, x) < 1 \; \forall n > M$

Put $K = \max\{d(x_1, x), d(x_2, x), \ldots, d(x_M, x), 1\}$

$\therefore \forall n \in \mathbb{N}, d(x_n, x) \leq K$

$\therefore$ the sequence $\{x_n\}$ is bounded. $\square$

Remark 2.1.4. In normed space the definition of convergent and bounded sequence will be:

converge sequence: A sequence $\{x_n\}$ in a normed space $\mathcal{X}$ **converges to** $x \in \mathcal{X}$ (or the sequence $\{x_n\}$ is **convergent**) if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_n - x\| < \varepsilon$, for all $n > N$.

bounded sequence: A sequence $\{x_n\}$ in a normed space $\mathcal{X}$ is called **bounded** if there exists $M \in \mathbb{N}$ such that $\|x_n\| < M$ for all $n \in \mathbb{N}$.

So the proposition (2.1.3) is still true in normed space.

Proposition 2.1.5. *Let $\mathcal{X}$ be an inner product space and suppose that for any pair of a convergent sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$, with $x_n \longrightarrow x$ and $y_n \longrightarrow y$. Then*

$$\langle x_n, y_n\rangle \longrightarrow \langle x, y\rangle \ \text{for } n \longrightarrow \infty.$$

Proof.

$$\begin{aligned}
|\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle| \\
&\leq |\langle x_n, y_n\rangle - \langle x_n, y\rangle| + |\langle x_n, y\rangle - \langle x, y\rangle| \\
&= |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \\
&\leq \|x_n\|\,\|y_n - y\| + \|x_n - x\|\,\|y\|
\end{aligned}$$

Since the sequence $\{x_n\}$ is convergent, then by (2.1.3) the sequence $\{x_n\}$ is bounded, $\therefore \|x_n\|$ is bounded.

So the right hand side of this inequality lends to zero as $n \to \infty$.

$\therefore \langle x_n, y_n\rangle \longrightarrow \langle x, y\rangle$ for $n \longrightarrow \infty$. □

Definition 2.1.6. Let $\mathcal{X}$ be a metric space. A sequence $\{x_n\}$ in $\mathcal{X}$ is said to be a **Cauchy sequence** if it has the following property: Given any $\epsilon > 0$ there exists N such that if $n, m \geq N$, then $d(x_n, x_m) < \epsilon$.

Proposition 2.1.7. *Every convergent sequence in a metric space (X, d) is a Cauchy sequence.*

Proof. Let $\{x_n\}$ be a sequence in X that converges to the limit x. Let $\epsilon > 0$.
$\because x_n \to x$
$\therefore \exists N$ such that $\forall n > N, d\left(x_n, x\right) < \dfrac{\epsilon}{2}$
$\therefore$ if $m > N$ and $n > N$
$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} = \epsilon$ □

Example 2.1.8 (The Converse of the proportion (2.1.7) is not true.). If $X = \mathbb{R} - \{0\}$, and define $d : \mathbb{R} \times \mathbb{R}$ as

$$d(x, y) = |x - y|$$

Then the sequence $\left\{\dfrac{1}{n}\right\}$ is Cauchy sequence but it is not convergent.

Definition 2.1.9. A metric space is said to be **complete** if every Cauchy sequence in $\mathcal{X}$ converges in $\mathcal{X}$.

Definition 2.1.10. The complete normed space is called **Banach space**.

Example 2.1.11. the space $C[a, b]$ is a Banach space.

Definition 2.1.12. The complete inner product space is called **Hilbert space**.

Examples 2.1.13.

1. Every finite-dimensional inner product space is a Hilbert space.

2. ℓ^2 with the standard inner product is a Hilbert space.

Remark 2.1.14. Since every inner product space has an induced norm, then every Hilbert space is a Banach space.

the converse is not true unless that one satisfied parallelogram law (1.4.7).

Definition 2.1.15. A set X is **closed** if and only if for all sequence $\{x_n\}$ in X such that $x_n \to x$, then $x \in X$.

Proposition 2.1.16. *If H is a Hilbert space and $Y \subset H$ is a linear subspace, then Y is a Hilbert space if and only if Y is closed in H.*

Proof. $\Rightarrow$) Let Y be a Hilbert space

Claim: Y is closed in H:

Let $\{x_n\}$ be a sequence in Y such that $x_n \to x$.

Assume that x is not in Y

$\therefore \{x_n\}$ is Cauchy sequence but not converge in Y.

$\therefore Y$ is not complete, which is contraction with Y is Hilbert space

$\therefore Y$ is closed.

$\Leftarrow$): Let Y be a closed subspace in H.

Claim: Y is Hilbert space:

Let $\{x_n\}$ be a Cauchy sequence in Y.

$\therefore \{x_n\}$ is Cauchy sequence in H

$\because H$ is a Hilbert space

$\therefore H$ is complete

$\therefore$ the Cauchy sequence $\{x_n\}$ is convergent in H, say x

$\because Y$ is closed

$\therefore x \in Y$

$\therefore$ the sequence $\{x_n\}$ is convergent in Y

$\therefore Y$ is a Hilbert space. $\square$

2.2 ORTHOGONALITY

Definition 2.2.1. Let $\mathcal{X}$ be an inner product space. The vectors $x, y \in \mathcal{X}$ are said to be **orthogonal** if $\langle x, y\rangle = 0$, this is symbolically written $x \perp y$.

Definition 2.2.2. For subsets M and N of an inner product space $\mathcal{X}$, one says that M, N are **orthogonal**, written $M \perp N$ if $\langle x, y\rangle = 0$ for every $x \in M$, $y \in N$. In addition **the orthogonal complement** of such M is defined as

$$M^{\perp} := \{y \in \mathcal{X} | \langle x, y\rangle = 0 \text{ for all } x \in M\}.$$

Example 2.2.3. If $\mathcal{X} = \mathbb{R}^3$ and $A = \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\}$, then $A^{\perp} = \{(0, 0, x_3) : x_3 \in \mathbb{R}\}$.

Solution. Let $x = (x_1, x_2, x_3) \in A^{\perp}$

$\therefore \forall a = (a_1, a_2, 0) \in A$ with $a_1, a_2 \in \mathbb{R}$,

$$\langle x, a\rangle = \langle (x_1, x_2, x_3), (a_1, a_2, 0)\rangle = x_1 a_1 + x_2 a_2 = 0.$$

Putting $a_1 = x_1$, $a_2 = x_2$,

$x_1^2 + x_2^2 = 0$

$x_1 = x_2 = 0$.

$\therefore x = (0, 0, x_3) \in A^{\perp}$.

Proposition 2.2.4. *If $\mathcal{X}$ is an inner product space and $A \subset \mathcal{X}$, then $0 \in A^{\perp}$.*

Proof.

$\because \langle 0, x \rangle = 0 \quad \forall x \in \mathcal{X}$

$\therefore 0 \in A^{\perp}$. □

Proposition 2.2.5. *If $\mathcal{X}$ is an inner product space and $0 \in A \subset \mathcal{X}$, then $A \cap A^{\perp} = \{0\}$, otherwise $A \cap A^{\perp} = \emptyset$.*

Proof.

If $0 \in A$, and let $x \in A \cap A^{\perp}$

$\therefore x \in A$ and $x \in A^{\perp}$

$\therefore \langle x, x \rangle = 0$

$\therefore x = 0$

$\therefore A \cap A^{\perp} = \{0\}$

Now, if $0 \notin A$,

$A \cap A^{\perp} = \emptyset$. □

Proposition 2.2.6. *If $\mathcal{X}$ is an inner product space and $A \subset \mathcal{X}$, then $\{0\}^{\perp} = X$; $X^{\perp} = \{0\}$.*

Proof.

$\because \langle x, 0 \rangle = 0 \quad \forall x \in \mathcal{X}$

$\therefore \{0\}^{\perp} = X$.

Let $y \in X^{\perp}$

$\therefore \langle y, z \rangle = 0 \quad \forall z \in X$

$\therefore \langle y, y \rangle = 0$

$\therefore y = 0.$

$\therefore X^{\perp} = \{0\}.$ □

Proposition 2.2.7. *If $\mathcal{X}$ is an inner product space and $B \subseteq A \subset \mathcal{X}$, then $A^{\perp} \subseteq B^{\perp}$.*

Proof.

Let $x \in A^{\perp}$

$\therefore \langle x, y \rangle = 0 \quad \forall y \in A$

$\because B \subseteq A$

$\therefore \langle x, y \rangle = 0 \quad \forall y \in B$

$\therefore x \in B^{\perp}$

$A^{\perp} \subseteq B^{\perp}.$ □

Proposition 2.2.8. *If $\mathcal{X}$ is an inner product space and $A \subset \mathcal{X}$ then $A \subseteq (A^{\perp})^{\perp}$.*

Proof.

Let $x \in A$

$\therefore x \in \mathcal{X}$ and $\langle x, y \rangle = 0 \quad \forall y \in A^{\perp}$

$\because (A^{\perp})^{\perp} = \left\{x \in X : \langle x, y \rangle = 0 \quad \forall y \in A^{\perp}\right\}$

$\therefore x \in (A^{\perp})^{\perp}$

$\therefore A \subseteq (A^{\perp})^{\perp}.$ □

Proposition 2.2.9. *If $\mathcal{X}$ is an inner product space and $A \subset \mathcal{X}$ then $A^{\perp}$ is a closed linear subspace of $\mathcal{X}$.*

Proof.

Let $x, y \in A^{\perp}$ and $\alpha, \beta \in \mathbb{F}$, let $z \in A$

$$\begin{aligned} \langle \alpha x + \beta y, z \rangle &= \langle \alpha x, z \rangle + \langle \beta y, z \rangle \\ &= \alpha \langle x, z \rangle + \beta \langle y, z \rangle = 0 \end{aligned}$$

$\therefore \alpha x + \beta y \in A^{\perp}$

$\because 0 \in A^{\perp}$

$\therefore A^{\perp}$ is linear space.

Now let $\{x_n\}$ be a sequence in $A^{\perp}$ such that $x_n \to x$, and let $w \in A$

$\therefore \langle x_n, w \rangle \to \langle x, w \rangle$

$\therefore 0 \to \langle x, w \rangle$

$\therefore \langle x, w \rangle = 0$

$\therefore x \in A^{\perp}$. $\square$

Proposition 2.2.10 (Pythagorean theorem)**.** *If $x_1, x_2, \ldots, x_n$ are pairwise orthogonal vectors in a Hilbert space, then*

$$\|x_1 + x_2 + \cdots + x_n\|^2 = \|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2 .$$

Proposition 2.2.11. *Let Y be a linear subspace of an inner product space $\mathcal{X}$. Then*

$$x \in Y^{\perp} \text{ if and only if } \|x\| \leq \|x - y\|, \text{ for all } y \in Y.$$

2.2.1 Orthonormal Bases

We now wish to extend the idea of an orthonormal basis to infinite-dimensional spaces.

Definition 2.2.12. An **orthonormal set** in a Hilbert space $\mathbb{H}$ is a set E with the properties:

1. For every $e \in E$, $\|e\| = 1$,
2. For distinct vectors e and x in E , $\langle e, x \rangle = 0$.

Definition 2.2.13. An **orthonormal sequence** in a Hilbert space $\mathbb{H}$ is a sequence $\{e_i\}_{i=1}^{\infty}$ with the properties:

1. $\|e_i\| = 1$, for every i
2. $\langle e_i, e_j \rangle = 0$, for every $i \neq j$.

Example 2.2.14. For an easy example of an orthonormal set (sequence) in the Hilbert space ℓ^2, take the set E of vectors $\{e_j\}_{j=1}$ where e_j has a 1 in the j–th coordinate and zeros elsewhere. (Check?)

Definition 2.2.15. An **orthonormal basis** for a Hilbert space $\mathbb{H}$ is a maximal orthonormal set;

that is, an orthonormal set that is not properly contained in any orthonormal set.

Example 2.2.16. In the ℓ^2 example above, the set $\{e_j\}_{j=1}$ is an orthonormal basis.

When is an orthonormal set in a Hilbert space an orthonormal basis?

Theorem 2.2.17 (Gram–Schmidt process)**.** *Let* $\{v_i : i = 1.2, 3 \dots\}$ *be a sequence of vectors of* $\mathbb{H}$. *Then there exists an orthonormal sequence* $\{e_i : i = 1, 2, 3, \dots\}$ *such* *that, for each integer k*

$$\mathit{span}\,\{e_1, e_2, e_3, \ldots, e_k\} \supseteq \mathit{span}\,\{v_1, v_2, v_3, \ldots, v_k\}\,.$$

If $\{v_i : i = 1.2, 3\ldots\}$ *is a linearly independent set, then the above inclusion is an equality for each* k*.*

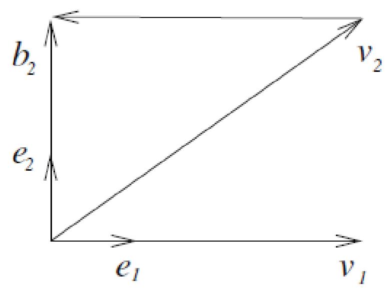

Figure 2.1: Gram–Schmidt algorithm, at stage $k = 2$

Proof. Define recursively

$$e_1 = \frac{v_1}{\|v_1\|},\ e_2 = \frac{v_2 - \langle v_2, e_1\rangle\, e_1}{\|v_2 - \langle v_2, e_1\rangle\, e_1\|}$$

and if we assume that $e_1, e_2, \ldots, e_j$ are defined,

$$e_{j+1} = \frac{v_{j+1} - \sum_{k=1}^{j} \langle v_{j+1}, e_k\rangle\, x_k}{\left\| v_{j+1} - \sum_{k=1}^{j} \langle v_{j+1}, e_k\rangle\, x_k \right\|}$$

Then the set $\{e_k\}$ is orthonormal by construction and it satisfies the requirement about span. □

Proposition 2.2.18 (Bessels Inequality). *Let* $\mathcal{X}$ *be an inner product space and let* $\{e_i\}$ *be an orthonormal set in* $\mathcal{X}$*. For any* $h \in \mathcal{X}$ *the (real) series* $\sum_{i=1}^{\infty} |\langle h, e_i\rangle|^2$ *converges and*

$$\sum_{i=1}^{\infty} |\langle h, e_i\rangle|^2 \le \|h\|^2\,.$$

Proof. Let $h \in \mathbb{H}$, Then

$$
\begin{aligned}
0 \le \left\| h - \sum_{i=1}^{n} \langle h, e_i \rangle e_i \right\|^2 &= \left\langle h - \sum_{i=1}^{n} \langle h, e_i \rangle e_i, h - \sum_{i=1}^{n} \langle h, e_i \rangle e_i \right\rangle \\
&= \|h\|^2 - \left\langle h, \sum_{i=1}^{n} \langle h, e_i \rangle e_i \right\rangle - \left\langle \sum_{i=1}^{n} \langle h, e_i \rangle e_i, h \right\rangle \\
&\quad + \sum_{i,j=1}^{n} \langle h, e_i \rangle \overline{\langle h, e_j \rangle} \langle e_i, e_j \rangle \\
&= \|h\|^2 - \left\langle h, \sum_{i=1}^{n} \langle h, e_i \rangle e_i \right\rangle - \left\langle \sum_{i=1}^{n} \langle h, e_i \rangle e_i, h \right\rangle + \sum_{i=1}^{n} |\langle h, e_i \rangle|^2 \\
&= \|h\|^2 - \sum_{i=1}^{n} |\langle h, e_i \rangle|^2 - \sum_{i=1}^{n} |\langle h, e_i \rangle|^2 + \sum_{i=1}^{n} |\langle h, e_i \rangle|^2 \\
&= \|h\|^2 - \sum_{i=1}^{n} |\langle h, e_i \rangle|^2
\end{aligned}
$$

Thus $\|h\|^2 \ge \sum_{i=1}^{n} |\langle h, e_i \rangle|^2$, and hence this sequence of partial sums is increasing and bounded above so the result follows. ☐

Remarks 2.2.19.

1. In above theorem, the case $n = 1$ is the Cauchy–Schwartz inequality

2. The geometric meaning of Bessel's inequality is that the orthogonal projection of an element h on the linear span of the elements,$\{e_i\}$, has a norm which does not exceed the norm of h (i.e. the hypothenuse in a right-angled triangle is not shorter than one of the other sides).

Lemma 2.2.20. *If $\{e_i\}_1^\infty$ is an orthonormal sequence, then for any $h \in \mathbb{H}$,*

$$\sum_{i=1}^{\infty} \langle h, e_i \rangle e_i$$

converges to a vector h_0 such that $\langle h - h_0, e_i \rangle = 0$ for all i.

Theorem 2.2.21. *If $\{e_i\}_1^\infty$ is an orthonormal sequence in a Hilbert space $\mathbb{H}$, then the following conditions are equivalent:*

1. *$\{e_i\}_1^\infty$ is an orthonormal basis.*
2. *If $h \in \mathbb{H}$ and $h \perp e_i$ for all i, then $h = 0$.*
3. *(Fourier expansion) For every $h \in \mathbb{H}$, $h = \sum_1^\infty \langle h, e_i\rangle e_i$; equality here means the convergence in the norm of $\mathbb{H}$ of the partial sums to h.*
4. *(Parsevals relation) For all h and g in $\mathbb{H}$, $\sum_1^\infty \langle h, e_i\rangle \langle e_i, g\rangle = \langle h, g\rangle$.*
5. *For every $h \in \mathbb{H}$, $\sum_1^\infty |\langle h, e_i\rangle|^2 = \|h\|^2$.*

Proof.

(1) $\Longrightarrow$ (2): If (2) is false then adding $\frac{h}{\|h\|}$ to the set $\{e_n\}_1^\infty$ gives a larger orthonormal set, contradicting (1).

(2) $\Longrightarrow$ (3): Let $h_0 = \sum_{j=1}^\infty \langle h, e_j\rangle e_j$ (this exists, by Lemma 2.2.20). Then for all i

$$\begin{aligned}\langle h - h_0, e_i\rangle &= \left\langle h - \sum_{j=1}^\infty \langle h, e_j\rangle e_j, e_i\right\rangle = \langle h, e_i\rangle - \left\langle \sum_{j=1}^\infty \langle h, e_j\rangle e_j, e_i\right\rangle \\ &= \langle h, e_i\rangle - \sum_{j=1}^\infty \langle h, e_j\rangle \langle e_j, e_i\rangle = \langle h, e_i\rangle - \langle h, e_i\rangle = 0\end{aligned}$$

and so $h = h_0$ by (2)

(3) $\Longrightarrow$ (4): Let $h_r = \sum_{i=1}^r \langle h, e_i\rangle e_i$ and $g_s = \sum_{i=1}^s \langle g, e_i\rangle e_i$. Then

$$\langle h_r, g_s\rangle = \sum_{i=1}^{\min[r,s]} \langle h, e_i\rangle \overline{\langle g, e_i\rangle}.$$

Let $r \longrightarrow \infty$ and $s \longrightarrow \infty$. Using the continuity of the inner product, it follows that

$$\langle h, g\rangle = \sum_{i=1}^\infty \langle h, e_i\rangle \overline{\langle g, e_i\rangle}.$$

(4) $\Longrightarrow$ (5): Put $g = h$ in (4).

(5) $\Longrightarrow$ (1): If $\{e_i\}_1^\infty$ is not maximal and can be enlarged by adding z, then $\langle z, e_i \rangle = 0$ for all i but also

$$1 = \|z\|^2 = \sum_{i=1}^{\infty} |\langle z, e_i \rangle|^2 = 0$$

which give a contradiction. □

Remark 2.2.22. Let $\mathbb{H}$ be a Hilbert space and let $\{e_n\}$ be an orthonormal sequence in $\mathbb{H}$. Then $\{e_n\}$ is called an **orthonormal basis for** $\mathbb{H}$ if any of the conditions in Theorem (2.2.21) hold.

Definition 2.2.23. A Hilbert space is called **separable** if it contains a countable, dense subset

Examples 2.2.24.

1. the space $\mathbb{R}$ is separable since the set of rational numbers is countable and dense in $\mathbb{R}$
2. $\mathbb{C}$ is separable since the set of complex numbers of the form p+iq, with p and q rational, is countable and dense in $\mathbb{C}$.
3. Finite dimensional normed vector spaces are separable.
4. The Hilbert space ℓ^2 is separable.

Theorem 2.2.25.

An infinite-dimensional Hilbert space $\mathbb{H}$ is separable if and only if it has an orthonormal basis.

Chapter 3

Hilbert Space Geometry

3.1 Nearest Point Property

Definition 3.1.1. A subset A of a vector space $\mathcal{X}$ is **convex** if, for all $x, y \in A$ and all $t \in [0,1]$, $tx + (1-t)y \in A$.

In other words, A is convex if, for any two points x, y in A, the line segment joining x and y also lies in A,

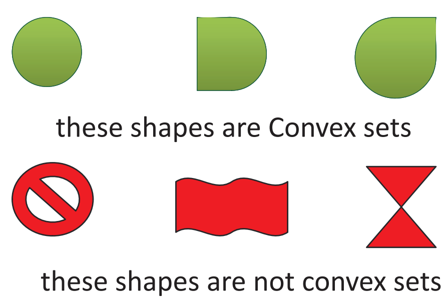

Figure 3.1: Convex and non-convex planar regions

Examples 3.1.2.

1. Every subspace is convex.

2. Every ball in a normed linear space is convex

3. Any translate $x + S := \{x + s : s \in S\}$ of a convex set S is convex. (Check?)

Proposition 3.1.3 (Nearest Point Property)**.** *Every nonempty, closed convex set* $\mathbb{K}$ *in a Hilbert space* $\mathbb{H}$ *contains a unique element of smallest norm.*

Moreover, given any $h \in \mathbb{H}$ *, there is a unique* k_0 *in* $\mathbb{K}$ *such that*

$$\|h - k_0\| = dist(h, \mathbb{K}) = \inf \{\|h - k\| : k \in \mathbb{K}\}.$$

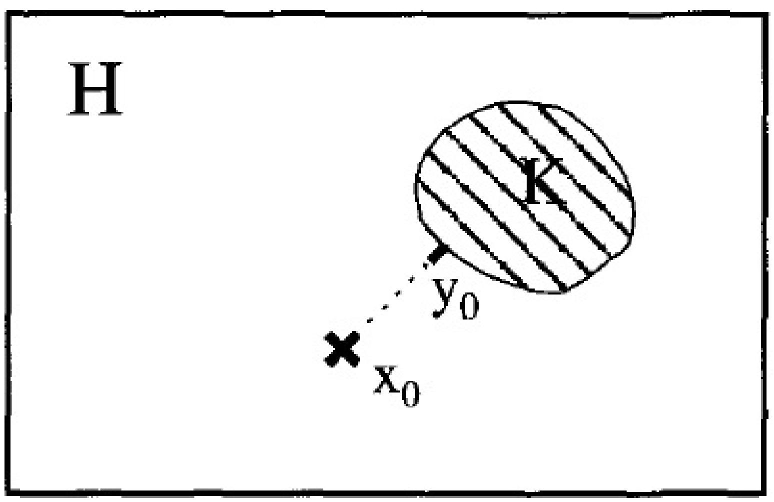

Figure 3.2: Convex and non-convex planar regions

Proof. Let $\mathbb{K}$ be a nonempty, closed convex set in a Hilbert space $\mathbb{H}$

Claim: $\mathbb{K}$ contains a unique element of smallest norm:

Let $d = \inf \{\|y\| : y \in \mathbb{K}\}$.

$\because$ $\mathbb{K}$ is closed,

$\therefore$ there is a sequence of vectors $\{x_n\}$ in $\mathbb{K}$ with $\|x_n\| \to d$.

Thus, by The parallelogram equality, for any n, m we have

$$\|x_n - x_m\|^2 + \|x_n + x_m\|^2 = 2\left(\|x_n\|^2 + \|x_m\|^2\right),$$

$$\|x_n - x_m\|^2 = 2\left(\|x_n\|^2 + \|x_m\|^2\right) - 4\left\|\frac{x_n + x_m}{2}\right\|^2,$$

$\because$ $\mathbb{K}$ is convex,

$$\frac{1}{2}x_n + \frac{1}{2}x_m \in \mathbb{K},$$

$\therefore \left\|\frac{x_n+x_m}{2}\right\|^2 \geq d^2$.

$\therefore -4\left\|\frac{x_n+x_m}{2}\right\|^2 \leq -4d^2$

$\therefore 0 \leq \|x_n - x_m\|^2 \leq 2\left(\|x_n\|^2 + \|x_m\|^2\right) - 4d^2$.

as $n, m \longrightarrow \infty$, $\|x_n - x_m\|^2 \longrightarrow 2d^2 + 2d^2 - 4d^2 = 0$

$\therefore$ $\{x_n\}$ is a Cauchy sequence,

$\because$ $\mathbb{H}$ is complete

$\{x_n\}$ converge to some $x \in \mathbb{H}$.

$\because$ $\mathbb{K}$ is closed,

$\therefore x \in \mathbb{K}$.

$\because$ the norm is continuous map

$\therefore \|x_n\| \longrightarrow \|x\|$,

$\therefore \|x\| = d$.

This gives us the existence part of the first statement.

For uniqueness: suppose $\|z\| = \|x\| = d$ for some z in $\mathbb{K}$.

$\therefore \frac{1}{2}x + \frac{1}{2}z \in \mathbb{K}$.

$\because \left\|\frac{x+z}{2}\right\| \geq d$

By the parallelogram equality,

$$\|x-z\|^2 = 2(\|x\|^2 + \|z\|^2) - 4\left\|\frac{x+z}{2}\right\|^2 \leq 4d^2 - 4d^2 = 0,$$

which forces $x = z$.

This completes the proof of the first statement.

The second statement is obtained by translation: $\because h - \mathbb{K}$ is closed and convex, then by the first part, there is a unique element in $h - \mathbb{K} := \{h - k : k \in \mathbb{K}\}$, namely x, with smallest norm, i.e. $x \in h - \mathbb{K}$ with $\|x\| = \inf\{\|y\| : y \in h - \mathbb{K}\}$.

$\therefore$ there is a unique $k_0 \in \mathbb{K}$ such that $\|h - k_0\| = dist(h, \mathbb{K}) = \inf\{\|h - k\| : k \in \mathbb{K}\}$.

□

Remarks 3.1.4.

- **The Nearest Point Property** fails to be true if we omit either the requirement that $\mathbb{K}$ be closed or convex, or change Hilbert space to Banach space in the statement.

- The last theorem say: $\forall\, x, \in \mathbb{H}\ \exists\, y \in \mathbb{K}$ such that $\|x - y\| = \inf_{a\in\mathbb{K}} \|x - a\|$

Corollary 3.1.5. *If $\mathbb{K}$ is a closed linear subspace of $\mathbb{H}$, $h \in \mathbb{H}$, and k_0 is a unique element of $\mathbb{K}$ such that $\|h - k_0\| = dist\,(h, \mathbb{K})$, then $h - k_0 \perp \mathbb{K}$.*

Conversely, if $k_0 \in \mathbb{K}$ such that $h - k_0 \perp \mathbb{K}$, then $\|h - k_0\| = dist\,(h, \mathbb{K})$.

3.2 Projection Theorems

Definition 3.2.1. For orthogonal subspaces M and N,

i.e. $M \perp N$, **the orthogonal sum** is defined as $M \oplus N$ where

$$M \oplus N := \{x + y : x \in M, y \in N\}.$$

Hence any vector $z \in M \oplus N$ has a decomposition $z = x + y$ with $x \in M$ and $y \in N$.

and

$$\langle x_1 + y_1, x_2 + y_2 \rangle = \langle x_1 + x_2 \rangle + \langle y_1 + y_2 \rangle$$

Remarks 3.2.2.

1. $\|z\|^2 = \|x + y\|^2 = \langle x + y, x + y \rangle = \langle x, x \rangle + \langle y, y \rangle = \|x\|^2 + \|y\|^2$

2. The decomposition $z = x + y$ with $x \in M$ and $y \in N$ is unique.

   Proof. Let $z \in M \oplus N$, such that $z = x_1 + y_1 = x_2 + y_2$ where $x_1, x_2 \in M$ and $y_1, y_2 \in N$

   $\therefore x_1 + y_1 = x_2 + y_2$

   $\therefore x_1 - x_2 = y_2 - y_1 \in M \cap N = \{0\}$

   $\therefore x_1 = x_2$ and $y_1 = y_2$

   □

3. If M and N are subspaces in $\mathbb{H}$, then $M \oplus N$ is a subspace too.

   Proof. Let $z_1, z_2 \in M \oplus N$ and $\alpha, \beta \in \mathbb{C}$

$\therefore z_1 = x_1 + y_1$ and $z_2 = x_2 + y_2$ where $x_1, x_2 \in M$ and $y_1, y_2 \in N$.

$$\begin{aligned} \alpha z_1 + \beta z_2 &= \alpha x_1 + \alpha y_1 + \beta x_2 + \beta y_2 \\ &= (\alpha x_1 + \beta x_2) + (\alpha y_1 + \beta y_2) \\ &\in M \oplus N \end{aligned}$$

$\because 0 \in M$ and $0 \in N$,

$\therefore 0 \in M \oplus N$

$\therefore M \oplus N$ is a subspace. □

4. If M and N are closed in $\mathbb{H}$, then $M \oplus N$ is a closed subspace too.

Proof. Let $z_n \in M \oplus N$ such that $z_n \to z$.

Claim: $z \in M \oplus N$;

Let $\epsilon > 0$

$\because \{z_n\}$ is converge,

$\therefore \{z_n\}$ is Cauchy sequence,

$\therefore \forall n, m \in \mathbb{N}, \|z_n - z_m\| \leq \epsilon$

$\because \|z_n - z_m\|^2 = \|x_n - x_m\|^2 + \|y_n - y_m\|^2 \leq \epsilon^2$

$\therefore \|x_n - x_m\| \leq \epsilon$ and $\|x_n - x_m\| \leq \epsilon$

$\therefore \{x_n\}$ is Cauchy sequence in M and $\{y_n\}$ is Cauchy sequence in N which are subspace in $\mathbb{H}$.

$\therefore \{x_n\}, \{y_n\}$ are convergent sequence in $\mathbb{H}$.

$\therefore x_n \to x$ and $y_n \to y$

$\because M, N$ are closed

$\therefore x \in M$, $y \in N$ and $z_n = x_n + y_n \to x + y$

But the limit point is unique,

$\therefore z = x + y$ which is in $M \oplus N$.

$\square$

Theorem 3.2.3 (the Projection Theorem I)**.** *Let M be a closed subspace of a Hilbert space $\mathbb{H}$. Then there is an orthogonal sum*

$$H = M \oplus M^{\perp}.$$

Proof. Let $x \in \mathbb{H}$ there is a $y \in M$ such that $\|x - y\| \leq \|x - v\|$ for all $v \in M$, by Proposition (3.1.3). Let $z = x - y$

Claim: $z \in M^{\perp}$:

For $\lambda \in \mathbb{F}$ and $v \in M$ with $\|v\| = 1$,

$\because y, v \in M$ and M is subspace,

$\therefore y + \lambda v \in M$

therefore

$$\begin{aligned} \|z\|^2 = \|x - y\|^2 &\leq \|x - (y + \lambda v)\|^2 \\ &= \|z - \lambda v\|^2 \\ &= \langle z - \lambda v, z - \lambda v \rangle \\ &\leq \langle z, z \rangle - \langle z, \lambda v \rangle - \langle \lambda v, z \rangle + \langle \lambda v, \lambda v \rangle \\ &= \langle z, z \rangle - (\langle z, \lambda v \rangle + \overline{\langle z, \lambda v \rangle}) + |\lambda|^2 \langle v, v \rangle \\ &= \|z\|^2 - 2Re\overline{\lambda} \langle z, v \rangle + |\lambda|^2 \|v\|^2 \\ &= \|z\|^2 - 2Re\overline{\lambda} \langle z, v \rangle + |\lambda|^2 \end{aligned}$$

$\therefore 2Re\overline{\lambda} \langle z, v \rangle \leq |\lambda|^2 \quad \forall \lambda \in \mathbb{F}$

So take $\lambda = \langle z, v \rangle$

$\therefore 2|\lambda|^2 = 2Re\overline{\lambda}\lambda \leq |\lambda|^2$

$\therefore |\lambda|^2 \leq 0$

$\therefore |\lambda| = 0;$

$\therefore \langle z, v\rangle = 0$ for any $v \in M$.

$\therefore z \in \mathcal{M}^{\perp}$

$\therefore x = y + z \in M \oplus M^{\perp}$

$\therefore M \oplus M^{\perp} = H.$ □

Corollary 3.2.4. *For every closed subspace $M \subset \mathbb{H}$, $M = M^{\perp\perp}$.*

Proof. By theorem (2.2.8) $M \subseteq M^{\perp\perp}$.

Now let $x \in M^{\perp\perp}$

$\therefore x \in \mathbb{H}$

By theorem (3.2.3), $\mathbb{H} = M \oplus M^{\perp}$

$\therefore x = y + z$ for $y \in M$ and $z \in M^{\perp}$,

$\because M \subseteq M^{\perp\perp}$ and $M^{\perp\perp}$ is subspace

$\therefore y \in M^{\perp\perp}$ and $z = x - y \in M^{\perp\perp}$

But $z \in M^{\perp}$

$\therefore z \in M^{\perp} \cap M^{\perp\perp} = \{0\}$

$\therefore z = 0$

$\therefore x = y \in M.$

$\therefore M^{\perp\perp} \subseteq M$

$\therefore M^{\perp\perp} = M$

□

Definition 3.2.5. Let $\mathbb{K}$ be a closed linear subspace of $\mathbb{H}$, A function

$$P : \mathbb{H} \longrightarrow \mathbb{K}$$

can be defined by

$$Ph = k_0$$

where $h - k_0 \perp \mathbb{K}$

is called **the orthogonal projector mapping**.

Remark 3.2.6. Since for every $h \in \mathbb{H}$, then there is a unique element $k_0 \in \mathbb{K}$ such that $h - k_0 \in \mathbb{K}^{\perp}$, the orthogonal projector mapping is well defined.

Proposition 3.2.7 (Projection Theorem II)**.** *Let $\mathbb{K}$ be a closed subspace of a Hilbert space $\mathbb{H}$. There is a unique pair of mappings $P : \mathbb{H} \longrightarrow \mathbb{K}$ and $Q : \mathbb{H} \longrightarrow \mathbb{K}^{\perp}$ such that $x = Px + Qx$ for all $x \in \mathbb{H}$.*

Furthermore, P and Q have the following additional properties:

1. *$x \in \mathbb{K}$ then $Px = x$ and $Qx = 0$.*
2. *$x \in \mathbb{K}^{\perp}$, then $Px = 0$ and $Qx = x$.*
3. *Px is the closest vector in $\mathbb{K}$ to x.*
4. *Qx is the closest vector in $\mathbb{K}^{\perp}$ to x.*
5. *$\|Px\|^2 + \|Qx\|^2 = \|x\|^2$ for all x.*
6. *P and Q are linear maps.*

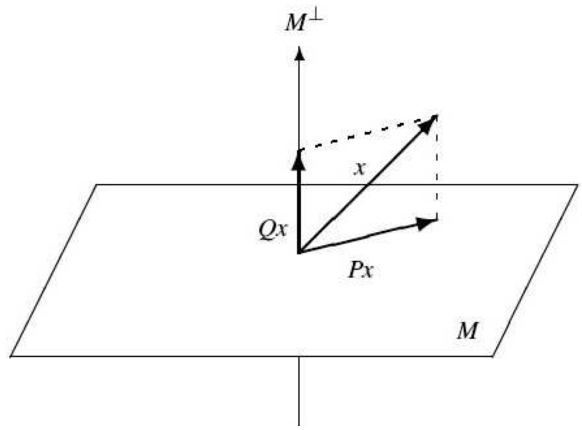

Figure 3.3: The projections P and Q

Chapter 4

Linear Operators

4.1 Bounded Linear Operators

Definition 4.1.1. If $\mathcal{X}$ and $\mathcal{Y}$ are normed linear spaces, a map $T : \mathcal{X} \longrightarrow \mathcal{Y}$ is **linear** if $T(\alpha x_1 + \beta x_2) = \alpha(Tx_1) + \beta T(x_2)$ for all x_1, x_2 in $\mathcal{X}$ and scalars α and β.

Definition 4.1.2. f is **continuous at some point** $x_0 \in \mathcal{X}$ if and only if for any neighborhood $\mathcal{V}$ of $f(x_0)$, there is a neighborhood $\mathcal{U}$ of x_0 such that $f(\mathcal{U}) \subset \mathcal{V}$

i.e. $\forall \varepsilon > 0, \exists \delta > 0$ such that, if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$

Definition 4.1.3. A function is **continuous** if it is continuous everywhere.

Remarks 4.1.4.

1. A function $f : \mathcal{X} \longrightarrow \mathcal{Y}$ between two topological spaces $\mathcal{X}$ and $\mathcal{Y}$ is continuous if for every open set $V \subset \mathcal{Y}$, the inverse image

$$f^{-1}(V) = \{x \in \mathcal{X} \mid f(x) \in V\}$$

is an open subset of $\mathcal{X}$.

2. f is continuous if and only if $x_n \longrightarrow x_0$ then $f(x_n) \longrightarrow f(x_0)$.

Definition 4.1.5. We say the linear map T is a **bounded linear operator** from $\mathcal{X}$ to $\mathcal{Y}$ if there is a finite constant k such that $\|Tx\|_{\mathcal{Y}} \leq k\|x\|_{\mathcal{X}}$ for all x in $\mathcal{X}$.

Proposition 4.1.6. *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed linear spaces and let $T : \mathcal{X} \longrightarrow \mathcal{Y}$ be a linear operator. The following are equivalent:*

1. *T is continuous;*
2. *T is continuous at 0;*
3. *there exists a positive real number k such that $\|T(x)\| \leq k$ whenever $x \in \mathcal{X}$ and $\|x\| \leq 1$;*
4. *T is bounded.*

Proof. (1)$\Longrightarrow$ (2):

$\because T$ is continuous

$\therefore T$ is continuous everywhere

$\therefore T$ is continuous at 0.

(2) $\Longrightarrow$ (3):

taking $\epsilon = 1$

$\because T$ is continuous at 0,

$\therefore \exists\delta > 0$ such that $\|T(x)\| < 1$ when $x \in \mathcal{X}$ and $\|x\| < \delta$.

Let $w \in \mathcal{X}$ with $\|w\| \leq 1$.

$\because \left\|\frac{\delta w}{2}\right\| = \frac{\delta}{2}\|w\| \leq \frac{\delta}{2} < \delta,$

$\therefore \left\|T\left(\frac{\delta w}{2}\right)\right\| < 1$

$\because T$ is a linear operator

$\therefore T\left(\frac{\delta w}{2}\right) = \frac{\delta}{2}T(w).$

$\therefore \frac{\delta}{2}\left\|T(w)\right\| < 1$

$\therefore \left\|T(w)\right\| < \frac{2}{\delta}$.

Therefore condition (2) holds with $k = \frac{2}{\delta}$.

(3) $\Longrightarrow$ (4):

$\because T$ is linear operator,

$\therefore T(0) = 0$

$\therefore \left\|T(0)\right\| \leq k\left\|0\right\|$. Then the proof have done.

Let $x \in \mathcal{X}$ with $x \neq 0$.

$\therefore \left\|\frac{x}{\|x\|}\right\| = 1$

By condition (3), $\exists k > 0$ such that $\left\|T(x)\right\| \leq k$.

$\therefore \left\|T\left(\frac{x}{\|x\|}\right)\right\| \leq k$.

$\because T$ is a linear operator

$\frac{1}{\|x\|}\left\|T(x)\right\| = \left\|\left(\frac{1}{\|x\|}\right)T(x)\right\| = \left\|T\left(\frac{x}{\|x\|}\right)\right\| \leq k,$

$\therefore \left\|T(x)\right\| \leq k\left\|x\right\|$.

(4)$\Longrightarrow$ (1):

Let $\epsilon > 0$ and let $\delta = \frac{\epsilon}{k}$.

Then when $x, y \in \mathcal{X}$ such that $\left\|x - y\right\| < \delta$

$\because T$ is a linear operator,

$\therefore \left\|T(x) - T(y)\right\| = \left\|T(x - y)\right\| \leq k\left\|x - y\right\| < k\left(\frac{\epsilon}{k}\right) = \epsilon$.

$\therefore T$ is continuous. □

Example 4.1.7. The linear operator $T : C_{\mathbb{F}}[0,1] \longrightarrow \mathbb{F}$ defined by $T(f) = f(0)$ is continuous.

Solution. Let $f \in C_{\mathbb{C}}[0,1]$. Then

$$|T(f)| = |f(0)| \leq \sup\left\{|f(x)| : x \in [0,1]\right\} = \left\|f\right\|.$$

$\therefore T$ is bounded with $k = 1$.

$\therefore T$ is continuous (by Proposition (4.1.6)).

Example 4.1.8. Let P be the linear subspace of $C_{\mathbb{C}}[0, 1]$ consisting of all polynomial functions. If $T : P \longrightarrow P$ is the linear operator defined by

$$T(p) = p',$$

where p' is the derivative of p, then T is not continuous.

Solution. Let $p_n \in P$ be defined by $p_n(t) = t^n$.

$\therefore \|p_n\| = \sup\{|p_n(t)| : t \in [0, 1]\} = \sup\{|t^n| : t \in [0, 1]\} = 1,$

while

$$\|T(p_n)\| = \|p_n'\| = \sup\{|p_n'(t)| : t \in [0, 1]\} = \sup\left\{\left|nt^{n-1}\right| : t \in [0, 1]\right\} = n.$$

$\therefore \nexists k \geq 0$ such that $\|T(p)\| \leq k\,\|p\|$ for all $p \in P$,

$\therefore T$ is not bounded.

$\therefore T$ is not continuous (by Proposition (4.1.6)).

4.2 The Norm of a Bounded Linear Operators

Definition 4.2.1. Let $\mathcal{X}, \mathcal{Y}$ be normed spaces, an **operator norm** of a linear operator $T : \mathcal{X} \longrightarrow \mathcal{Y}$ is

$$\|T\| := \inf \left\{ k \in \mathbb{R}^+ : \|Tx\| \leq k \|x\| \text{ for all } x \in \mathcal{X} \right\}$$

Proposition 4.2.2. *Let $\mathcal{X}, \mathcal{Y}$ be normed spaces, then*

$$\begin{aligned} \|T\| &:= \inf \left\{ k \in \mathbb{R}^+ : \|Tx\| \leq x \|x\| \text{ for all } x \in \mathcal{X} \right\} \\ &= \sup \{ \|Tx\| : x \in \mathcal{X}, \|x\| \leq 1 \} \\ &= \sup \{ \|Tx\| : x \in \mathcal{X}, \|x\| = 1 \} . \end{aligned}$$

Example 4.2.3. If $T : C_{\mathbb{C}}[0,1] \longrightarrow \mathbb{F}$ is the bounded linear operator defined by $T(f) = f(0)$, then $\|T\| = 1$.

Solution. It was shown in Example (4.1.7) that $|T(f)| \leq \|f\|$ for all $f \in C_{\mathbb{C}}[0,1]$.
$\therefore \|T\| = \inf \{ k : \|T(f)\| \leq k \|f\| \text{ for all } f \in C_{\mathbb{C}}[0,1] \} \leq 1$.
On the other hand,
if $g : [0,1] \longrightarrow \mathbb{C}$ is defined by $g(x) = 1$ for all $x \in \mathcal{X}$
$\therefore g \in C_{\mathbb{C}}[0,1]$ with $\|g\| = \sup \{ |g(x)| : x \in [0,1] \} = 1$ and $|T(g)| = |g(0)| = 1$.
$\therefore 1 = |T(g)| \leq \|T\| \|g\| = \|T\|$.
$\therefore \|T\| = 1$.

4.3 The Space $B(\mathcal{X}, \mathcal{Y})$

Definition 4.3.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed linear spaces. The set of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $B(\mathcal{X}, \mathcal{Y})$. Elements of $B(\mathcal{X}, \mathcal{Y})$ are also called **bounded linear operators**.

Proposition 4.3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed linear spaces. Then $B(\mathcal{X}, \mathcal{Y})$ is a vector space.*

Proof. Let $\alpha, \beta \in \mathbb{F}$ and $T, S \in B(\mathcal{X}, \mathcal{Y})$

$\therefore T, S$ are linear bounded operators.

$\therefore \exists k_1, k_2 \geq 0$ such that $\|Tx\| \leq k_1 \|x\|$ and $\|Sx\| \leq k_2 \|x\|$

Now,

$$\begin{aligned} \|(\alpha T + \beta S)x\| &= \|\alpha Tx + \beta Sx\| \\ &\leq \|\alpha Tx\| + \|\beta Sx\| \quad \text{(By the property of norm)} \\ &= |\alpha| \|Tx\| + |\beta| \|Sx\| \quad \text{(Since } T \text{ is linear)} \\ &\leq |\alpha| k_1 \|x\| + |\beta| k_2 \|x\| \\ &= (|\alpha| k_1 + |\beta| k_2) \|x\| \end{aligned}$$

$\therefore \alpha T + \beta S \in B(\mathcal{X}, \mathcal{Y})$. □

Proposition 4.3.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed linear spaces. Then $B(\mathcal{X}, \mathcal{Y})$ is a normed space, where the norm on $B(\mathcal{X}, \mathcal{Y})$ is the operator norm.*

Proof. Let $S, T \in B(\mathcal{X}, \mathcal{Y})$ and let $\lambda \in \mathbb{F}$.

1. $\because \|Tx\| \geq 0$ for all $x \in \mathcal{X}$

   $\therefore \|T\| = \sup \{\|Tx\| : \|x\| = 1\} \geq 0$

2. Let $x \in \mathcal{X}$

   $\|Tx\| = 0 \iff Tx = 0 \iff T = 0$

   $\therefore \|T\| = \sup \{\|Tx\| : \|x\| = 1\} = 0 \iff T = 0.$

3. As $\|T(x)\| \leq \|T\| \, \|x\|$ we have $\|(\lambda T)(x)\| \leq |\lambda| \, \|T\| \, \|x\|$ for all $x \in \mathcal{X}$.

   $\therefore \|\lambda T\| \leq |\lambda| \, \|T\|$.

   If $\lambda = 0$

   $\therefore \|\lambda T\| = |\lambda| \, \|T\| = 0$

   while if $\lambda \neq 0$

   $\therefore \|T\| = \left\|\frac{\lambda}{\lambda} T\right\| \leq \frac{1}{|\lambda|} \|\lambda T\| \leq \frac{1}{|\lambda|} |\lambda| \, \|T\| = \|T\|$.

   $\therefore \|T\| = \frac{1}{|\lambda|} \|\lambda T\|$

   $\therefore \|\lambda T\| = |\lambda| \, \|T\|$.

4.

$$\begin{aligned} \|(S+T)(x)\| &\leq \|S(x)\| + \|T(x)\| \\ &\leq \|S\| \, \|x\| + \|T\| \, \|x\| \\ &= (\|S\| + \|T\|) \, \|x\| \end{aligned}$$

   $\therefore \|S + T\| \leq \|S\| + \|T\|$.

□

Proposition 4.3.4. *If $\mathcal{X}$ is normed space and $\mathcal{Y}$ is a Banach space, then $B(\mathcal{X}, \mathcal{Y})$ is a Banach space.*

Proof. By remark (4.3.3) $B(\mathcal{X}, \mathcal{Y})$ is a normed space.

We have to show that $B(\mathcal{X}, \mathcal{Y})$ is a complete normed space. let $\{T_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $B(\mathcal{X}, \mathcal{Y})$.

Claim: $\{T_n x\}$ is Cauchy sequence in $\mathcal{Y}$: Let $x \in \mathcal{X}$ and $\epsilon > 0$

$\because \{T_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

$\therefore \|T_n - T_m\| \leq \epsilon$

$\because \|T_n(x) - T_m(x)\| = \|(T_n - T_m)(x)\| \leq \|T_n - T_m\| \, \|x\| \leq \epsilon \, \|x\|$.

$\therefore \{T_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{Y}$,

$\because \mathcal{Y}$ is Banach space,

$\therefore \mathcal{Y}$ is complete,

$\therefore \{T_n(x)\}_{n\in\mathbb{N}}$ converges

so we may define $T : \mathcal{X} \longrightarrow \mathcal{Y}$ by

$$T(x) = \lim_{n\to\infty} T_n(x)$$

We now need to show that $T \in B(\mathcal{X}, \mathcal{Y})$ and that T is the required limit in $B(\mathcal{X}, \mathcal{Y})$, so that $B(\mathcal{X}, \mathcal{Y})$ is a Banach space.

The first step is to show that T is linear: Let $\alpha, \beta \in \mathbb{F}$

$$\begin{aligned} T(\alpha x + \beta y) &= \lim_{n\to\infty} T_n(\alpha x + \beta y) \\ &= \lim_{n\to\infty} (T_n(\alpha x) + T_n(\beta y)) \\ &= \alpha \lim_{n\to\infty} T_n x + \beta \lim_{n\to\infty} T_n y \\ &= \alpha T x + \beta T y \end{aligned}$$

The second step is to show that $T \in B(\mathcal{X}, \mathcal{Y})$:

$\because \{T_n\}_{n\in\mathbb{N}}$ is a bounded set (Since every Cauchy sequence is bounded).

$\therefore \exists M > 0$ such that $\|T_n\| \le M$ for all $n \in \mathbb{N}$

$\because \|Tx\| = \lim_{n\to\infty} \|T_n x\| \le \lim_{n\to\infty} \|T_n\| \|x\| \le M \|x\|$

$\therefore T$ is bounded.

$\therefore T \in B(\mathcal{X}, \mathcal{Y})$.

Finally, we have to show that $T = \lim_{n\to\infty} T_n$:

let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that for $n, m \ge N$

$$\|T_n - T_m\| \le \frac{\epsilon}{2}.$$

$\therefore$ for any $x \in \mathcal{X}$ such that $\|x\| \leq 1$ and for any $n, m \geq N$,

$$\begin{aligned}
\|T_n(x) - T_m(x)\| &= \|(T_n - T_m)(x)\| \\
&\leq \|T_n - T_m\| \|x\| \\
&\leq \frac{\epsilon}{2} \|x\| \\
&\leq \frac{\epsilon}{2}
\end{aligned}$$

$\because T(x) = \lim_{n\to\infty} T_n(x)$,

$\therefore \exists N_1 \in \mathbb{N}$ such that when $m \geq N_1$,

$$\|Tx - T_m x\| \leq \frac{\epsilon}{2}.$$

Then when $n \geq N$ and $m \geq N_1$,

$$\begin{aligned}
\|T(x) - T_n(x)\| &= \|T(x) - T_m(x) + T_m(x) - T_n(x)\| \\
&\leq \|T(x) - T_m(x)\| + \|T_m(x) - T_n(x)\| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \|x\| \\
&\leq \epsilon \qquad \forall x \in \mathcal{X}
\end{aligned}$$

$\therefore \|T_n - T\| \leq \epsilon$ when $n \geq N$

$\therefore \lim_{n\longrightarrow\infty} T_n = T$

$\therefore \{T_n\}$ converges to T in $B(\mathcal{X}, \mathcal{Y})$.

$\therefore B(\mathcal{X}, \mathcal{Y})$ is complete, hence it is a Banach space. $\square$