ميكانيك تحليلي I

الفصل الأول

المحاضرة الآولى

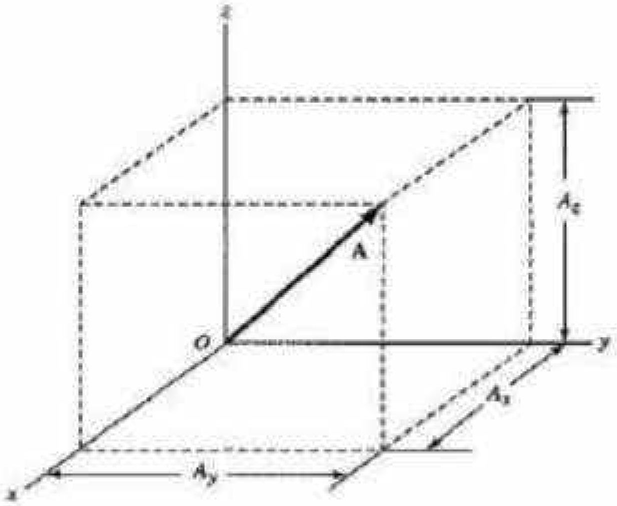
أ.د. رعد الحداد

1.3 Vectors

A- Component of a Vector

$\vec{A} = (A_x, A_y, A_z)$ ………….. (1 − 1)

Set of three component



B- Vector Addition

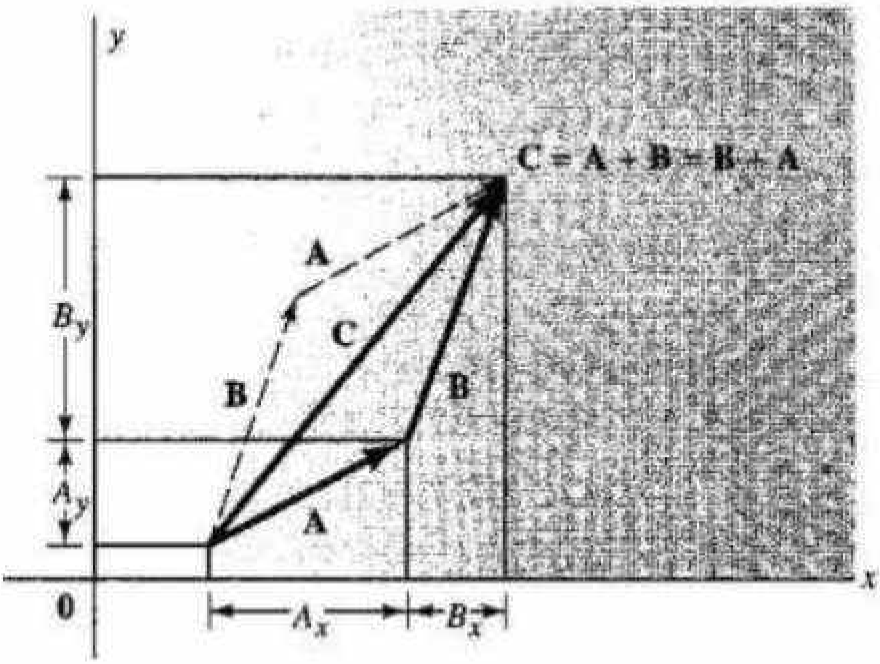
The addition of two vectors is defined by the equation:-

$$\vec{A} + \vec{B} = (A_x, A_y, A_z) + (B_x, B_y, B_z) \quad \ldots\ldots\ldots\ldots (1-2)$$

$$\vec{A} + \vec{B} = (A_x + B_x, A_y + B_y, A_z + B_z) \ldots\ldots\ldots\ldots (1-3)$$

$$\vec{A} + \vec{B} = (\vec{C}_x, \vec{C}_y, \vec{C}_z) \ldots\ldots\ldots\ldots (1-4)$$

$$\vec{C} = \vec{A} + \vec{B} = \vec{B} + \vec{A} \ldots\ldots\ldots\ldots (1-5)$$



C- Multiplication by a Scalar

If **c** is a scalar and $\vec{A}$ is a vector,

$$c\vec{A} = c(A_x, A_y, A_z) = (cA_x, cA_y, cA_z) = \vec{A}c \quad \dots\dots\dots\dots (1-6)$$

D- Magnitude of a Vector

The magnitude of a vector $\vec{A}$, denoted by $|\vec{A}|$ or by $\vec{A}$, is defined as the square root of the sum of the squares of the components, namely,

$$\vec{A} = |\vec{A}| = (A_x^2 + A_y^2 + A_z^2)^{1/2} = \sqrt{A_x^2 + A_y^2 + A_z^2} \quad \dots\dots\dots\dots (1-7)$$

E- Unit Coordinate Vectors

A unit vector is a vector whose magnitude is unity. Unit vectors are often designated by the symbol $\hat{\boldsymbol{e}}$.

$$\hat{e}_x = (1,0,0) \qquad \hat{e}_y = (0,1,0) \qquad \hat{e}_z = (0,0,1) \quad \dots\dots\dots\dots (1-8)$$

The three unit vectors are called **unit coordinate vectors** or **basis vectors**. In terms of basis vectors, any vector can be expressed as a vector sum of components as follows:

$$\vec{A} = (A_x, A_y, A_z) = (A_x, 0, 0) = (0, A_y, 0) = (0,0, A_z)$$

$$\vec{A} = (A_x, A_y, A_z) = A_x(1,0,0) = A_y(0,1,0) = A_z(0,0,1)$$

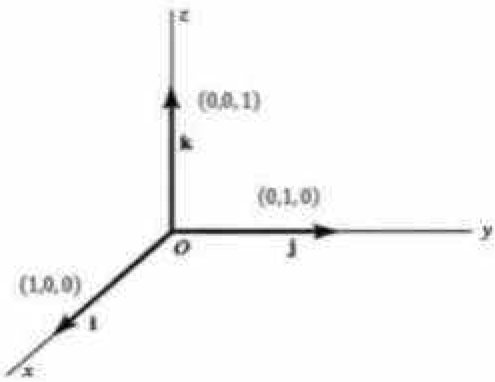
$$\vec{A} = (A_x, A_y, A_z) = A_x\hat{e}_x = A_x\,\hat{e}_y = A_z\,\hat{e}_z \quad \dots\dots\dots\dots (1-9)$$

F- Cartesian unit vectors

$$\hat{\imath} = \hat{e}_x \qquad \hat{\jmath} = \hat{e}_y \qquad \hat{k} = \hat{e}_z \quad \dots\dots (1-10)$$

$$\vec{A} = \hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z \quad \dots\dots\dots (1-11)$$

$$\vec{B} = \hat{\imath}B_x + \hat{\jmath}B_y + \hat{k}B_z \quad \dots\dots\dots (1-12)$$



EXAMPLE 1.3.1

Find the sum and the magnitude of the sum of the two vectors $\mathbf{A} = (1,0,2)$ and $\mathbf{B} = (0,1,1)$.

Solution:

Adding components, we have $\mathbf{A} + \mathbf{B} = (1,0,2) + (0,1,1) = (1,1,3)$.

$$|\mathbf{A} + \mathbf{B}| = (1+1+9)^{1/2} = \sqrt{11}$$

EXAMPLE 1.3.2

For the above two vectors, express the difference in **ijk** form.

Solution:

Subtracting components, we have

$$\mathbf{A} - \mathbf{B} = (1,-1,1) = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

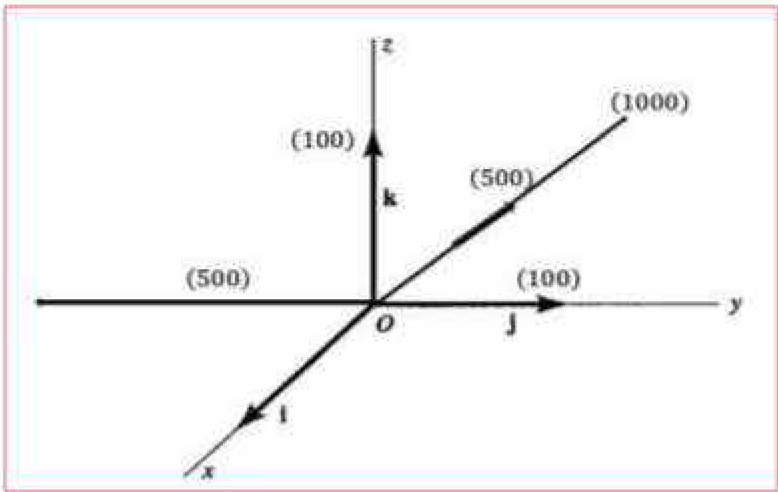
EXAMPLE 1.3.3

A helicopter flies 100 m vertically upward, then 500 m horizontally east, then 1000 m horizontally north. How far is it from a second helicopter that started from the same point and flew 200 m upward, 100 m west, and 500 m north?

Solution:

Choosing up, east, and north as basis directions, the final position of the first helicopter is expressed vectorially as $\mathbf{A} = (100,500,1000)$ and the second as $\mathbf{B} = (200,-100,500)$, in meters. Hence, the distance between the final positions is given by the expression

$$\begin{aligned} |\mathbf{A} - \mathbf{B}| &= |((100-200), (500+100), (1000-500))| \text{ m} \\ &= (100^2 + 600^2 + 500^2)^{1/2} \text{ m} \\ &= 787.4 \text{ m} \end{aligned}$$



1.4 The Scalar Product ("dot" product)

Given two vectors **A** and **B**, the scalar product or "dot" product, **A. B**, is the scalar defined by the equation:-

$$\vec{A}\cdot\vec{B} = \left(A_xB_x + A_yB_y + A_zB_z\right) \ldots\ldots\ldots\ldots (1-13)$$

From equation (1-11) and equation (1-12) we get

$$\vec{A}\cdot\vec{B} = \left(\hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z\right)\cdot\left(\hat{\imath}B_x + \hat{\jmath}B_y + \hat{k}B_z\right)$$

$$\vec{A}\cdot\vec{B} = (\hat{\imath}\cdot\hat{\imath})A_xB_x + (\hat{\imath}\cdot\hat{\jmath})A_xB_y + \cdots \ldots\ldots\ldots\ldots (H.W)$$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Note:-

$$\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$$

$$\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} = 0 \quad (orthogonal)$$

$$\vec{A} = (A_x, 0, 0) \quad ; \quad \vec{B} = (B_x, B_y, 0)$$

$$\vec{A}\cdot\vec{B} = A_xB_x = A(B\cos\theta) = |A||B|\cos\theta$$

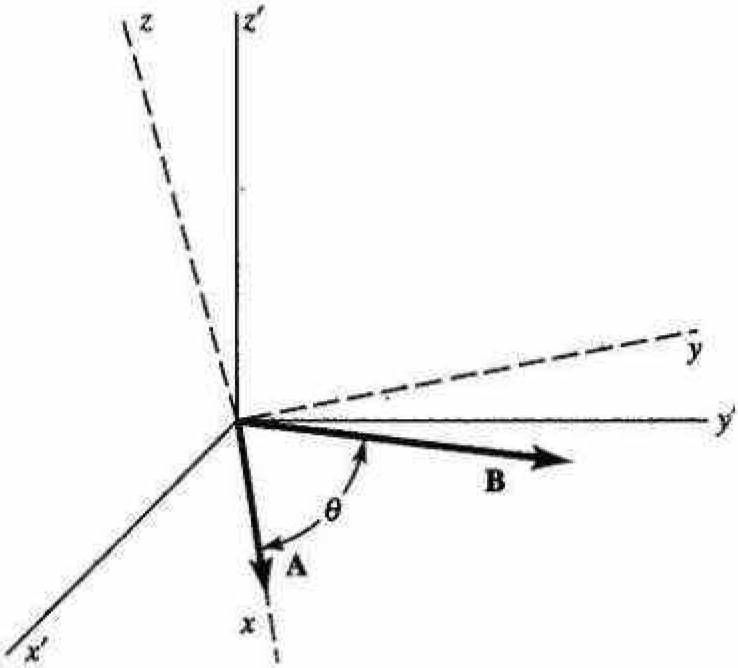


Figure 1.4.1 Evaluating a dot product between two vectors.

In general:-

$$\vec{A}\cdot\vec{B} = AB\cos\theta \quad \ldots\ldots(1-14)$$

Where θ is the angle between $\vec{A}$ and $\vec{B}$

Expressing Any Vector as the Product of Its Magnitude by a Unit Vector: Projection

From equation (1-11) we have

$$\vec{A} = \left(\hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z\right)$$

Multiply and divide on the right by the magnitude of A

$$\vec{A} = A\left(\hat{\imath}\frac{A_x}{A} + \hat{\jmath}\frac{A_y}{A} + \hat{k}\frac{A_z}{A}\right)$$

$$NOW \Rightarrow \cos\alpha = \frac{A_x}{A} \qquad \cos\gamma = \frac{A_y}{A} \qquad \cos\beta = \frac{A_z}{A}$$

Where α , γ *and* β *are direction of angles*

$$\therefore \vec{A} = A\left(\hat{\imath}\cos\alpha + \hat{\jmath}\cos\gamma + \hat{k}\cos\beta\right) \ldots\ldots\ldots\ldots(1-15)$$

$$\therefore \vec{A} = A\hat{n} \ldots\ldots\ldots\ldots(1-16)$$

Where $\hat{n}$ a unit vector is whose components are $\cos\alpha$, $\cos\gamma$, $\cos\beta$.

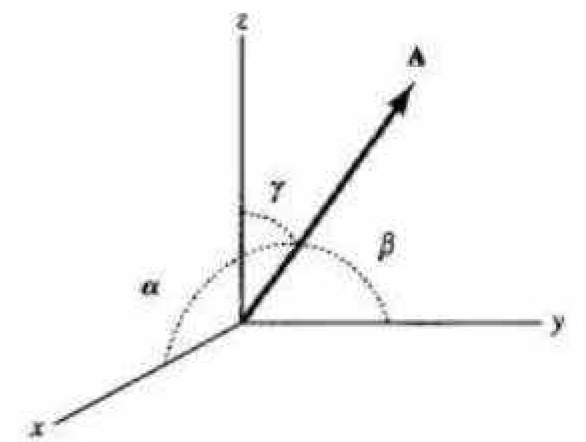


Figure 1.4.2 Direction angles α, β, γ of a vector.

Clearly, the projection of **B** on **A** is just

$$B\cos\theta = \vec{B}\cdot\hat{n} \ldots\ldots(1-17)$$

EXAMPLE 1.4.1

Component of a Vector: Work

As an example of the dot product, suppose that an object under the action of a constant force[5] undergoes a linear displacement $\Delta\mathbf{s}$, as shown in Figure 1.4.3. By definition, the *work* ΔW done by the force is given by the product of the component of the force $\mathbf{F}$ in the direction of $\Delta\mathbf{s}$, multiplied by the magnitude Δs of the displacement; that is,

$$\Delta W = (F\cos\theta)\,\Delta s$$

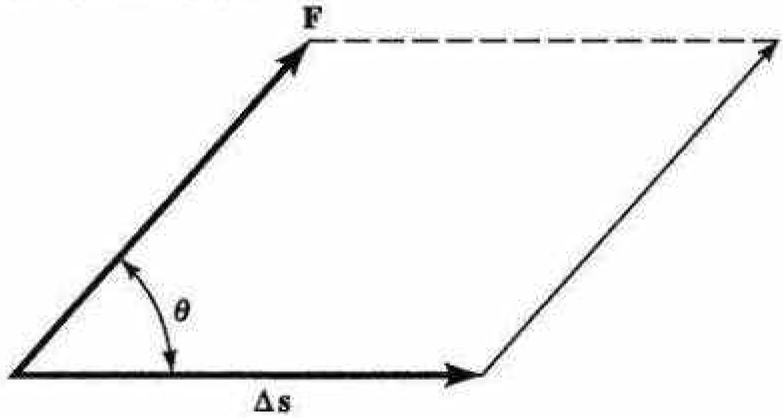


Figure 1.4.3 A force acting on a body undergoing a displacement.

where θ is the angle between $\mathbf{F}$ and $\Delta\mathbf{s}$. But the expression on the right is just the dot product of $\mathbf{F}$ and $\Delta\mathbf{s}$, that is,

$$\Delta W = \mathbf{F}\cdot\Delta\mathbf{s}$$

EXAMPLE 1.4.2

Law of Cosines

Consider the triangle whose sides are **A**, **B**, and **C**, as shown in Figure Then

$\vec{C} = \vec{A} + \vec{B}$

$\vec{C}\cdot\vec{C} = \left(\vec{A}+\vec{B}\right)\cdot\left(\vec{A}+\vec{B}\right)$

$\vec{C}\cdot\vec{C} = \vec{A}\cdot\vec{A} + 2\vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{B}$

Replace **A.B** with **AB** $\cos\theta$ to obtain

$C^2 = A^2 + 2AB\cos\theta + B^2$

Which is the familiar law of cosines.

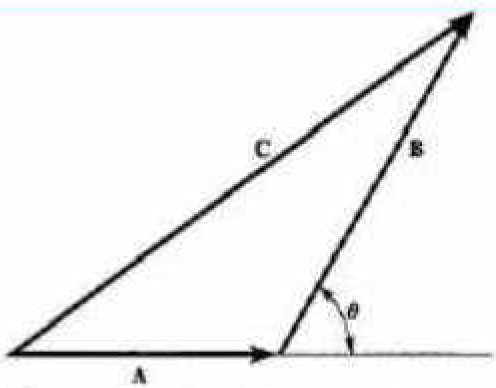


Figure 1.4.3 A force acting on a body undergoing a displacement.

EXAMPLE 1.4.3

Find the cosine of the angle between a long diagonal and an adjacent face diagonal of a cube.

Solution:

We can represent the two diagonals in question by the vectors $\mathbf{A} = (1,1,1)$ and $\mathbf{B} = (1,1,0)$.

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{AB} = \frac{1+1+0}{\sqrt{3}\sqrt{2}} = \sqrt{\frac{2}{3}} = 0.8165$$

EXAMPLE 1.4.4

The vector $a\mathbf{i} + \mathbf{j} - \mathbf{k}$ is perpendicular to the vector $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. What is the value of a?

Solution:

If the vectors are perpendicular to each other, their dot product must vanish ($\cos 90° = 0$).

$$(a\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = a + 2 + 3 = a + 5 = 0$$

Therefore,

$$a = -5$$

1.5 The Vector Product ("cross" product)

Given two vectors **A** and **B**, the vector product or cross product can be written as

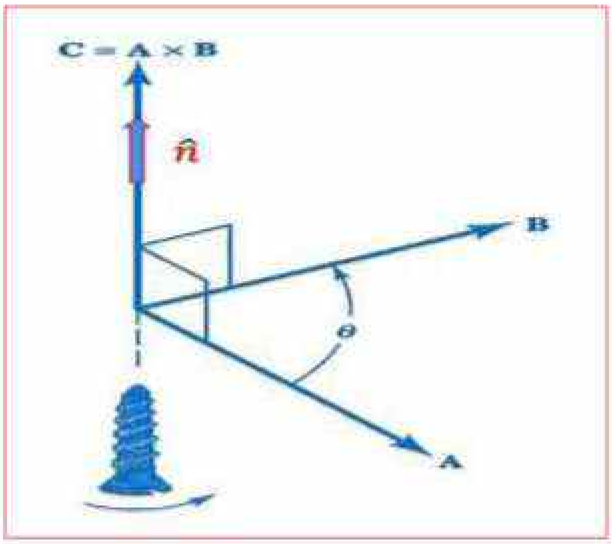
$\vec{A} \times \vec{B} = \vec{C}$ ……………..(1 − 18)

$\left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right| \sin\theta$ …………(1 − 19)

$\vec{A} \times \vec{B} = (AB\sin\theta)\hat{n}$ ………(1 − 20)

Where $\hat{n}$ is unit vector in direction of $\vec{C}$

$\vec{B} \times \vec{A} = -\vec{A} \times \vec{B} = -\vec{C}$ ………(1 − 21)

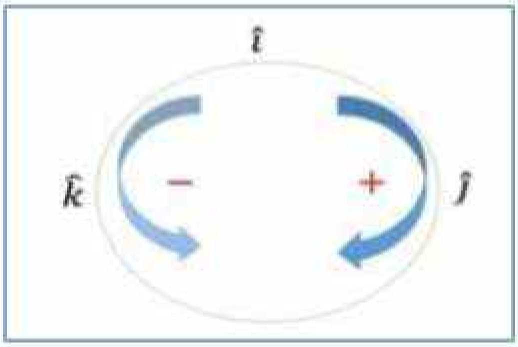


Note:-

$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$

$\hat{\imath} \times \hat{\jmath} = \hat{k} \qquad \hat{\jmath} \times \hat{k} = \hat{\imath} \qquad \hat{k} \times \hat{\imath} = \hat{\jmath}$

$\hat{\jmath} \times \hat{\imath} = -\hat{k} \qquad \hat{k} \times \hat{\jmath} = -\hat{\imath} \qquad \hat{\imath} \times \hat{k} = -\hat{\jmath}$



From equation (1-11) and (1-12) we have

$$\vec{A} \times \vec{B} = \left(\hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z\right) \times \left(\hat{\imath}B_x + \hat{\jmath}B_y + \hat{k}B_z\right) \quad \dots\dots\dots (1-22) \quad \textbf{H.W}$$

$$\vec{A} \times \vec{B} = \hat{\imath}\left(A_yB_z - A_z B_y\right) + \hat{\jmath}\left(A_zB_x - A_xB_z\right) + \hat{k}\left(A_xB_y - A_yB_x\right) \quad (1-23)$$

Each term in parentheses is equal to a determinant,

$$\vec{A} \times \vec{B} = \hat{\imath}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} + \hat{\jmath}\begin{vmatrix} A_z & A_x \\ B_z & B_x \end{vmatrix} + \hat{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \quad \dots\dots\dots (1-24)$$

And finally

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad \dots\dots\dots (1-25)$$

Let us calculate the magnitude of the cross product. From eq. (1-23) we have

$$\left|\vec{A} \times \vec{B}\right|^2 = \left(A_yB_z - A_z B_y\right)^2 + \left(A_zB_x - A_xB_z\right)^2 + \left(A_xB_y - A_yB_x\right)^2$$

$$\left|\vec{A} \times \vec{B}\right|^2 = \left(A_x^2 + A_y^2 + A_z^2\right)\left(B_x^2 + B_y^2 + B_z^2\right) - \left(A_xB_x + A_yB_y + A_zB_z\right)^2 \quad (1-26)$$

Or from the definition of the dot product

$$\left|\vec{A} \times \vec{B}\right|^2 = A^2B^2 - (A \cdot B)^2 \quad \dots\dots\dots (1-27)$$

$$\left|\vec{A} \times \vec{B}\right|^2 = A^2B^2 - A^2 B^2 \cos^2\theta$$

$$\left|\vec{A} \times \vec{B}\right|^2 = A^2B^2(1 - \cos^2\theta)$$

$$\left|\vec{A} \times \vec{B}\right| = AB(1 - \cos^2\theta)^{1/2}$$

$$\left|\vec{A} \times \vec{B}\right| = AB\sin\theta \ldots\ldots (1-28)$$

Where θ is the angle between $\vec{A}$ and $\vec{B}$

EXAMPLE 1.5.1

Given the two vectors $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{B} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, find $\mathbf{A} \times \mathbf{B}$.

Solution:

In this case it is convenient to use the determinant form

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i}(2-1) + \mathbf{j}(-1-4) + \mathbf{k}(-2-1)$$

$$= \mathbf{i} - 5\mathbf{j} - 3\mathbf{k}$$

EXAMPLE 1.5.2

Find a unit vector normal to the plane containing the two vectors **A** and **B** above.

Solution:

$$\mathbf{n} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = \frac{\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}}{[1^2 + 5^2 + 3^2]^{1/2}}$$

$$= \frac{\mathbf{i}}{\sqrt{35}} - \frac{5\mathbf{j}}{\sqrt{35}} - \frac{3\mathbf{k}}{\sqrt{35}}$$

1.6 An Example of the Cross Product Moment of a Force (Torque)

The moment N of force, or the torque N, about a given point O is defined as the cross product:-

$$\vec{N} = \vec{r} \times \vec{F} \ldots\ldots\ldots\ldots (1-29)$$

$$|\vec{N}| = rF\sin\theta \ldots\ldots (1-30)$$

Thus, the moment of a force about a point is a **vector quantity** having a **magnitude** and a **direction**.

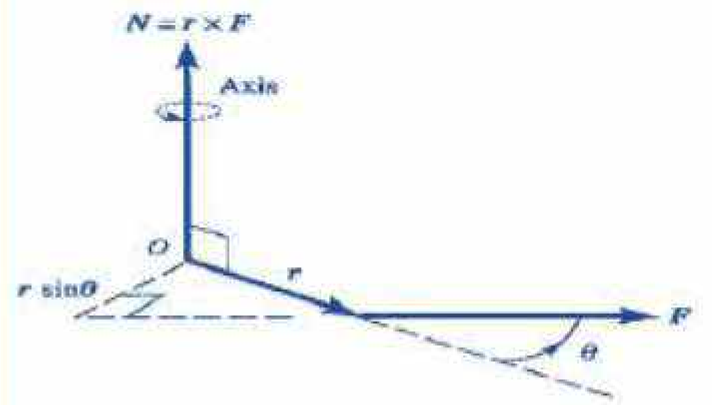


Figure 1.6.1 Illustration of the moment of a force about a point *O*.

1.7 Change of Coordinate System: The Transformation Matrix

In this section we show how to represent a vector in different coordinate systems. From eq. (1-11) we have the vector **A** expressed relative to the triad **ijk**:-

$$\vec{A} = \hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z \quad \dots\dots\dots\dots(1-11)$$

Relative to a new triad $\boldsymbol{i'j'k'}$ having a different orientation from that of $\boldsymbol{ijk}$, the same vector **A** is expressed as

$$\vec{A'} = \hat{\imath}'A_{x'} + \hat{\jmath}'A_{y'} + \hat{k}'A_{z'} \quad \dots\dots\dots\dots(1-31)$$

Now by using the dot product

$$\left.\begin{aligned} A_{x'} &= \vec{A}\cdot\hat{\imath}' = (\hat{\imath}\cdot\hat{\imath}')A_x + (\hat{\jmath}\cdot\hat{\imath}')A_y + (\hat{k}\cdot\hat{\imath}')A_z \\ A_{y'} &= \vec{A}\cdot\hat{\jmath}' = (\hat{\imath}\cdot\hat{\jmath}')A_x + (\hat{\jmath}\cdot\hat{\jmath}')A_y + (\hat{k}\cdot\hat{\jmath}')A_z \\ A_{z'} &= \vec{A}\cdot\hat{k}' = (\hat{\imath}\cdot\hat{k}')A_x + (\hat{\jmath}\cdot\hat{k}')A_y + (\hat{k}\cdot\hat{k}')A_z \end{aligned}\right\} \quad \dots\dots(1-32)$$

The scalar products $(\hat{\imath}\cdot\hat{\imath}')$, $(\hat{\imath}\cdot\hat{\jmath}')$ and so on are called **<u>the coefficients of transformation</u>.** The equations (1-32) can be expressed in matrix notation. Thus

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} \hat{\imath}\cdot\hat{\imath}' & \hat{\jmath}\cdot\hat{\imath}' & \hat{k}\cdot\hat{\imath}' \\ \hat{\imath}\cdot\hat{\jmath}' & \hat{\jmath}\cdot\hat{\jmath}' & \hat{k}\cdot\hat{\jmath}' \\ \hat{\imath}\cdot\hat{k}' & \hat{\jmath}\cdot\hat{k}' & \hat{k}\cdot\hat{k}' \end{pmatrix} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \quad \dots\dots\dots\dots(1-33)$$

- ✓ $Row\ (3) \times Colume\ (1) = (3\times 3)\times(3\times 1) = 3$
- ✓ The three by three matrix in Equation (1-33) is called the transformation matrix.

<u>Ex:</u> - Find the transformation matrix for a rotation about a different coordinate axis—say, the y-axis through an angle θ

Sol/

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

EXAMPLE 1.8.1

Express the vector $\mathbf{A} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ in terms of the triad $\mathbf{i'j'k'}$, where the $x'y'$–axes are rotated 45° around the z-axis, with the z- and z'-axes coinciding, as shown in Figure 1.8.1. Referring to the figure, we have for the coefficients of transformation $\mathbf{i}\cdot\mathbf{i'} = \cos 45°$ and so on; hence,

$$\begin{array}{lll} \mathbf{i}\cdot\mathbf{i'} = 1/\sqrt{2} & \mathbf{j}\cdot\mathbf{i'} = 1/\sqrt{2} & \mathbf{k}\cdot\mathbf{i'} = 0 \\ \mathbf{i}\cdot\mathbf{j'} = -1/\sqrt{2} & \mathbf{j}\cdot\mathbf{j'} = 1/\sqrt{2} & \mathbf{k}\cdot\mathbf{j'} = 0 \\ \mathbf{i}\cdot\mathbf{k'} = 0 & \mathbf{j}\cdot\mathbf{k'} = 0 & \mathbf{k}\cdot\mathbf{k'} = 1 \end{array}$$

These give

$$A_{x'} = \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{5}{\sqrt{2}} \qquad A_{y'} = \frac{-3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{-1}{\sqrt{2}} \qquad A_{z'} = 1$$

so that, in the primed system, the vector $\mathbf{A}$ is given by

$$\mathbf{A} = \frac{5}{\sqrt{2}}\mathbf{i'} - \frac{1}{\sqrt{2}}\mathbf{j'} + \mathbf{k'}$$

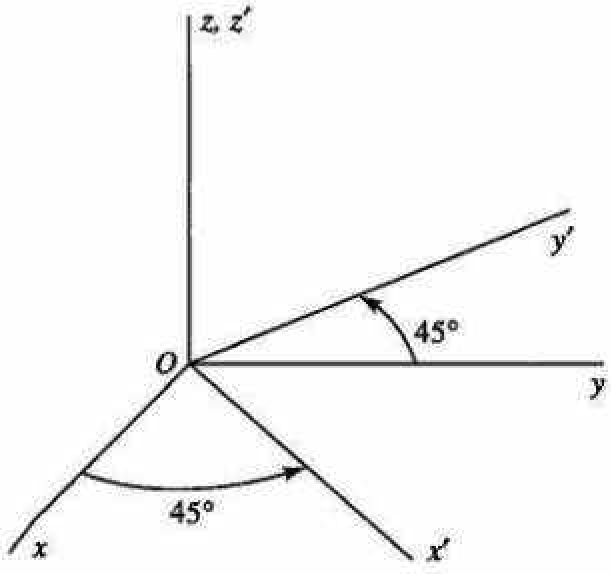


Figure 1.8.1 Rotated axes.

H.W ⇒ 1.7 1.14 ⇐

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المحاضرة الثانيه

أ.د. رعد الحداد

1.7 Change of Coordinate System: The Transformation Matrix

In this section we show how to represent a vector in different coordinate systems. From eq. (1-11) we have the vector **A** expressed relative to the triad **ijk**:-

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Now by using the dot product

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The scalar products $(\hat{\imath}\cdot\hat{\imath}'), (\hat{\imath}\cdot\hat{\jmath}')$ and so on are called **the coefficients of transformation**. The equations (1-32) can be expressed in matrix notation. Thus

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} \hat{\imath}\cdot\hat{\imath}' & \hat{\jmath}\cdot\hat{\imath}' & \hat{k}\cdot\hat{\imath}' \\ \hat{\imath}\cdot\hat{\jmath}' & \hat{\jmath}\cdot\hat{\jmath}' & \hat{k}\cdot\hat{\jmath}' \\ \hat{\imath}\cdot\hat{k}' & \hat{\jmath}\cdot\hat{k}' & \hat{k}\cdot\hat{k}' \end{pmatrix} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \quad \dots\dots\dots\dots(1-33)$$

- ✓ $Row\ (3) \times Colume\ (1) = (3\times 3)\times(3\times 1) = 3$
- ✓ The three by three matrix in Equation (1-33) is called the transformation matrix.

Ex: - Find the transformation matrix for a rotation about a different coordinate axis—say, the y-axis through an angle θ

Sol/

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

EXAMPLE 1.8.1

Express the vector $\mathbf{A} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ in terms of the triad $\mathbf{i}'\mathbf{j}'\mathbf{k}'$, where the $x'y'$–axes are rotated 45° around the z-axis, with the z- and z'-axes coinciding, as shown in Figure 1.8.1. Referring to the figure, we have for the coefficients of transformation $\mathbf{i}\cdot\mathbf{i}' = \cos 45°$ and so on; hence,

$$\begin{array}{lll} \mathbf{i}\cdot\mathbf{i}' = 1/\sqrt{2} & \mathbf{j}\cdot\mathbf{i}' = 1/\sqrt{2} & \mathbf{k}\cdot\mathbf{i}' = 0 \\ \mathbf{i}\cdot\mathbf{j}' = -1/\sqrt{2} & \mathbf{j}\cdot\mathbf{j}' = 1/\sqrt{2} & \mathbf{k}\cdot\mathbf{j}' = 0 \\ \mathbf{i}\cdot\mathbf{k}' = 0 & \mathbf{j}\cdot\mathbf{k}' = 0 & \mathbf{k}\cdot\mathbf{k}' = 1 \end{array}$$

These give

$$A_{x'} = \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{5}{\sqrt{2}} \qquad A_{y'} = \frac{-3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{-1}{\sqrt{2}} \qquad A_{z'} = 1$$

so that, in the primed system, the vector $\mathbf{A}$ is given by

$$\mathbf{A} = \frac{5}{\sqrt{2}}\mathbf{i}' - \frac{1}{\sqrt{2}}\mathbf{j}' + \mathbf{k}'$$

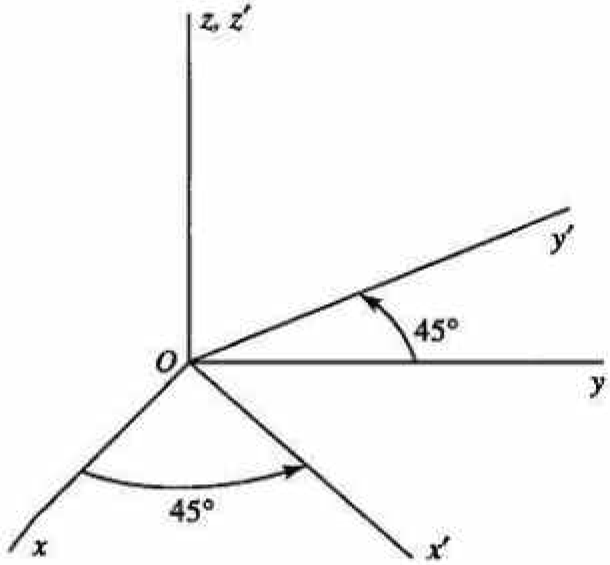


Figure 1.8.1 Rotated axes.

H.W ⇒ 1.7   1.14 ⇐

1.8 Derivative of a Vector

- In this section, we start to study of the calculus of vectors and its use in the description of the motion of particles.
- Consider a vector **A**, whose components are functions of a single variable **u**. which can be represented by this equation:-

$$\vec{A}(u) = \hat{\imath}A_x(u) + \hat{\jmath}A_y(u) + \hat{k}A_z(u) \quad \ldots\ldots\ldots\ldots(1-34)$$

Now take the derivative of up equation with limited $\Delta u \to 0$

$$\frac{d\vec{A}}{du} = \lim_{\Delta u\to 0}\frac{\Delta\vec{A}}{\Delta u} = \lim_{\Delta u\to 0}\left(\hat{\imath}\frac{\Delta A_x}{\Delta u} + \hat{\jmath}\frac{\Delta A_y}{\Delta u} + \hat{k}\frac{\Delta A_z}{\Delta u}\right) \quad \ldots\ldots\ldots\ldots(1-35)$$

$$\frac{d\vec{A}}{du} = \hat{\imath}\frac{dA_x}{du} + \hat{\jmath}\frac{dA_y}{du} + \hat{k}\frac{dA_z}{du} \quad \ldots\ldots\ldots\ldots(1-35)$$

It follows from eq. (1-35) that the derivative of the sum of two vectors is equal to the sum of the derivatives, namely

$$\frac{d\left(\vec{A}+\vec{B}\right)}{du} = \frac{d\vec{A}}{du} + \frac{d\vec{B}}{du} \quad \ldots\ldots\ldots\ldots(1-36)$$

The rules for differentiating vector products obey similar rules of vector calculus. For example,

$$\frac{d\left(n\vec{A}\right)}{du} = \frac{dn}{du}\vec{A} + n\frac{d\vec{A}}{du} \quad \ldots\ldots\ldots\ldots\ldots(1-37)$$

$$\frac{d\left(\vec{A}\cdot\vec{B}\right)}{du} = \frac{d\vec{A}}{du}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{du} \quad \ldots\ldots\ldots\ldots(1-38)$$

$$\frac{d\left(\vec{A}\times\vec{B}\right)}{du} = \frac{d\vec{A}}{du}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{du} \quad \ldots\ldots\ldots\ldots(1-39)$$

1.9 Position Vector of a Particle: Position, Velocity and Acceleration in Rectangular Coordinates

- ✓ The **position** of a particle can be specified by a single vector, namely, the **displacement of the particle relative to the origin of the coordinate system.** This vector is called the **position vector of the particle**. the position vector is simply

$$\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z \;\dots\dots\dots\dots(1-40)$$

The components of the position vector of a moving particle are functions of the time, namely,

$$x(t) \qquad y(t) \quad and \quad z(t)$$

In particular, the derivative of **r** with respect to t is called the **velocity**, which we shall denote by

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{\imath}\frac{d\vec{x}}{dt} + \hat{\jmath}\frac{d\vec{y}}{dt} + \hat{k}\frac{d\vec{z}}{dt} \;\dots(1-41)$$

$$\vec{v} = \hat{\imath}\dot{x} + \hat{\jmath}\dot{y} + \hat{k}\dot{z} \;\dots\dots\dots\dots(1-42)$$

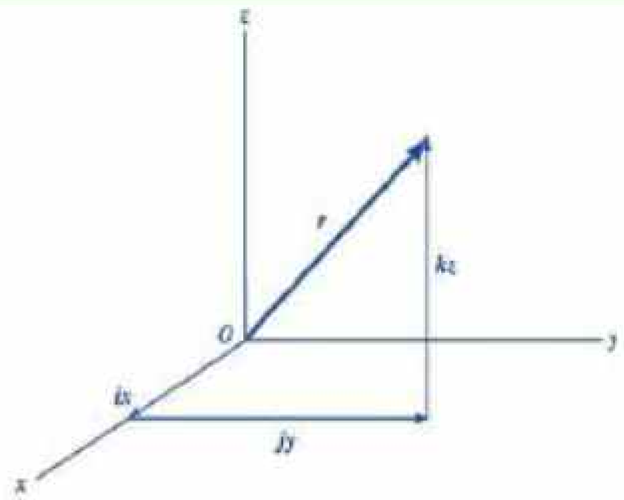


Figure (1.10.1) the position vector **r** and its components in a Cartesian coordinate system

Where the **dots** indicate differentiation with respect to t.

The magnitude of the velocity is called the **speed**. In rectangular components the speed is just

$$\vec{v} = |v| = \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)^{1/2} = \sqrt{\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)} \;\dots\dots\dots\dots(1-43)$$

The time derivative of the velocity is called the **acceleration**. Denoting the acceleration with **a**, we have

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \;\dots(1-44)$$

In rectangular components, the acceleration is just:-

$\vec{a} = \hat{\imath}\ddot{x} + \hat{\jmath}\ddot{y} + \hat{k}\ddot{z}$ ……………. (1 − 45)

EXAMPLE 1.10.1

Projectile Motion

Let us examine the motion represented by the equation

$$\mathbf{r}(t) = \mathbf{i}bt + \mathbf{j}\left(ct - \frac{gt^2}{2}\right) + \mathbf{k}0$$

This represents motion in the *xy* plane, because the *z* component is constant and equal to zero. The velocity **v** is obtained by differentiating with respect to *t*, namely,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}b + \mathbf{j}(c - gt)$$

The acceleration, likewise, is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\mathbf{j}g$$

Thus, **a** is in the negative *y* direction and has the constant magnitude *g*. The path of motion is a parabola, as shown in Figure 1.10.3. The speed *v* varies with *t* according to the equation

$$v = [b^2 + (c - gt)^2]^{1/2}$$

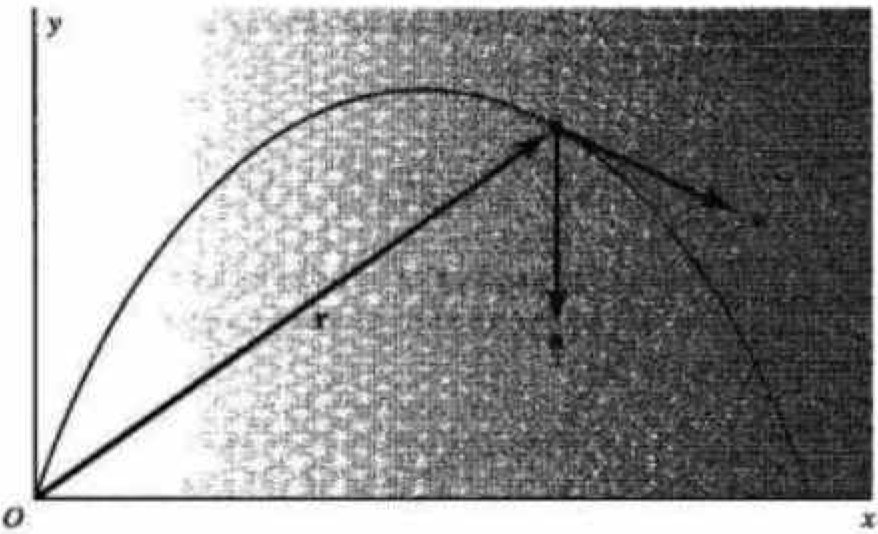


Figure 1.10.3 Position, velocity, and acceleration vectors of a particle (projectile) moving in a parabolic path.

EXAMPLE 1.10.2

Circular Motion

Suppose the position vector of a particle is given by

$$\mathbf{r} = \mathbf{i}b \sin \omega t + \mathbf{j}b \cos \omega t$$

where ω is a constant.

Let us analyze the motion. The distance from the origin remains constant:

$$|\mathbf{r}| = r = (b^2 \sin^2 \omega t + b^2 \cos^2 \omega t)^{1/2} = b$$

So the path is a circle of radius b centered at the origin. Differentiating $\mathbf{r}$, we find the velocity vector

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}b\omega \cos \omega t - \mathbf{j}b\omega \sin \omega t$$

The particle traverses its path with constant speed:

$$v = |\mathbf{v}| = (b^2\omega^2 \cos^2 \omega t + b^2\omega^2 \sin^2 \omega t)^{1/2} = b\omega$$

The acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\mathbf{i}b\omega^2 \sin \omega t - \mathbf{j}b\omega^2 \cos \omega t$$

In this case the acceleration is perpendicular to the velocity, because the dot product of $\mathbf{v}$ and $\mathbf{a}$ vanishes:

$$\mathbf{v} \cdot \mathbf{a} = (b\omega \cos \omega t)(-b\omega^2 \sin \omega t) + (-b\omega \sin \omega t)(-b\omega^2 \cos \omega t) = 0$$

Comparing the two expressions for $\mathbf{a}$ and $\mathbf{r}$, we see that we can write

$$\mathbf{a} = -\omega^2 \mathbf{r}$$

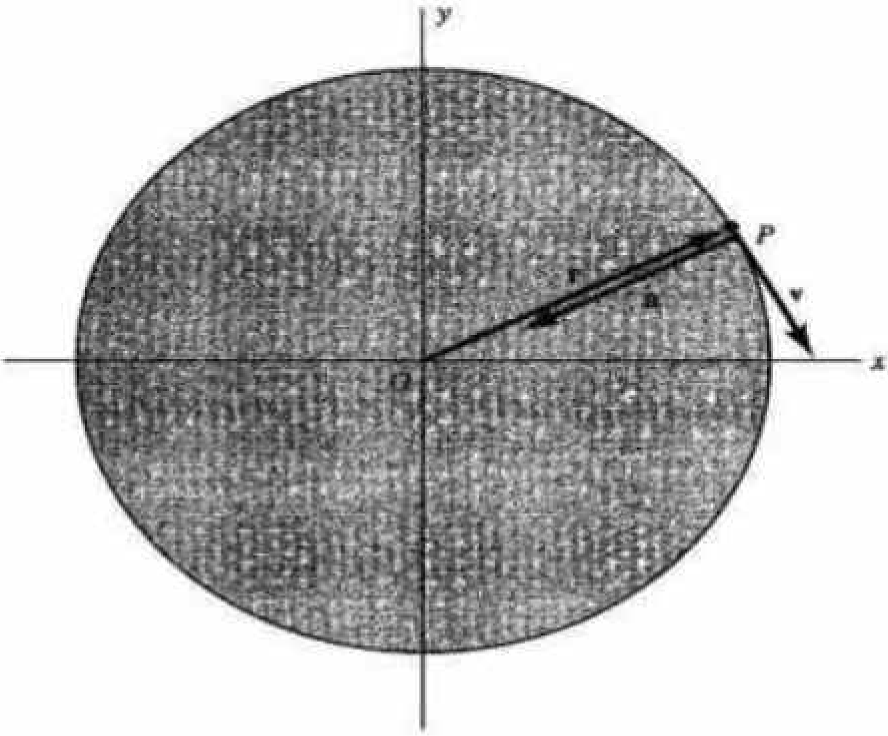


Figure 1.10.4 A particle moving in a circular path with constant speed.

so **a** and **r** are oppositely directed; that is, **a** always points toward the center of the circular path (Fig. 1.10.4).

EXAMPLE 1.10.3

Rolling Wheel

Let us consider the following position vector of a particle P:

$$\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$$

in which

$$\mathbf{r}_1 = \mathbf{i}b\omega t + \mathbf{j}b$$
$$\mathbf{r}_2 = \mathbf{i}b \sin \omega t + \mathbf{j}b \cos \omega t$$

Now $\mathbf{r}_1$ by itself represents a point moving along the line $y = b$ at constant velocity, provided ω is constant; namely,

$$\mathbf{v}_1 = \frac{d\mathbf{r}_1}{dt} = \mathbf{i}b\omega$$

The second part, $\mathbf{r}_2$, is just the position vector for circular motion, as discussed in Example 1.10.2. Hence, the vector sum $\mathbf{r}_1 + \mathbf{r}_2$ represents a point that describes a circle of radius b about a moving center. This is precisely what occurs for a particle on the rim of a rolling wheel, $\mathbf{r}_1$ being the position vector of the center of the wheel and $\mathbf{r}_2$ being

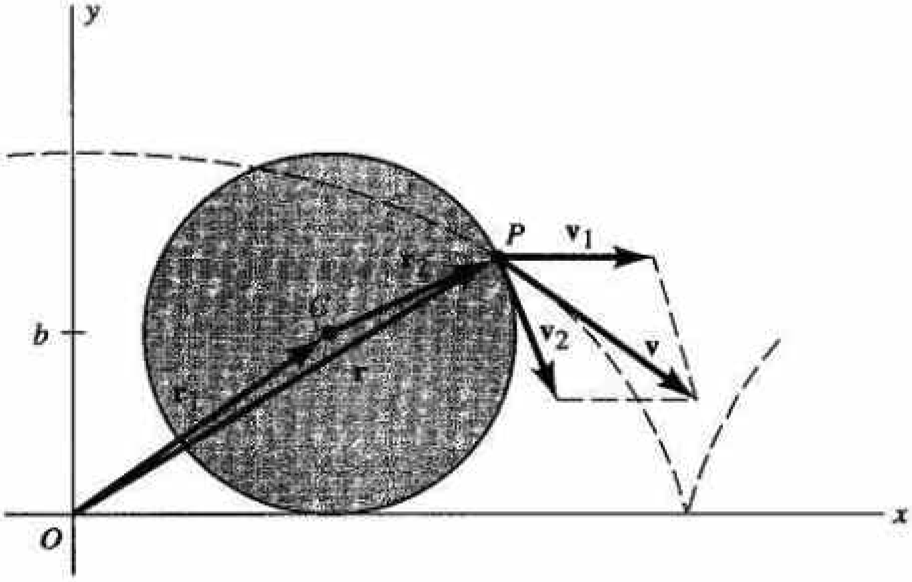


Figure 1.10.5 The cycloidal path of a particle on a rolling wheel.

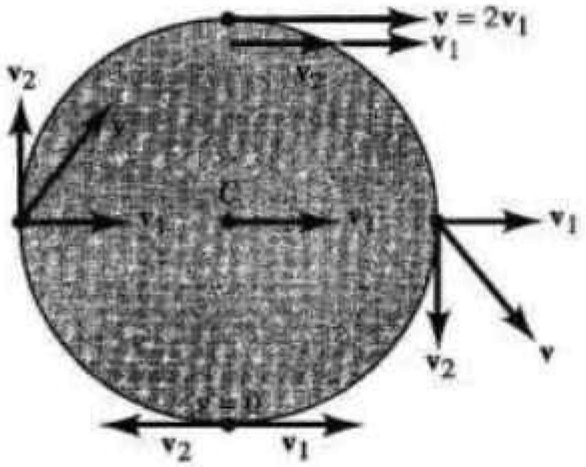


Figure 1.10.6 Velocity vectors for various points on a rolling wheel.

the position vector of the particle P *relative* to the moving center. The actual path is a *cycloid*, as shown in Figure 1.10.5. The velocity of P is

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{i}(b\omega + b\omega \cos \omega t) - \mathbf{j}b\omega \sin \omega t$$

In particular, for $\omega t = 0, 2\pi, 4\pi, \ldots$, we find that $\mathbf{v} = \mathbf{i}2b\omega$, which is just twice the velocity of the center C. At these points the particle is at the uppermost part of its path. Furthermore, for $\omega t = \pi, 3\pi, 5\pi, \ldots$, we obtain $\mathbf{v} = 0$. At these points the particle is at its lowest point and is instantaneously in contact with the ground. See Figure 1.10.6.

ميكانيك تحليلي I

الفصل الأول

المحاضرة الثالثه

أ.د. رعد الحداد

1.10 Velocity and Acceleration in Plane Polar Coordinates

- ✓ In this section, we used polar coordinates $(r\,,\theta)$ to express the position of a particle moving in a plane.
- ✓ Vectorially, the position of the particle can be written as the product of the **radial distance $\vec{r}$** by a **unit radial vector e_r**:

$$\vec{r} = r\hat{e}_r \quad \ldots\ldots\ldots\ldots\ldots.(1-46)$$

If we differentiate with respect to **t**, we have

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{e}_r + r\frac{d\hat{e}_r}{dt} \quad \ldots\ldots\ldots\ldots\ldots.(1-47)$$

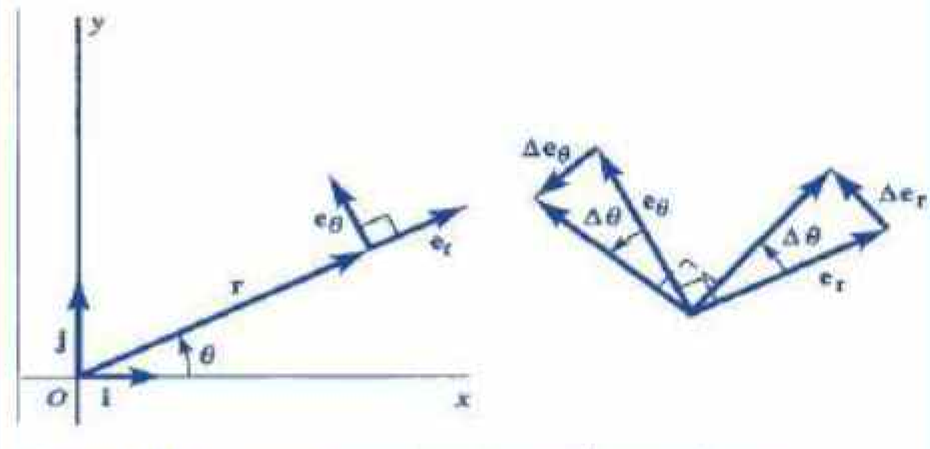


Figure 1.11.1 Unit vectors for plane polar coordinates.

Let us introduce another unit vector, $\hat{e}_\theta$, whose direction is perpendicular to $\hat{e}_r$. Then we have:-

$$\Delta\hat{e}_r \approx \hat{e}_\theta \Delta\theta \quad \ldots\ldots\ldots\ldots\ldots.(1-48)$$

If we divide by Δt

$$\frac{\Delta\hat{e}_r}{\Delta t} = \hat{e}_\theta \frac{\Delta\theta}{\Delta t} \quad \ldots\ldots\ldots\ldots\ldots.(1-49)$$

And take the limit $\Delta t \to 0$, we get

$$\frac{d\hat{e}_r}{dt} = \hat{e}_\theta \frac{d\theta}{dt} \quad \ldots\ldots\ldots\ldots\ldots.(1-50)$$ ***OR***

$$\frac{d\hat{e}_r}{dt} = \hat{e}_\theta \dot{\theta} \quad \ldots\ldots\ldots\ldots\ldots.(1-51)$$

By sub eq. (1-51) in eq. (1-47) we can finally write the equation for the velocity as

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \; \dots\dots\dots\dots\dots\dots(1-52)$$

Thus, $\dot{r}$ is the **radial component** of the velocity vector, and $r\dot{\theta}$ is the **transverse component**.

H.W: **Find the acceleration vector**

$$\vec{a}_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) \dots\dots\dots\dots\dots\dots(1-53)$$

1.11 Velocity and Acceleration in Cylindrical and Spherical Coordinates

A. Cylindrical Coordinates

In the case of three-dimensional motion, the position of a particle can be described in cylindrical coordinates R, ϕ, z. The position vector is then written as

$$\vec{r} = R\hat{e}_r + z\hat{e}_z \; \dots\dots\dots\dots\dots\dots(1-54)$$

Where $\hat{e}_r$ a unit is radial vector in the xy plane and $\hat{e}_z$ is the unit vector in the z direction.

$$\left.\begin{aligned} x &= R\cos\phi \\ y &= R\sin\phi \\ z &= z \end{aligned}\right\} \dots\dots(1-55)$$

The velocity and acceleration vectors are found by differentiating, as before. This again involves derivatives of the unit vectors.

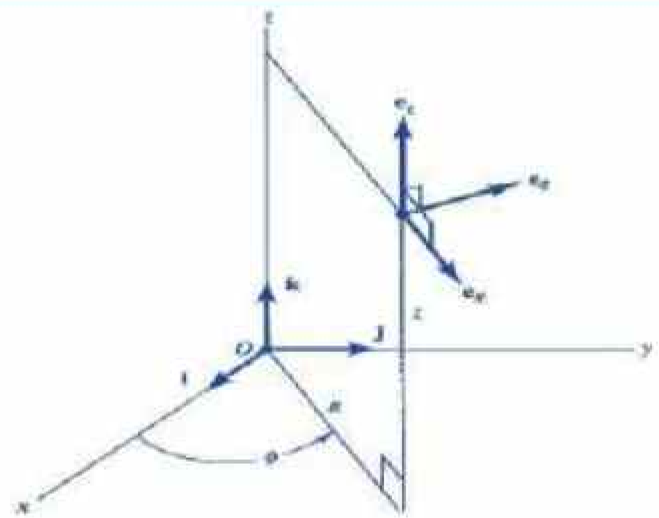


Figure 1.12.1 Unit vectors for cylindrical coordinates.

H.W: Find bothof velocity andacceleration vectors

$$\vec{v} = \dot{R}\hat{e}_R + R\dot{\phi}e_\phi + \dot{z}\hat{e}_z \quad \dots\dots\dots\dots\dots\dots.(1-56)$$

$$\vec{a} = \left(\ddot{R} - R\dot{\phi}^2\right)\hat{e}_R + \left(2\dot{R}\dot{\phi} + R\ddot{\phi}\right)e_\phi + \ddot{z}\hat{e}_z \quad \dots\dots\dots\dots\dots\dots.(1-57)$$

An alternative way of obtaining the derivatives of the unit vectors is to differentiate the following equations, which are the relationships between the fixed unit triad ***ijk*** and the rotated triad $(\hat{e}_R, e_\phi, \hat{e}_z)$:

$$\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z \quad \dots\dots\dots\dots.(1-40)$$

By sub equations (1-55) in eq. (1-40), we get

$$\vec{r} = \hat{\imath}R\cos\phi + \hat{\jmath}R\sin\phi + \hat{k}z$$

$$\vec{r} = \hat{\imath}\dot{R}\hat{e}_R + \hat{k}z$$

$$\therefore \hat{e}_R = \hat{\imath}\cos\phi + \hat{\jmath}\sin\phi \quad \dots\dots\dots\dots\dots\dots.(1-58)$$

Derivative eq. (1-56)

$$\vec{v} = \hat{\imath}\dot{R}\cos\phi - \hat{\imath}R\sin\phi\,\dot{\phi} - \hat{\jmath}\dot{R}\cos\phi + \hat{\jmath}R\cos\phi\,\dot{\phi} + \hat{k}\dot{z}$$

$$\vec{v} = \dot{R}\hat{e}_R + e_\phi R\dot{\phi} + \hat{k}\dot{z}$$

$$\vec{v} = \dot{R}\hat{e}_R - \hat{\imath}\sin\phi + \hat{\jmath}\cos\phi + \hat{k}\dot{z}$$

$$\therefore \hat{e}_\phi = -\hat{\imath}\sin\phi + \hat{\jmath}\cos\phi \quad \dots\dots\dots\dots\dots\dots.(1-59)$$

$$\therefore \hat{e}_z = \hat{k} \quad \dots\dots\dots\dots\dots\dots.(1-60)$$

Example: - From the above equation obtain the derivative of $\hat{e}_R$ and e_ϕ

Sol/

$$\frac{d\hat{e}_R}{dt} = -\hat{\imath}\sin\phi\,\dot{\phi} + \hat{\jmath}\cos\phi\,\dot{\phi}$$

$$\frac{d\hat{e}_R}{dt} = e_\phi \dot{\phi}$$

$$\frac{d\hat{e}_\phi}{dt} = -\hat{\imath}\cos\phi\,\dot{\phi} - \hat{\jmath}\sin\phi\,\dot{\phi}$$

$$\frac{d\hat{e}_\phi}{dt} = -\hat{e}_R\dot{\phi}$$

B. Spherical Coordinates

In case of spherical coordinates we used r, θ, ϕ to describe the position of a particle, the position vector is written as the product of the radial distance r and the unit radial vector $\hat{e}_r$, as with plane polar coordinates. Thus,

$$\vec{r} = r\,\hat{e}_r \quad \dots\dots\dots\dots\dots\dots.(1-61)$$

The direction of $\hat{e}_r$ is now specified by the two angles θ and ϕ.

The velocity is

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{e}_r + r\frac{d\hat{e}_r}{dt} \quad \dots\dots\dots.(1-47)$$

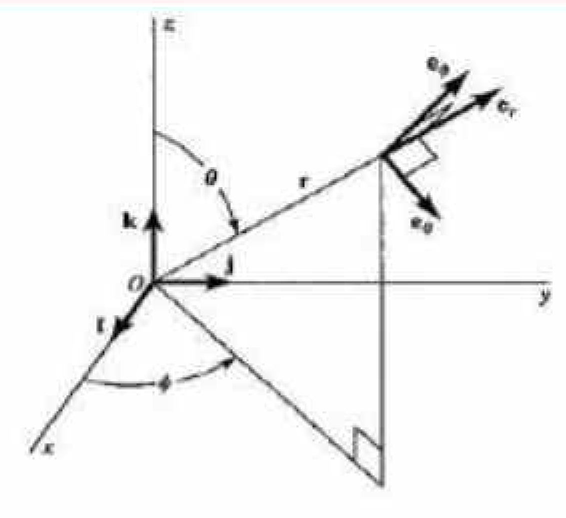


Figure 1.12.2 Unit vectors for spheric coordinates

Any vector $\vec{A}$ can be expressed in terms its projections $(\hat{\imath}, \hat{\jmath}, \hat{k})$ on to the $x, y, z,$ coordinate axes.

$$\vec{A} = \hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z \quad \dots\dots\dots\dots.(1-62)$$

$$\hat{\imath}\cdot\vec{A} = A_x \qquad \hat{\jmath}\cdot\vec{A} = A_y \qquad \hat{k}\cdot\vec{A} = A_z$$

$$\vec{A} = \hat{\imath}\left(\hat{\imath}\cdot\vec{A}\right) + \hat{\jmath}\left(\hat{\jmath}\cdot\vec{A}\right) + \hat{k}\left(\hat{k}\cdot\vec{A}\right)$$

$$\therefore \hat{e}_r = \hat{\imath}(\hat{\imath}\cdot\hat{e}_r) + \hat{\jmath}(\hat{\jmath}\cdot\hat{e}_r) + \hat{k}(\hat{k}\cdot\hat{e}_r) \quad \dots\dots\dots\dots\dots\dots(1-63)$$

$\hat{\imath}\cdot\hat{e}_r$ is the projection of the unit vector $\hat{e}_r$ directly onto the unit vector $\hat{\imath}$

$$\left.\begin{aligned} x &= r\,sin\,\theta\cos\phi \\ y &= r\,sin\,\theta\,sin\,\phi \\ z &= r\cos\theta \end{aligned}\right\} \quad \dots\dots(1-64)$$

$$\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z \quad \dots\dots\dots\dots(1-40)$$

$$\hat{\imath}\cdot\vec{r} = x = r\,sin\,\theta\cos\phi$$

The magnitude of the projection obtained in this way is the desired dot product:-

$$\hat{\imath}\cdot\hat{e}_r = x = sin\,\theta\cos\phi \quad \dots\dots\dots\dots\dots\dots(1-65)$$

$$\hat{\jmath}\cdot\hat{e}_r = y = sin\,\theta\,sin\,\phi \quad \dots\dots\dots\dots\dots\dots(1-66)$$

$$\hat{k}\cdot\hat{e}_r = z = \cos\theta \quad \dots\dots\dots\dots\dots\dots(1-67)$$

$$\therefore \hat{e}_r = \hat{\imath}\,sin\,\theta\cos\phi + \hat{\jmath}\,sin\,\theta\,sin\,\phi + \hat{k}\;cos\,\theta \quad \dots\dots\dots\dots\dots\dots(1-68)$$

✓ Orthogonal Coordinates

$$\vec{r} = \vec{r}(u_1, u_2, u_3) = \vec{r}(x, y, z) = \vec{r}(r, \theta, \phi)$$

The tangent to u_1 is

$$\frac{\partial\vec{r}}{\partial u_i} \Longrightarrow \quad \hat{u}_i = \frac{\partial\vec{r}/\partial u_i}{|\partial\vec{r}/\partial u_i|}$$

The unit vector is

$$\frac{\partial\vec{r}}{\partial u_1} \Longrightarrow \hat{u}_1 = \frac{\partial\vec{r}/\partial u_1}{|\partial\vec{r}/\partial u_1|}$$

$$\hat{e}_\theta = \frac{\partial\vec{r}/\partial e_\theta}{|\partial\vec{r}/\partial e_\theta|} \qquad \hat{e}_\phi = \frac{\partial\vec{r}/\partial e_\phi}{|\partial\vec{r}/\partial e_\phi|}$$

$$\hat{e}_\theta = \frac{\partial\vec{r}/\partial\theta}{|\partial\vec{r}/\partial\theta|} = \frac{\hat{\imath}r\cos\theta\cos\phi + \hat{\jmath}r\cos\theta\sin\phi + \hat{k}\sin\theta}{r\sqrt{cos^2\theta\cos^2\phi + cos^2\theta\, sin^2\phi + sin^2\theta}}$$

$$\hat{e}_\theta = \frac{\partial\vec{r}/\partial\theta}{|\partial\vec{r}/\partial\theta|} = \frac{r(\hat{\imath}\cos\theta\cos\phi + \hat{\jmath}\cos\theta\sin\phi) - \hat{k}\sin\theta}{r\sqrt{(\cos^2\phi + sin^2\phi)\,cos^2\theta + sin^2\theta}}$$

$$\hat{e}_\theta = \frac{\partial\vec{r}/\partial\theta}{|\partial\vec{r}/\partial\theta|} = \frac{\hat{\imath}\cos\theta\cos\phi + \hat{\jmath}\cos\theta\sin\phi - \hat{k}\sin\theta}{\sqrt{cos^2\theta + sin^2\theta}}$$

$$\therefore\ \hat{e}_\theta = \hat{\imath}\cos\theta\cos\phi + \hat{\jmath}\cos\theta\sin\phi - \hat{k}\sin\theta \quad \ldots\ldots\ldots\ldots\ldots(1-69)$$

$$\hat{e}_\phi = \frac{\partial\vec{r}/\partial\phi}{|\partial\vec{r}/\partial\phi|} = \frac{-\hat{\imath}r\sin\theta\sin\phi + \hat{\jmath}r\sin\theta\cos\phi}{r\sqrt{sin^2\theta\, sin^2\phi + cos^2\theta\, cos^2\phi}}$$

$$\hat{e}_\phi = \frac{\partial\vec{r}/\partial\phi}{|\partial\vec{r}/\partial\phi|} = \frac{r\sin\theta\,(-\hat{\imath}\sin\phi + \hat{\jmath}\cos\phi)}{r\sqrt{sin^2\theta\,(sin^2\phi + cos^2\phi)}}$$

$$\hat{e}_\phi = \frac{\partial\vec{r}/\partial\phi}{|\partial\vec{r}/\partial\phi|} = \frac{\sin\theta\,(-\hat{\imath}\sin\phi + \hat{\jmath}\cos\phi)}{\sin\theta}$$

$$\therefore\ \hat{e}_\phi = -\hat{\imath}\sin\phi + \hat{\jmath}\cos\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1-70)$$

- ✓ Last three equations are express the unit vectors of the rotated triad in terms of the fixed triad ***ijk***.
- ✓ Let us differentiate eq. (1-68) with respect to time. The result is

$$\frac{d\hat{e}_r}{dt} = \hat{\imath}(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi) + \hat{\jmath}(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi) - \hat{k}\,\dot{\theta}\sin\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots(1-71)$$

$$\frac{d\hat{e}_r}{dt} = \hat{\imath}\dot{\theta}\cos\theta\cos\phi - \hat{\imath}\dot{\phi}\sin\theta\sin\phi + \hat{\jmath}\dot{\theta}\cos\theta\sin\phi + \hat{\jmath}\dot{\phi}\sin\theta\cos\phi - \hat{k}\,\dot{\theta}\sin\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1-72)$$

Rearrange eq. (1-72)

$$\frac{d\hat{e}_r}{dt} = \hat{\imath}\dot{\theta}\cos\theta\cos\phi + \hat{\jmath}\dot{\theta}\cos\theta\sin\phi - \hat{k}\,\dot{\theta}\sin\theta - \hat{\imath}\dot{\phi}\sin\theta\sin\phi + \hat{\jmath}\dot{\phi}\sin\theta\cos\phi$$

$$\frac{d\hat{e}_r}{dt} = \dot{\theta}\left(\hat{\imath}\cos\theta\cos\phi + \hat{\jmath}\cos\theta\sin\phi - \hat{k}\,\sin\theta\right) + \dot{\phi}\sin\theta\left(-\hat{\imath}\sin\phi + \hat{\jmath}\cos\phi\right)$$

$$\frac{d\hat{e}_r}{dt} = \dot{\theta}\hat{e}_\theta + \dot{\phi}\sin\theta\,\hat{e}_\phi$$

$$\therefore \frac{d\hat{e}_r}{dt} = \hat{e}_\phi\dot{\phi}\sin\theta + \hat{e}_\theta\,\dot{\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1-73)$$

The other two derivatives are found through a similar procedure. The results are

$$\therefore \frac{d\hat{e}_\theta}{dt} = -\hat{e}_r\dot{\theta} + \hat{e}_\phi\,\dot{\phi}\cos\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1-74)$$

$$\therefore \frac{d\hat{e}_\phi}{dt} = -\hat{e}_r\dot{\phi}\sin\theta - \hat{e}_\phi\,\dot{\phi}\cos\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1-75)$$

To find the velocity, we sub eq. (1-73) in eq. (1-46), thus

$$\vec{v} = \dot{r}\hat{e}_r + \hat{e}_\phi r\dot{\phi}\sin\theta + \hat{e}_\theta r\dot{\theta} \quad \ldots\ldots\ldots\ldots(1-76)$$

To find the acceleration, we differentiate the eq. (1-76) with respect to time. This gives

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$\vec{a} = \ddot{r}\hat{e}_r + \dot{r}\frac{d\hat{e}_r}{dt} + \hat{e}_\phi \frac{d(r\dot{\phi}\sin\theta)}{dt} + r\dot{\phi}\sin\theta\frac{d\hat{e}_\phi}{dt} + \hat{e}_\theta\frac{d(r\dot{\theta})}{dt} + r\dot{\theta}\frac{d\hat{e}_\theta}{dt}$$

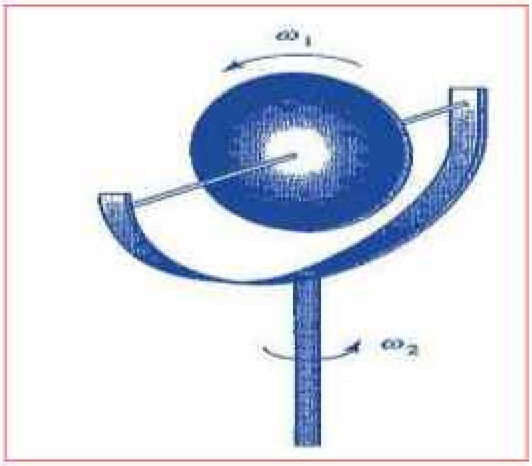
$$\vec{a} = (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\theta}^2\sin\theta\cos\theta)\hat{e}_\theta$$
$$+ (r\ddot{\theta}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\hat{e}_\phi \ldots\ldots (1-77)$$ ***H.W***

H. W

1) A bead slides on a wire bent into the form of a helix, the motion of the bead being given in cylindrical coordinates by $R = b, \phi = wt,\ z = Ct$. Find the velocity and acceleration vectors as functions of time.

2) A wheel of radius *b* is placed in a gimbal mount and is made to rotate as follows. The wheel spins with constant angular speed ω_1 about its own axis, which in turn rotates with constant angular speed ω_2 about a vertical axis in such a way that the axis of the wheel stays in a horizontal plane and the center of the wheel is motionless. Use spherical coordinates to find the acceleration of any point on the rim of the wheel. In particular, find the acceleration of the highest point on the wheel.

H. W $\Rightarrow$ 1.16 1.17 1.28 $\Leftarrow$



ميكانيك تحليلي I

الفصل الثاني

المحاضرة الأولى

أ.د. رعد الحداد

2.1 Newton's First Law: Inertial Reference Systems

- The first law describes a common property of matter, namely, **inertia**.

Inertia: is the resistance of all matter to having its motion changed.

- A mathematical description of the motion of a particle requires the selection of a **frame of reference.** (المحاور المرجعيه)

Frame of reference: A set of coordinates in configuration space that can be used to specify the **position**, **velocity**, and **acceleration** of the particle at any instant of time.

- A frame of reference in which Newton's first law of motion is valid is called an **inertial frame of reference**.

2.2 Newton's Second and Third Laws: Mass and Force

Newton's Second law: the time rate of change of an object's linear momentum ($m\vec{v}$) is proportional to the impressed force, **F**. Thus, this law can be written as:-

$$\vec{F} = \frac{d(m\vec{v})}{dt} = ma \dots\dots\dots\dots\dots\dots (2-1)$$

- The force F on the left side of Equation (2-1) is the net force acting upon the mass m; that is, it is the vector sum of all of the individual forces acting upon m.

Newton's third law: which state that two interacting bodies exert equal and opposite forces upon one another. This law can be written as:-

$$\vec{F}_1 = -\vec{F}_2 \dots\dots\dots\dots\dots\dots (2-2) \quad \textit{Newton's third law}$$

Linear Momentum

- Linear momentum represent as product of both mass $\boldsymbol{m}$ with velocity $\vec{\boldsymbol{v}}$

$$\vec{p} = m\vec{v} \quad \dots\dots\dots\dots\dots\dots (2-3) \quad in\ general$$

Therefore, Newton 2nd law in eq. (2-1) can be written in term eq. (2-4) as:-

$$\vec{F} = \frac{d\vec{p}}{dt} \dots\dots\dots\dots\dots\dots\dots\dots\dots (2-4) \quad in\ general$$

- Which represent the **time rate of change of a** **Linear momentum**.

It can be expressing better with the Newton 3rd law in term of liner momentum. So the mutual effect between two particles can be expressed as:-

$$m\vec{v}_1 = -m\vec{v}_2 \quad \dots\dots\dots\dots\dots\dots (2-5)$$

$$\vec{p}_1 = -\vec{p}_2 \quad \dots\dots\dots\dots\dots\dots (2-6)$$

$$\vec{p}_1 + \vec{p}_2 = 0 \quad \dots\dots\dots\dots\dots\dots (2-7)$$

$$\vec{F} = \frac{d(\vec{p}_1 + \vec{p}_2)}{dt} = 0 \quad \dots\dots\dots\dots\dots\dots (2-8)$$

$$\therefore \quad \vec{p}_1 + \vec{p}_2 = constant \dots\dots\dots\dots\dots\dots\dots\dots (2-9)$$

- In other words, Newton 3rd law implies that the total momentum of two mutually interacting bodies is a **constant**.
- This constancy is a special case of **the more general** situation in which **the total linear momentum of an isolated system (a system subject to no net externally applied forces) is a conserved quantity.**
- The law of ***linear momentum conservation*** is one of the most fundamental laws of physics and is valid even in situations in which Newtonian mechanics fails.

EXAMPLE 2.1.2

A spaceship of mass M is traveling in deep space with velocity $v_i = 20$ km/s relative to the Sun. It ejects a rear stage of mass $0.2\,M$ with a relative speed $u = 5$ km/s (Figure 2.1.4). What then is the velocity of the spaceship?

Solution:

The system of spaceship plus rear stage is a closed system upon which no external forces act (neglecting the gravitational force of the Sun); therefore, the total linear momentum is conserved. Thus

$$\mathbf{P}_f = \mathbf{P}_i$$

where the subscripts i and f refer to initial and final values respectively. Taking velocities in the direction of the spaceship's travel to be positive, before ejection of the rear stage, we have

$$P_i = Mv_i$$

Let U be the velocity of the ejected rear stage and v_f be the velocity of the ship after ejection. The total momentum of the system after ejection is then

$$P_f = 0.20\,MU + 0.80\,Mv_f$$

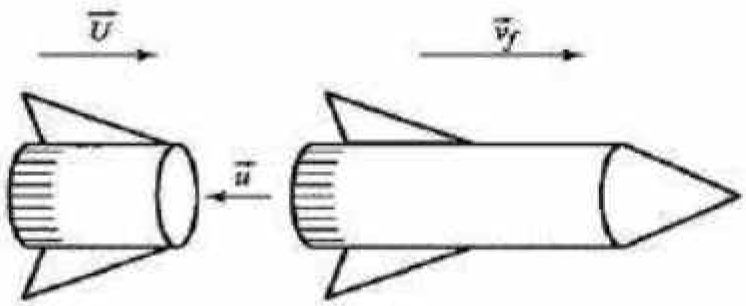


Figure 2.1.4 Spaceship ejecting a rear stage.

The speed u of the ejected stage relative to the spaceship is the difference in velocities of the spaceship and stage

$$u = v_f - U$$

or

$$U = v_f - u$$

Substituting this latter expression into the equation above and using the conservation of momentum condition, we find

$$0.20\,M(v_f - u) + 0.8\,Mv_f = Mv_i$$

which gives us

$$v_f = v_i + 0.2\,u = 20 \text{ km/s} + 0.20\,(5 \text{ km/s}) = 21 \text{ km/s}$$

Motion of a Particle

- ✓ Equation (2-1) **is the fundamental equation of motion for a particle subject to the influence of a net force, F**.
- ✓ In this section, we write the net force ***F*** as $\boldsymbol{F_{net}}$, the vector sum of all the forces acting on the particle.

$$\vec{F}_{net} = \sum \vec{F}_{net} = m\frac{d^2\vec{r}}{dt^2} = m\vec{a} \ldots\ldots\ldots\ldots\ldots(2-10)$$

The usual problem of dynamics can be expressed in the following way:

1) Given a knowledge of the forces acting on a particle (or system of particles), calculate the acceleration of the particle.
2) Knowing the acceleration, calculate the velocity and position as functions of lime.
3) This process involves solving the second-order differential equation of motion represented by Equation of motion.
4) A complete solution requires a knowledge of the initial conditions of the problem, such as the values of the position and velocity of the particle at time $t = 0$.
5) The subsequent motion of the particle can be obtained. (Not always some time the computer program are needed for approximation solutions).

2.2 Rectilinear Motion: Uniform Acceleration Under a Constant Force

- When a moving particle remains on a single straight line, the motion is said to be **rectilinear**. (الحركة على خط مستقيم)
- In this case, we need to choose only one component such the $i-axis$ as the line of motion, without loss of generality. Therefore, the general equation of motion is writing as:-

$$F_x(x,\dot{x},t) = m\ddot{x} \ldots\ldots\ldots\ldots\ldots (2-11)$$

$$\ddot{x} = \frac{dv}{dt} = \frac{F}{m} = constant = a \ldots\ldots\ldots..\ldots\ldots\ldots (2-12)$$

$$dv = adt \Longrightarrow \int_{v_0}^{v} dv = a\int_{0}^{t} dt$$

$$[v]_{v_0}^{v} = at \quad \Longrightarrow \quad v - v_0 = at$$

$$\therefore v = v_0 + at \quad \ldots\ldots\ldots\ldots (2-13)$$

By simplest and integrate the eq. (2-13)

$$\frac{dx}{dt} = v_0 + at \quad \Longrightarrow \int_{x_0}^{x} dx = v_0\int_{0}^{t} dt + a\int_{0}^{t} tdt$$

$$[x]_{x_0}^{x} = v_0t + a\left[\frac{1}{2}t^2\right]_0^t$$

$$x - x_0 = v_0t + a\frac{1}{2}t^2$$

$$\therefore x = x_0 + v_0t + a\frac{1}{2}t^2 \quad \ldots\ldots\ldots\ldots (2-14)$$

By rearrange the eq. (2-13)

$$v - v_0 = at$$

$$t = \frac{(v - v_0)}{a} \quad \dots\dots\dots\dots (2-15)$$

Sub eq. (2-15) in eq. (2-14). We get

$$x = x_0 + v_0\left(\frac{v - v_0}{a}\right) + a\frac{1}{2}\left(\frac{v - v_0}{a}\right)^2$$

$$\therefore\ x = x_0 + \frac{v_0}{a}(v - v_0) + \frac{1}{2}\frac{(v - v_0)^2}{a} \quad \dots\dots\dots\dots (2-16)$$

By rearrange the eq. (2-16)

$$2xa = 2x_0a + 2v_0(v - v_0) + (v - v_0)^2$$

$$2xa - 2x_0a = 2(vv_0 - v_0^2) + v^2 - 2vv_0 + v_0^2$$

$$2a(x - x_0) = 2vv_0 - 2v_0^2 + v^2 - 2vv_0 + v_0^2$$

$$2a(x - x_0) = -v_0^2 + v^2$$

$$\therefore\ 2a(x - x_0) = v^2 - v_0^2 \quad \dots\dots\dots\dots (2-17)$$

EXAMPLE 2.2.1

Consider a block that is free to slide down a smooth, frictionless plane that is inclined at an angle θ to the horizontal, as shown in Figure 2.2.1(a). If the height of the plane is h and the block is released from rest at the top, what will be its speed when it reaches the bottom?

Solution:

We choose a coordinate system whose positive $x - axis$ points down the plane and whose $y - axis$ points "upward," perpendicular to the plane, as shown in the figure. The only force along the x direction is the component of gravitational force, $mg\sin\theta$ as shown in Figure 2.2.1(b). It is constant. Thus, Equations (2-12 to 2-17) the equations of motion، where

$$\ddot{x} = \frac{F_x}{m} = a$$

$$\ddot{x} = a = \frac{mg \sin\theta}{m}$$

$$\therefore\ \ddot{x} = a = g \sin\theta \ \ldots\ldots\ldots\ldots (1)$$

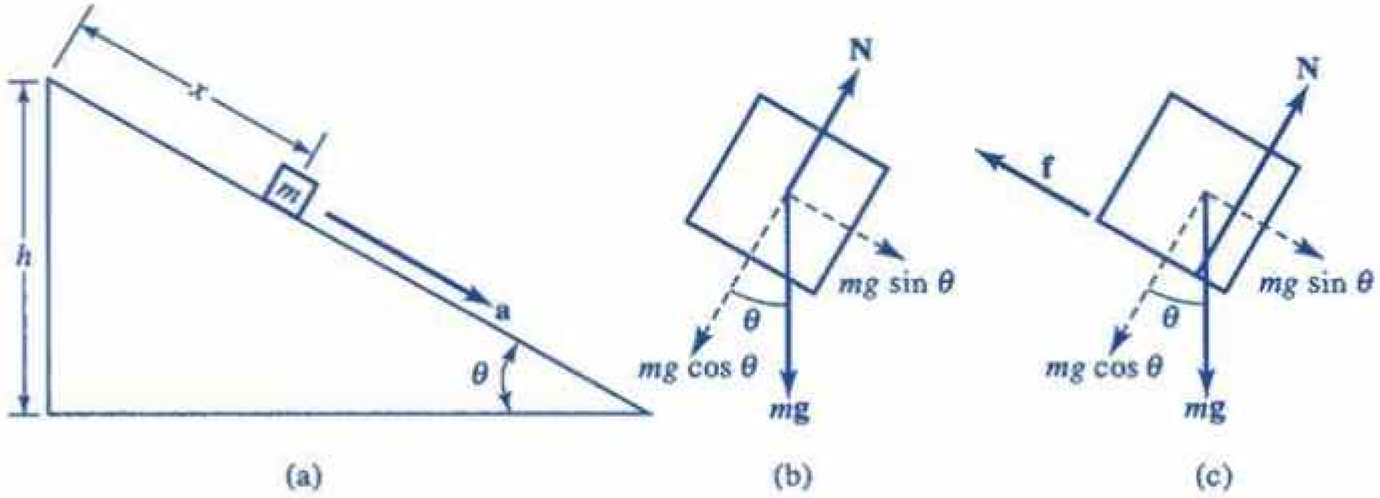


Figure 2.2.1 (a) A block sliding down an inclined plane. (b) Force diagram (no friction). (c) Force diagram (friction $f = \mu_k N$).

$$\therefore \dot{x} = g \sin\theta\ t + \dot{x}_0 \ \ldots\ldots\ldots\ldots (2)$$

Not that $\dot{x}_0 = 0$ *because start from rest*

$$\therefore v = \dot{x} = g \sin\theta\ t \ \ldots\ldots\ldots\ldots (3)$$

Rearrange and integrate eq. (3)

$$\frac{dx}{dt} = g \sin\theta\ t$$

$$\int dx = g \sin\theta \int t\, dt$$

$$\therefore x = \frac{1}{2} g \sin\theta\ t^2 \ \ldots\ldots\ldots\ldots (4)$$

at bottom $\qquad x = L \qquad$ *and* $\qquad \sin\theta = h/L$

$$\therefore L = \frac{1}{2} g \sin\theta\ t^2 \ \ldots\ldots\ldots\ldots (4)$$

$$t^2 = \frac{2L}{g\sin\theta} \quad \dots\dots\dots\dots(5)$$

$Eq.(5)$ *represent the time to reach the bottom. From* $eq.(3)$ *we have the velocity to reach the bottom*

$$\therefore\ v^2 = g^2\sin^2\theta\ t^2 \quad \dots\dots\dots\dots(6)$$

Sub $eq.(5)$ *in* $eq.(6)$ *we get:* –

$$v^2 = g^2\sin^2\theta\frac{2L}{g\sin\theta} \quad \dots\dots\dots\dots(7)$$

$$\therefore\ v^2 = 2Lg\sin\theta \quad \dots\dots\dots\dots\dots\dots(8)$$

$\because\ \sin\theta = h/L$

$$v^2 = 2Lg\ \frac{h}{L}$$

$$\therefore\ v^2 = 2gh \quad \dots\dots\dots\dots\dots\dots\dots(9)$$

Suppose that, instead of being smooth, the plane is rough; that is, it exerts a frictional force **f** on the particle. Then the net force in the x direction, (see Figure 2.2.1(c)), is equal to $mg\sin\theta - f$. Now, for sliding contact it is found that the magnitude of the frictional force is proportional to the magnitude of the normal force N; that is,

$$mg\sin\theta - f_r = ma \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(10)$$

$$mg\sin\theta - \mu_k N = ma \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(11)$$

$$mg\sin\theta - \mu_k mg\cos\theta = ma \quad \dots\dots\dots\dots\dots\dots\dots(12)$$

$$g\sin\theta - \mu_k g\cos\theta = a$$

$$a = \ddot{x} = g\left(\sin\theta - \mu_k\cos\theta\right) \quad \dots\dots\dots\dots\dots\dots\dots(13)$$

Similarly,

$$a = \frac{d\dot{x}}{dt} = g\left(\sin\theta - \mu_k\cos\theta\right)$$

$$\int d\dot{x} = g\left(\sin\theta - \mu_k\cos\theta\right)\int dt$$

$$\therefore \quad \dot{x} = g\,(sin\,\theta - \mu_k\,cos\,\theta)t \;\ldots\ldots\ldots\ldots(14)$$

$$v = \dot{x} \Longrightarrow \frac{dx}{dt} = g\,(sin\,\theta - \mu_k\,cos\,\theta)t \;\ldots\ldots\ldots..(15)$$

$$\int dx = g\,(sin\,\theta - \mu_k\,cos\,\theta)\int t\,dt$$

$$x = \frac{1}{2}g\,(sin\,\theta - \mu_k\,cos\,\theta)t^2 \quad \ldots\ldots\ldots\ldots(16)$$

$Not\ that \quad x = L\,, \quad and\ by\ \ Rearrange\ eq.(16)\ we\ get$

$$t^2 = \frac{2L}{g\,(sin\,\theta - \mu_k\,cos\,\theta)} \quad \ldots\ldots\ldots\ldots(17)$$

Take eq. (15) and square both side

$$v^2 = g^2\,(sin\,\theta - \mu_k\,cos\,\theta)^2 t^2 \;\ldots\ldots\ldots..(18)$$

$Sub\ eq.(17)\ in\ eq.(18)\ we\ get: -$

$$v^2 = g^2\,(sin\,\theta - \mu_k\,cos\,\theta)^2 \frac{2L}{g\,(sin\,\theta - \mu_k\,cos\,\theta)}$$

$$v^2 = 2g\,L(sin\,\theta - \mu_k\,cos\,\theta)$$

$$v^2 = 2g\,L\,sin\,\theta\left(1 - \mu_k\frac{cos\,\theta}{sin\,\theta}\right)\ldots\ldots\ldots\ldots(19)$$

$\because\ sin\,\theta = h/L$

$$v^2 = 2g\,L\frac{h}{L}(1 - \mu_k\cot\theta)$$

$$v^2 = 2gh(1 - \mu_k\cot\theta)\;\ldots\ldots\ldots..\ldots(20)$$

ميكانيك تحليلي I

الفصل الثاني

المحاضرة الثانيه

أ.د. رعد الحداد

2.3 Forces that Depend on Position: The Concepts of Kinetic and Potential Energy

A. Kinetic Energy

- In many instances, the effect of force on a particle depends on the particle's position with respect to other bodies. For example, this case applies to of earth attraction (gravitational forces) and the electrostatic forces.
- It also applies to forces of elastic tension or compression.
- If the force is independent of velocity or time, then the differential equation for rectilinear motion is simply

$$F(x) = m\ddot{x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2-18)$$

$$F(x) = m\frac{d\dot{x}}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2-19)$$

This type of differential equation can be solve by using the **chain rule** to write the acceleration in the following way:

$$\ddot{x} = \frac{dx}{dt}\frac{d\dot{x}}{dx} \quad \ldots\ldots\ldots\ldots\ldots(2-20)$$

$$\therefore \quad \ddot{x} = v\frac{dv}{dx} \quad \ldots\ldots\ldots\ldots\ldots(2-21)$$

So the differential equation of motion may be written as

$$F(x) = mv\frac{dv}{dx}$$

$$F(x) = \frac{1}{2}m\frac{d}{dx}(v^2) \quad \ldots\ldots\ldots\ldots(2-22)$$

Where $T = mv^2/2$ is the kinetic energy of the particle.

$$F(x) = \frac{dT}{dx} \quad \ldots\ldots\ldots\ldots.\ldots\ldots(2-23)$$

We can now express Equation (2-22) in integral form:

$$W = \int_{x_0}^{x} F(x)\,dx = \int_{T_0}^{T} dT \quad \ldots\ldots\ldots\ldots\ldots(2-23)$$

$$W = \int_{x_0}^{x} F(x)dx = (T - T_0) \quad \ldots\ldots\ldots\ldots\ldots(2-24)$$

✓ The integral of $\int_{x_0}^{x} F(x)dx$ is the work W done on the particle by the impressed force $F(x)$.

$$\therefore\ W = (T - T_0) \quad \ldots\ldots\ldots\ldots\ldots(2-25)$$

✓ The work is equal to the change in the kinetic energy of the particle.

B. Potential Energy

The force $F(x)$ can be written in term of the function $V(x)$

$$F(x) = -\frac{dV(x)}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2-26)$$

The function $V(x)$ is called the **potential energy**; it is defined only to within an arbitrary additive constant. In terms of $V(x)$, the work integral is

$$W = \int_{x_0}^{x} F(x)dx = -\int_{x_0}^{x} dV(x) \quad \ldots\ldots\ldots\ldots\ldots(2-27)$$

$$W = -[V(x)]_{x_0}^{x} = -[V(x) - V(x_0)]$$

$$W = -V(x) + V(x_0) \quad \ldots\ldots\ldots\ldots\ldots(2-28)$$

$$T - T_0 = -V(x) + V(x_0) \ldots\ldots\ldots\ldots\ldots(2-29)$$

$$T + V(x) = T_0 + V(x_0) = constant = E \quad \ldots\ldots\ldots\ldots(2-30)$$

Eq. (2-30) is conservation of mechanical energy. E is total energy

- ✓ It is called total energy. In other words, if the force acting is a function of the position only of the motion in a straight line, then the sum of the kinetic and potential energy remains constant during the movement. This is called **conservative**. As the non-conservative force, meaning they don't have potential energy, they are normal of the type of dissipation, such as friction.

C. Conservative Force

- ✓ The force is a function of position (x) and can be derivative from a corresponding potential energy function $V(x)$.
- ✓ The motion of the particle can be found by solving the energy equation (2-30)

$$E = T + V(x) \quad \ldots\ldots\ldots\ldots (2-30)$$

$$T = E - V(x) \quad \ldots\ldots\ldots\ldots (2-31)$$

$$\frac{1}{2}mv^2 = E - V(x)$$

$$v^2 = \frac{2}{m}\left(E - V(x)\right)$$

$$\therefore v = \pm\sqrt{\frac{2}{m}\left(E - V(x)\right)} \quad \ldots\ldots\ldots\ldots (2-32)$$

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\left(E - V(x)\right)}$$

$$dt = \frac{dx}{\pm\sqrt{\frac{2}{m}\left(E - V(x)\right)}}$$

$$\int_{t_0}^{t} dt = \int_{x_0}^{x} \frac{dx}{\pm\sqrt{\frac{2}{m}\left(E - V(x)\right)}}$$

$$t - t_0 = \int_{x_0}^{x} \frac{dx}{\pm\sqrt{\frac{2}{m}(E - V(x))}}$$

$$\therefore t = \int_{x_0}^{x} \frac{dx}{\pm\sqrt{\frac{2}{m}(E - V(x))}} + t_0 \qquad \dots\dots\dots\dots(2-33)$$

EXAMPLE 2.3.1

Free Fall

The motion of a freely falling body is an example of conservative motion. If we choose the x direction to be positive upward, then the gravitational force is equal to $-mg$. Therefore, eq. (2-26) become

$$-mg = -\frac{dV(x)}{dx} \quad \dots\dots\dots\dots\dots\dots\dots(1)$$

Rearrange and integrate this equation

$$\int dV(x) = mg\int dx$$

$$V(x) = mgx + C \dots\dots\dots\dots\dots\dots\dots(2)$$

The constant of integration C is arbitrary and merely depends on the choice of the reference level for measuring V. We can choose $C = 0$, which means that $V = 0$ when $x = 0$. The energy equation is then

$$V(x) = mgx \dots\dots\dots\dots\dots\dots\dots(3)$$

As we know the total energy equation is

$$E = T + V(x) \dots\dots\dots\dots(4)$$

Sub eq. (3) eq. (4), we get

$$E = T + mgx \quad \dots\dots\dots\dots(5)$$

$$E = \frac{1}{2}mv^2 + mgx \quad \dots\dots\dots\dots (6)$$

The energy constant E is determined from the initial conditions. For instance, let the body be projected upward with initial speed v_0 from the origin $x = 0$. These values give

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + mgx \quad \dots\dots\dots\dots (7)$$

$$\frac{1}{2}v_0^2 = \frac{1}{2}v^2 + gx$$

$$v_0^2 = v^2 + 2gx$$

$$v^2 = v_0^2 - 2gx \quad \dots\dots\dots\dots (8)$$

The turning point of the motion, which is in this case the *maximum height*, is given by setting $v = 0$. Therefore eq. (8) become

$$0 = v_0^2 - 2gx_{max}$$

$$v_0^2 = 2gx_{max} \quad \dots\dots\dots\dots (9)$$

Not that

$$x_{max} = h$$

$$v_0^2 = 2gh \quad \dots\dots\dots\dots (10)$$

$$\therefore h = \frac{v_0^2}{2g} \quad \dots\dots\dots\dots (11)$$

EXAMPLE 2.3.2

Variation of Gravity with Height

In Example 2.3.1 we assumed that g was constant. Actually, the force of gravity between two particles is inversely proportional to the square of the distance between them (Newton's law of gravity). Thus, the gravitational force that the Earth exerts on a body of mass m is given by

$$F_r = -\frac{GMm}{r^2} \dots\dots\dots\dots\dots\dots\dots (1)$$

Where

G is Newton's constant of gravitation

M is the mass of the Earth.

r is the distance from the center of the Earth to the body.

By definition, this force F_r is equal to the quantity $-mg$ when the body is at the surface of the Earth, so eq. (1) become

$$-mg = -G\frac{GMm}{r_e^2} \dots\dots\dots\dots\dots\dots\dots (2)$$

$$g = \frac{GM}{r_e^2} \dots\dots\dots\dots\dots\dots\dots (3)$$

Eq. (3) is the acceleration of gravity at the Earth's surface. Here r_e is the radius of the Earth (assumed to be spherical).

$$G = g\frac{r_e^2}{M} \dots\dots\dots\dots\dots\dots\dots (4)$$

Let x be the distance above the surface, so that $x = r_e + x$. Then, neglecting any other forces such as air resistance, we can write eq. (1) as

$$F_r = -G\frac{Mm}{x^2} \dots\dots\dots\dots\dots\dots\dots (5)$$

$$F_r = -G\frac{Mm}{(r_e + x)^2} \dots\dots\dots\dots\dots\dots\dots.(6)$$

Sub eq. (4) in eq. (6) we get

$$F_r = -g\frac{r_e^2}{M}\frac{Mm}{(r_e + x)^2}$$

$$F_r = -mg\frac{r_e^2}{(r_e + x)^2} = m\ddot{x} \dots\dots\dots\dots\dots\dots\dots.(7)$$

$$\ddot{x} = -g\frac{r_e^2}{(r_e + x)^2} \dots\dots\dots\dots\dots\dots\dots.(8)$$

For the differential equation of motion of a vertically falling (or rising) body with the variation of gravity taken into account. To integrate, we see

$$\ddot{x} = \frac{d\dot{x}}{dx} = v\frac{dv}{dx}$$

$$\frac{dv}{dx} = -g\frac{r_e^2}{(r_e + x)^2}$$

$$\int_{v_0}^{v} v\,dv = -gr_e^2\int_{x_0}^{x}\frac{dx}{(r_e + x)^2}$$

$$\int_{v_0}^{v} v\,dv = -gr_e^2\int_{x_0}^{x}(r_e + x)^{-2}dx$$

$$\frac{1}{2}[v^2]_{v_0}^{v} = -gr_e^2\left[\frac{(r_e + x)^{-1}}{-1}\right]_{x_0}^{x}$$

$$\frac{1}{2}[v^2 - v_0^2] = gr_e^2\left[\frac{1}{(r_e + x)}\right]_{x_0}^{x}$$

$$\frac{1}{2}v^2 - \frac{1}{2}v_0^2 = gr_e^2\left[\frac{1}{(r_e+x)} - \frac{1}{(r_e+x_0)}\right] \dots\dots\dots\dots (9)$$

Multiply both side of eq. (8) with m

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = mgr_e^2\left[\frac{1}{(r_e+x)} - \frac{1}{(r_e+x_0)}\right] \dots\dots\dots\dots (10)$$

This is just the energy equation in the form of Equation 2-25.

$$T - T_0 = mgr_e^2\left[\frac{1}{(r_e+x)} - \frac{1}{(r_e+x_0)}\right] \dots\dots\dots\dots (11)$$

$$T - mg\frac{r_e^2}{(r_e+x)} = T_0 - mg\frac{r_e^2}{(r_e+x_0)} \dots\dots\dots\dots (11)$$

$$T + V(x) = T_0 - V(x_0) \dots\dots\dots\dots (12)$$

By comparing between eq. (11) and eq. (12) we can conclude that the potential energy is

$$\therefore \quad V(x) = -mg\frac{r_e^2}{(r_e+x)} \quad \dots\dots\dots\dots (13)$$

H.W $\Rightarrow$ 2.10 2.11 2.12 $\Leftarrow$

ميكانيك تحليلي I

الفصل الثالث

المحاضرة الأولى

أ.د. رعد الحداد

General Motion of a Particle in Three Dimensions

3.1 General Principles:-

We now examine the general case of the motion of a particle in three dimensions. The vector form of the equation of motion for such a particle is

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3-1)$$

Where $\vec{p} = mv$ *is the linear momentum of the particle.*

$$\vec{F} = \frac{d}{dt}(m\vec{v}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3-2)$$

The above equation represents an abbreviation of the following three equations:-

$$\vec{F}_x = \frac{d}{dt}(m\dot{x}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3-3)$$

$$\vec{F}_y = \frac{d}{dt}(m\dot{y}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3-4)$$

$$\vec{F}_z = \frac{d}{dt}(m\dot{z}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3-5)$$

The Work Principle

- ✓ Work done on a particle causes it to **gain** or **lose kinetic energy**.

We first take the dot product of both sides of eq. (3-2) with the velocity $\vec{v}$:

$$\vec{v}\cdot\vec{F} = \vec{v}\cdot\frac{d}{dt}(m\vec{v}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3-6)$$

$$\vec{v}\cdot\vec{F} = \frac{d}{dt}\left(\frac{1}{2}m\vec{v}^2\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3-7)$$

$$\vec{F}\cdot\vec{v} = \frac{dT}{dt} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3-8)$$

Now rewrite and integrate eq. (3-8)

$$\vec{F}\cdot\frac{d\vec{r}}{dt}=\frac{dT}{dt}$$

$$\int \vec{F}\cdot d\vec{r}=\int_{T_i}^{T_f} dT$$

$$\int \vec{F}\cdot d\vec{r}=T_f-T_i=\Delta T \quad \ldots(3-9)$$

$\vec{F}_r\cdot d\vec{r}$: The component of $\vec{F}$ parallel to the particle's displacement vector $d\vec{r}$.

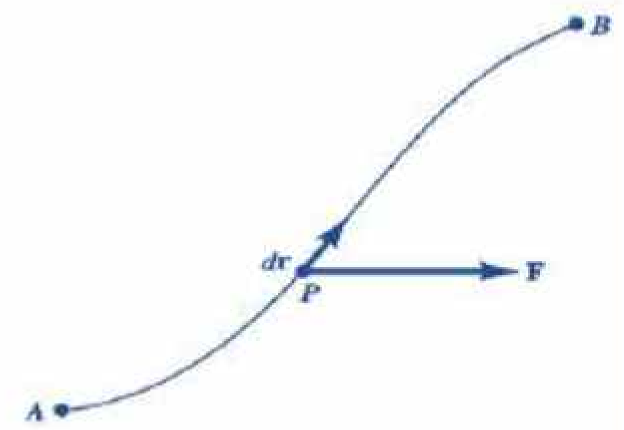


Figure 4.1.1 The work done by a force **F** is the line integral $\int_A^B \mathbf{F}\cdot d\mathbf{r}$.

$$\int \vec{F}\cdot d\vec{r}=T_B-T_A \quad \ldots\ldots\ldots\ldots(3-10)$$

- ✓ The integral is carried out along the trajectory of the particle from some initial point in space **A** to some final point **B** is equal to the net change in the kinetic energy of the particle.

Conservative Forces and Force Fields

The forces that they could be derived as the derivative of a scalar potential energy function

$$F(x)=-\frac{dV(x)}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3-11)$$

$$\int \vec{F}\cdot d\vec{x}=-\nabla V=V(A)-V(B)\ldots\ldots\ldots\ldots(3-12)$$

Thus, we no longer required a detailed knowledge of the motion of the particle from **A** to **B** to calculate the work done on it by a conservative force. We needed to know only that it started at point **A** and ended up at point **B**. The work done depended only upon the potential energy function evaluated at the endpoints of the motion.

$$\oint \vec{F}\cdot d\vec{r}=0 \quad \ldots\ldots\ldots\ldots(3-13) \quad \textit{The force is conservative}$$

Stokes' theorem

- ✓ The theorem states that the closed-loop line integral of any vector function $\vec{F}$ is equal to curl $\vec{F} \cdot \hat{n}\, da$ integrated over a surface **S** surrounded by the closed loop.
- ✓ The vector $\hat{n}$ is a unit vector normal to the surface-area integration element da.

$$\oint \vec{F} \cdot d\vec{r} = \int_S Curl\, \vec{F} \cdot \hat{n}\, da \;\ldots\ldots\ldots\ldots(3-14)$$

$$Curl\, \vec{F} = \hat{\imath}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial x}\right) \ldots\ldots(3-15)$$

- ✓ If $\vec{\nabla} \times \vec{F} = 0 \quad \Rightarrow The\ force\ is\ conservative$
- ✓ The **delta operator** $(\vec{\nabla})$ is used as a convening criterion for determining whether the field strength is **<u>conservative</u>** or **<u>non-conservative</u>**.

4.2 The Potential Energy Function in Three-Dimensional Motion:

The Del Operator $\vec{\nabla}$: $V = V(x, y, z)$

Assume that we have a test particle subject to some force whose curl vanishes. Then all the components of $curl\ \vec{F}$ in Equation (3-14) vanish. We can make certain that the curl vanishes if we derive $\vec{F}$ from a potential energy function $V(x, y, z)$ according to

$$\vec{F}_x = -\frac{\partial V}{\partial x} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3-16)$$

$$\vec{F}_y = -\frac{\partial V}{\partial y} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3-17)$$

$$\vec{F}_z = -\frac{\partial V}{\partial z} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3-18)$$

<u>**Note:**</u> If the force field is conserved so that the compounds are given in the form of partial derivatives of the potential energy function (as mentioned above), then we can represent the field by the vector algebra as follows:-

$$\vec{F} = \hat{\imath}\frac{\partial V}{\partial x} - \hat{\jmath}\frac{\partial V}{\partial y} - \hat{k}\frac{\partial V}{\partial z} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots..(3-19)$$

We can write eq. (3-19) briefly

$$\vec{F} = -\vec{\nabla}V \quad \dots\dots\dots\dots\dots\dots\dots\dots..(3-20)$$

Where the vector operator del is

$$\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} - \hat{\jmath}\frac{\partial}{\partial y} - \hat{k}\frac{\partial}{\partial z} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots..(3-21)$$

- ✓ $\vec{\nabla}V$ is called gradient of V.
- ✓ Physically, the negative gradient of the potential energy function gives the direction and magnitude of the force that acts on a particle located in a field created by other particles.
- ✓ The meaning of the negative sign is that the particle is urged to move in the direction of decreasing potential energy rather than in the opposite direction.

EXAMPLE 4.2.3

Is the force field $\mathbf{F} = \mathbf{i}xy + \mathbf{j}xz + \mathbf{k}yz$ conservative?

Sol/

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & xz & yz \end{vmatrix}$$

$$\vec{\nabla} \times \vec{F} = \hat{\imath}\begin{vmatrix} \partial/\partial y & \partial/\partial z \\ xz & yz \end{vmatrix} - \hat{\jmath}\begin{vmatrix} \partial/\partial x & \partial/\partial z \\ xy & yz \end{vmatrix} + \hat{k}\begin{vmatrix} \partial/\partial x & \partial/\partial y \\ xy & xz \end{vmatrix}$$

$$\vec{\nabla} \times \vec{F} = \hat{\imath}\left(\frac{\partial yz}{\partial y} - \frac{\partial xz}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial yz}{\partial x} - \frac{\partial xy}{\partial z}\right) + \hat{k}\left(\frac{\partial xz}{\partial x} - \frac{\partial xy}{\partial y}\right)$$

$$\vec{\nabla} \times \vec{F} = \hat{\imath}(z - x) - \hat{\jmath}(0 - 0) + \hat{k}(z - x)$$

$$\therefore \ \vec{\nabla} \times \vec{F} \neq 0 \quad \textit{the field is not conservative.}$$

EXAMPLE 4.2.4

For what values of the constants a, b, and c is the force $\mathbf{F} = \mathbf{i}(ax + by^2) + \mathbf{j}cxy$ conservative?

Sol/

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ax + by^2 & cxy & 0 \end{vmatrix}$$

$$\vec{\nabla} \times \vec{F} = \hat{\imath} \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ cxy & 0 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ ax + by^2 & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ ax + by^2 & cxy \end{vmatrix}$$

$$\vec{\nabla} \times \vec{F} = \hat{\imath}\left(0 - \frac{\partial cxy}{\partial z}\right) - \hat{\jmath}\left(0 - \frac{\partial(ax + by^2)}{\partial z}\right) + \hat{k}\left(\frac{\partial cxy}{\partial x} - \frac{\partial(ax + by^2)}{\partial y}\right)$$

$$\vec{\nabla} \times \vec{F} = \hat{\imath}(0 - 0) - \hat{\jmath}(0 - 0) + \hat{k}\left(cy - \frac{\partial ax}{\partial y} - \frac{\partial}{\partial y} by^2\right)$$

$$\vec{\nabla} \times \vec{F} = \hat{k}(cy - 0 - 2by)$$

$$\vec{\nabla} \times \vec{F} = \hat{k}(c - 2b)y$$

If $c = 2b$

$$\vec{\nabla} \times \vec{F} = \hat{k}(2b - 2b)y = 0$$

$\vec{\nabla} \times \vec{F} = 0$ *so the force is conservative. The value of* $\boldsymbol{a}$ *is immaterial.*

3.3 Constrained Motion of a Particle (حركة الجسيم المقيدة)

When a moving particle is restricted geometrically in the sense that it must stay on a certain definite surface or curve, the motion is said to be **constrained**.

The Energy Equation for Smooth Constraints

The total force acting on a particle moving under constraint can be expressed as the vector sum of **the net external force** $\vec{F}$ and **the force of constraint** $\vec{R}$. The latter force is the reaction of the constraining agent upon the particle. The equation of motion may, therefore, be written

$$m\frac{d\vec{v}}{dt} = \vec{F} + \vec{R} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3-22)$$

If we take the dot product with the velocity $\vec{v}$, we have

$$m\frac{d\vec{v}}{dt}\cdot\vec{v} = \vec{F}\cdot\vec{v} + \vec{R}\cdot\vec{v} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3-23)$$

Now in the case of a smooth constraint—for example, a frictionless surface—the reaction $\vec{R}$ **is normal to the surface or curve** while the velocity $\vec{v}$ is **tangent to the surface**. Hence, $\vec{R}$ is perpendicular to $\vec{v}$, and the dot product $\vec{R}\cdot\vec{v}$ vanishes ($\vec{R}\cdot\vec{v} = 0$). Equation (3-23) then reduces to

$$\frac{d}{dt}\left(\frac{1}{2}m\vec{v}\cdot\vec{v}\right) = \vec{F}\cdot\vec{v} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3-24)$$

$$d\left(\frac{1}{2}m\vec{v}\cdot\vec{v}\right) = \vec{F}\cdot\vec{v}\,dt$$

$$d\left(\frac{1}{2}m\vec{v}\cdot\vec{v}\right) = \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt$$

$$\therefore\ d\left(\frac{1}{2}m\vec{v}\cdot\vec{v}\right) = \vec{F}\cdot d\vec{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3-25)$$

H.W $\Rightarrow$ 4.3 4.4 4.5 4.21 4.22 $\Leftarrow$

ميكانيك تحليلي I

الفصل الرابع

المحاضرة الأولى

أ.د. رعد الحداد

Noninertial Reference Systems

4.1 Accelerated Coordinate Systems and Inertial Forces:-

- ✓ Often times it is preferable to use moving axes to describe the motion of a particle.
- ✓ Despite the movment of the earth and its rotation, it is prefers to use axes fixed in them to study the movment of the projectile.

Translation of coordinate system

Transitional axis movement is the simplest type motion.

From the figure (1):-

1. $Oxyz$ are the primary coordinate axes (assumed fixed).
2. $O'x'y'z'$ are the moving axes.
3. In the case of translational motion, the respective axes Ox and $O'x'$, and soon, remain **parallel**.

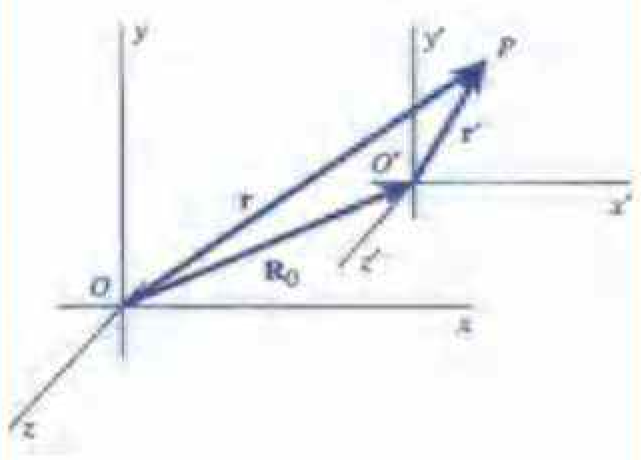


Figure (1) Relationship between the position vectors for two coordinate systems undergoing pure translation relative to each other.

If assume that $\vec{r}$ represents the position vector of a particle P in the fixed system, $\vec{r}'$ in the moving system and $\vec{R}_0$ represents the displacement of the origin from its original position OO'. Thus, from the triangle $OO'P$, we have

$$\vec{r} = \vec{R}_0 + \vec{r}' \quad \ldots\ldots\ldots\ldots\ldots\ldots (4-1)$$

Taking the first and second time derivatives gives

$$\vec{v} = \vec{V}_0 + \vec{v}' \quad \ldots\ldots\ldots\ldots\ldots\ldots (4-2)$$

$$\vec{a} = \vec{A}_0 + \vec{a}' \quad \ldots\ldots\ldots\ldots\ldots\ldots (4-3)$$

in which $\vec{V}_0$ and $\vec{A}_0$ are, respectively, the velocity and acceleration of the moving system, and $\vec{v}'$ and $\vec{a}'$ are the velocity and acceleration of the particle in the moving system.

In particular, if the moving system is not accelerating, so that $\vec{A}_0 = 0$, then

$$\vec{a} = \vec{a}' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4-4)$$

That is, the acceleration of point P is the same for two sets of axes, and this is true for only in the absence of rotational movement in the moving axes.

Inertial Forces (القوة الزائفه)

If the primary system ($x\ y\ z$) is **inertial**, Newton's second law in fixed axes:-

$$\vec{F} = m\vec{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4-5)$$

In the moving axes ($x'y'z'$), Newton's 2nd

$$\vec{F} = m\vec{a}' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4-6)$$

$\therefore$ $x'y'z'$ is also **an inertial** system moving with uniform velocity relative to the first (provided it is not rotating).

If the moving system is accelerating, then, the equation of motion (4-3) by using (4-6) is

$$\vec{F} = m\vec{A}_0 + m\vec{a}' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4-7)$$

OR

$$\vec{F} - m\vec{A}_0 = m\vec{a}' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4-8)$$

If we wish, we can write eq. (4-6) in the form

$$\vec{F}' = m\vec{a}' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4-9)$$

And sub eq. (4-9) in eq. (4-8) we get

$$\vec{F} - m\vec{A}_0 = \vec{F}' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4-10)$$

OR

$$\vec{F}' = \vec{F} - \left(-m\vec{A}_0\right) \quad \ldots\ldots\ldots\ldots\ldots (4-11)$$

- ✓ That is, an acceleration $\vec{A}_0$ of the reference system can be taken into account by adding an **inertial term** $(-m\vec{A}_0)$ to the force $\vec{F}$ and equating the result to the product of mass and acceleration in the moving system.
- ✓ **Inertial terms** in the equations of motion are sometimes called **inertial forces**, or **fictitious forces**. Such "forces" are not due to interactions with other bodies, rather, they stem from the acceleration of the reference system.

EXAMPLE 5.1.1

A block of wood rests on a rough horizontal table. If the table is accelerated in a horizontal direction, under what conditions will the block slip?

Solution:

Let μ_s be the coefficient of static friction between the block and the table top. Then the force of friction F has a maximum value of $\mu_s mg$, where m is the mass of the block. The condition for slipping is that the inertial force $-mA_0$ exceeds the frictional force, where A_0 is the acceleration of the table. Hence, the condition for slipping is

$$|-mA_0| > \mu_s mg$$

or

$$A_0 > \mu_s g$$

4.2 Rotating Coordinate Systems:-

- ✓ In the previous section, we showed how velocities, accelerations, and forces transform between an inertial frame of reference and a noninertial one that is accelerating at a constant rate.
- ✓ In this section and the following one, we show how these quantities transform between an inertial frame and a noninertial one that is rotating as well.

Let us consider, as before, the position of the particle in the reference axes with the symbol $\vec{r}$, and in the moving axes with the symbol $\vec{r}'$

$$\vec{r} = \hat{\imath}'x + \hat{\jmath}'y + \hat{k}'z \quad \ldots\ldots\ldots\ldots (4-12)$$

$$\vec{r}' = \hat{\imath}'x' + \hat{\jmath}'y' + \hat{k}'z' \quad \ldots\ldots\ldots\ldots (4-13)$$

Note: Because the coordinate axes of the two systems have the same origin, these vectors are equal, that is, $\vec{r} = \vec{r}'$, as shown in fig. (2).

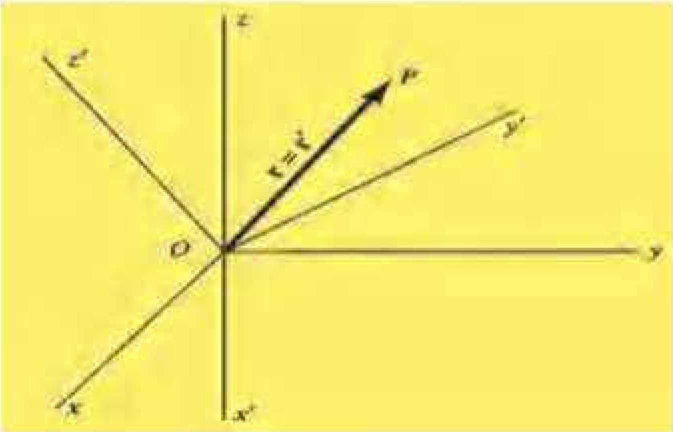


Figure (2) rotating coordinate system (primed system).

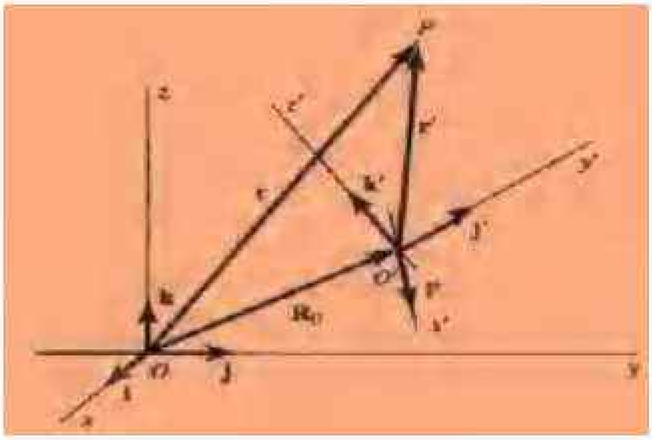


Figure (3) Geometry for the general case of translation and rotation of the moving coordinate system (primed system).

Whereupon, if $\vec{R}_0$ represents the position vector of the moving axes, as shown in fig. (3). Then

$$\vec{r} = \vec{R}_0 + \vec{r}' \qquad \ldots\ldots\ldots\ldots\ldots (4-14)$$

$$\vec{r} = \vec{R}_0 + \hat{\imath}'x' + \hat{\jmath}'y' + \hat{k}'z' \ldots (4-15)$$

By derivative eq. (4-15) with respect to the time

$$\frac{d\vec{r}}{dt} = \frac{d\vec{R}_0}{dt} + \hat{\imath}'\frac{dx'}{dt} + \hat{\jmath}'\frac{dy'}{dt} + \hat{k}'\frac{dz'}{dt} + x'\frac{d\hat{\imath}'}{dt} + y'\frac{d\hat{\jmath}'}{dt} + z'\frac{d\hat{k}'}{dt}$$

$$\vec{v} = \vec{V}_0 + \underline{\hat{\imath}'\dot{x}' + \hat{\jmath}'\dot{y}' + \hat{k}'\dot{z}'} + x'\frac{d\hat{\imath}'}{dt} + y'\frac{d\hat{\jmath}'}{dt} + z'\frac{d\hat{k}'}{dt} \ldots\ldots\ldots\ldots (4-16)$$

$$\therefore\ \vec{v} = \vec{V}_0 + \vec{v}' + x'\frac{d\hat{\imath}'}{dt} + y'\frac{d\hat{\jmath}'}{dt} + z'\frac{d\hat{k}'}{dt} \ldots\ldots\ldots\ldots\ldots\ldots (4-17)$$

- ✓ The last three terms on the right represent the velocity due to rotation of the primed coordinate system ($Oxyz$).
- ✓ We must now determine how the time derivatives of the basis vectors are related to the rotation ($d\hat{\imath}'/dt,\ d\hat{\jmath}'/dt\ and\ d\hat{k}'/dt$).

Let us now represent the direction of t axis of rotation in the axes by a unit vector, $\hat{n}$ and the instantaneous angu speed of the rotation around this axis w the symbol ω as shown in fig.(4). the product, $\hat{n}\omega$ is the angular velocity ($\vec{\omega}$) of the rotating system:

$$\vec{\omega} = \hat{n}\omega \quad \ldots\ldots\ldots\ldots(4-18)$$

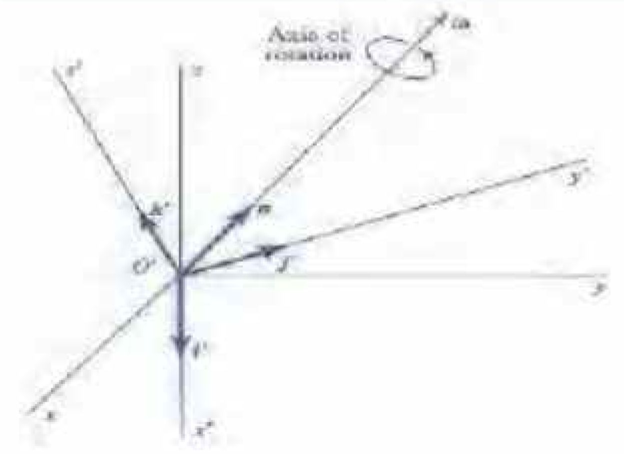


Figure (4) the angular velocity vector of a rotating coordinate system.

✓ The direction of the angular velocity vector is set using the right-hand rule

To find the time derivative of the bas vectors in terms of $\vec{\omega}$, see Fig.(5).

1) The change in unit vector ($\hat{\imath}'$) by ($\Delta\hat{\imath}'$) due to a small rotation ($\Delta\theta$) about the axis of rotation. (The vectors $\hat{\jmath}'$ and $\hat{k}'$ have been removed to simplify th solution).
2) From the figure we see that th magnitude of $\Delta\hat{\imath}'$ is given by the approximate relation

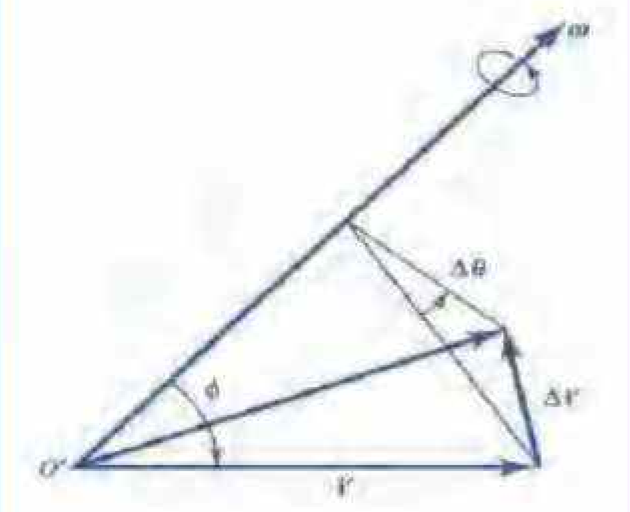


Figure (5) Change in the unit vector produced by a small rotation $\Delta\theta$.

$$|\Delta\hat{\imath}'| \cong (|\hat{\imath}'|\sin\phi)\Delta\theta = (\sin\phi)\Delta\theta \quad \ldots\ldots\ldots\ldots(4-19)$$

Where ϕ is the angle between $\hat{\imath}'$ and $\vec{\omega}$. $\Delta\theta$ represent the amount of rotation of the axes ($Oxyz$) that occurs in a certain time interval Δt. Then

$$\left|\frac{d\hat{\imath}'}{dt}\right| = \lim_{\Delta t\to 0}\left|\frac{\Delta\hat{\imath}'}{\Delta t}\right| = \sin\phi\,\frac{d\theta}{dt} = (\sin\phi)\omega \quad \ldots\ldots\ldots\ldots(4-20)$$

But the direction of $\Delta\hat{\imath}'$ is perpendicular to both $\vec{\omega}$ and $\hat{\imath}'$, and so we can express $d\hat{\imath}'/dt$ by the cross product of $\vec{\omega}$ and $\hat{\imath}'$, then we can write eq.(4-20) in vector form

$$\frac{d\hat{\imath}'}{dt} = \vec{\omega} \times \hat{\imath}' \quad \ldots\ldots\ldots\ldots\ldots (4-21)$$

Similarly, we find

$$\frac{d\hat{\jmath}'}{dt} = \vec{\omega} \times \hat{\jmath}' \quad \ldots\ldots\ldots\ldots\ldots (4-22)$$

$$\frac{d\hat{k}'}{dt} = \vec{\omega} \times \hat{k}' \quad \ldots\ldots\ldots\ldots\ldots (4-23)$$

We now apply the preceding result to the last three terms in eq. (4-17) as follows:

$$\therefore\ \vec{v} = \vec{V}_0 + \vec{v}' + x'(\vec{\omega} \times \hat{\imath}') + y'(\vec{\omega} \times \hat{\jmath}') + z'(\vec{\omega} \times \hat{k}') \ldots\ldots\ldots\ldots (4-24)$$

$$\therefore \vec{v} = \vec{V}_0 + \vec{v}' + \vec{\omega} \times (\hat{\imath}'x' + \hat{\jmath}'y' + \hat{k}'z') \quad \ldots\ldots\ldots\ldots\ldots\ldots (4-25)$$

$$\therefore \vec{v} = \vec{v}' + \vec{\omega} \times \vec{r}' + \vec{V}_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4-26)$$

The eq. (4-26) expresses the relationship between the time derivatives of the two position vectors of a moving particle in two axes, the first being considering fixed and the second moving and rotating. The limit $\vec{V}_0$ is show due to the transitional and rotation moving axes only and do not appear in the case of pure rotation.

✓ Now we calculate the acceleration of particle in the fixed frame:-

$$\left(\frac{d\vec{v}}{dt}\right)_{fixed} = \left(\frac{d\vec{V}_0}{dt}\right)_{fixed} + \left(\frac{d\vec{v}'}{dt}\right)_{fixed} + \dot{\vec{\omega}} \times \vec{r}' + \vec{\omega} \times \left(\frac{d\vec{r}'}{dt}\right)_{fixed}$$

$$\vec{a} = \vec{A}_0 + \left(\frac{d\vec{v}'}{dt}\right)_{fixed} + \dot{\vec{\omega}} \times \vec{r}' + \vec{\omega} \times \left(\frac{d\vec{r}'}{dt}\right)_{fixed} \quad \ldots\ldots\ldots\ldots (4-27)$$

The second term can be find by using

$$\left(\frac{dQ}{dt}\right)_{fixed} = \left(\frac{dQ}{dt}\right)_{rot} + \vec{\omega} \times Q$$

$$\left(\frac{d\vec{v}'}{dt}\right)_{fixed} = \left(\frac{d\vec{v}'}{dt}\right)_{rot} + \vec{\omega} \times \vec{v}' \quad \ldots\ldots\ldots\ldots\ldots (4-28)$$

The last term is

$$\vec{\omega} \times \left(\frac{d\vec{r}'}{dt}\right)_{fixed} = \vec{\omega} \times \left[\left(\frac{d\vec{r}'}{dt}\right)_{rot} + \vec{\omega} \times \vec{r}'\right]$$

$$\vec{\omega} \times \left(\frac{d\vec{r}'}{dt}\right)_{fixed} = \vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}'\ ) \dots\dots\dots\dots\dots (4-29)$$

Sub eq. (4-29) and (4-28) in eq. (4-27), we get

$$\vec{a} = \vec{A}_0 + \left(\frac{d\vec{v}'}{dt}\right)_{rot} + \underline{\vec{\omega} \times \vec{v}'} + \dot{\vec{\omega}} \times \vec{r}' + \underline{\vec{\omega} \times \vec{v}'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}'\ )$$

$$\therefore \vec{a} = \vec{a}' + \dot{\vec{\omega}} \times \vec{r}' + 2\vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}') + \vec{A}_0 \dots\dots\dots (4-30)$$

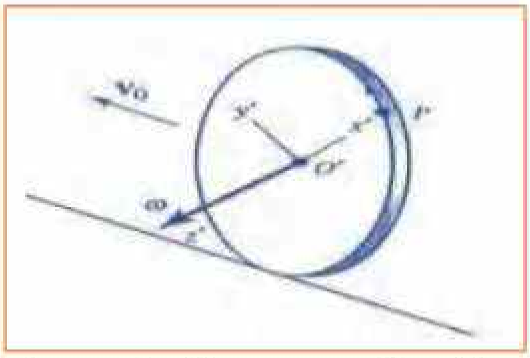
1. The term $(\dot{\vec{\omega}} \times \vec{r}')$ is called the **transverse acceleration**.
2. The term $(2\vec{\omega} \times \vec{v}')$ is known as the **Coriolis acceleration**.
3. the term $[\vec{\omega} \times (\vec{\omega} \times \vec{r}')]$ is called the **centripetal acceleration**.

EXAMPLE 5.2.1

A wheel of radius **b** rolls along the ground with constant forward speed V_0. Find the acceleration, relative to the ground, of any point on the rim.

Sol/

Let us choose a coordinate system fixed to the rotating wheel, and let the moving origin be at the center with the $x' - axis$ passing through the point in question, as shown in Figure. Then we have



$$\vec{r}' = \hat{\imath}'b \qquad \vec{v}' = \dot{\vec{r}}' = 0 \qquad \vec{a}' = \ddot{\vec{r}}' = 0$$

The angular velocity vector is given by

$$\vec{\omega} = \hat{k}'\omega = \hat{k}'\frac{V_0}{b}$$

for the choice of coordinates shown; therefore, all terms in the expression for acceleration **vanish** except the centripetal term:

$$\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r}') = \hat{k}'\omega \times (\hat{k}'\omega \times \hat{\imath}'b)$$

$$\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r}') = \hat{k}' \frac{V_0}{b} \times \left(\hat{k}' \frac{V_0}{b} \times \hat{\imath}' b \right)$$

$$\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r}') = \frac{V_0^2}{b^2} b \, \hat{k}' \times (\hat{k}' \times \hat{\imath}')$$

$$\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r}') = V_0^2/b \, (\hat{k}' \times \hat{\jmath}')$$

$$\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r}') = V_0^2/b \, (-\hat{\imath}')$$

Thus, a is of magnitude V_0^2/b and is always directed toward the center of the rolling wheel.

4.3 Dynamics of a Particle in a Rotating Coordinate System:-

The fundamental equation of motion of a particle in an inertial frame of reference is

$$\vec{F} = m\vec{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4-5)$$

By multiply both side of eq. (4-30) by m

$$m\vec{a} = m\vec{a}' + m\dot{\vec{\omega}} \times \vec{r}' + 2m\vec{\omega} \times \vec{v}' + m\vec{\omega} \times (\vec{\omega} \times \vec{r}') + m\vec{A}_0$$

$$m\vec{a} - m\vec{A}_0 - m\dot{\vec{\omega}} \times \vec{r}' - 2m\vec{\omega} \times \vec{v}' - m\vec{\omega} \times (\vec{\omega} \times \vec{r}') = m\vec{a}'$$

$$\vec{F} - m\vec{A}_0 - 2m\vec{\omega} \times \vec{v}' - m\dot{\vec{\omega}} \times \vec{r}' - m\vec{\omega} \times (\vec{\omega} \times \vec{r}') = \vec{F}' \quad \ldots\ldots\ldots (4-31)$$

The Coriolis force is

$$\vec{F}'_{Cor} = -2m\vec{\omega} \times \vec{v}' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4-32)$$

The transverse force is

$$\vec{F}'_{trans} = -m\dot{\vec{\omega}} \times \vec{r}' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4-33)$$

The centrifugal force is

$$\vec{F}'_{centrif} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}') \qquad \ldots\ldots\ldots\ldots (4-34)$$

$$\vec{F} - m\vec{A}_0 + \vec{F}'_{Cor} + \vec{F}'_{trans} + \vec{F}'_{centrif} = \vec{F}'$$

$$\vec{F}' = \vec{F}_{physical} + \vec{F}'_{Cor} + \vec{F}'_{trans} + \vec{F}'_{centrif} - m\vec{A}_0 \quad \ldots\ldots\ldots\ldots (4-35)$$

EXAMPLE 5.3.1

A bug crawls outward with a constant speed v' along the spoke of a wheel that is rotating with constant angular velocity ω about a vertical axis. Find all the apparent forces acting on the bug (see Figure 5.3.2).

Sol/

First, let us choose a coordinate system fixed on the wheel, and let the $x'-axis$ point along the spoke in question. Then we have

$$\vec{\dot{r}}' = \hat{\imath}'\dot{x}' = \hat{\imath}'\vec{v}' \qquad \vec{a}' = \vec{\ddot{r}}' = 0$$

The angular velocity vector is given by for the velocity and acceleration of the bug as described in the rotating system. If we

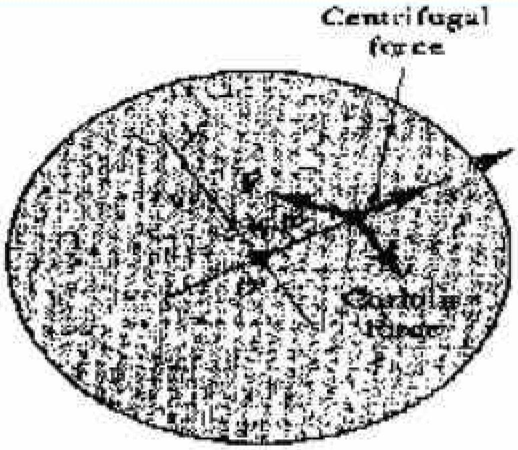


Figure 5.3.2 Forces on an insect crawling outward along a radial line on a rotating wheel.

$$\vec{\omega} = \hat{k}'\omega$$

The various forces are then given by the following:

$$\vec{F}'_{Cor} = -2m\vec{\omega}\times\vec{v}' = -2m\vec{\omega}\vec{v}'(\hat{k}'\times\hat{\imath}')$$

$$\vec{F}'_{Cor} = -2m\vec{\omega}\vec{v}'\hat{\jmath}'$$

$$\vec{F}'_{trans} = -m\vec{\dot{\omega}}\times\vec{r}' = 0 \qquad because\ \omega = constant$$

$$\vec{F}'_{centrif} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r}') = -m\omega^2[\hat{k}'\times(\hat{k}'\times\hat{\imath}'x')]$$

$$\vec{F}'_{centrif} = -m\omega^2[\hat{k}'\times\hat{\jmath}'x']$$

$$\vec{F}'_{centrif} = m\omega^2 x'\hat{\imath}'$$

ميكانيك تحليلي I

الفصل الرابع

المحاضرة الثانيه

أ.د. رعد الحداد

4.4 Effects of Earth's Rotation:-

The spin of the Earth about its rotation axis with angular frequency or the angular speed of Earth's rotation is

$$\omega = \frac{2\pi}{1\,day} = 7.3 \times 10^{-5} \ \ radians/sec$$

✓ We might expect the effects of such rotation to be relatively small.

Static Effects: The Plumb Line

✓ Let us now study the effects of the rotation of the earth on a "static" frame.

Let assume that the particle is at a rest at the surface of the Earth. To illustrate the example, we'll consider this particle to be a sphere at the end of the plumb bob and hangs perpendicular to the surface of the earth, as shown in fig. (1). Where

$\boldsymbol{\lambda}$ = Latitude (angle)

$\boldsymbol{\rho}$ = Distance of plumb bob from axis of rotation.

$\boldsymbol{r_e}$ = Distance of plumb bob from the center of the Earth.

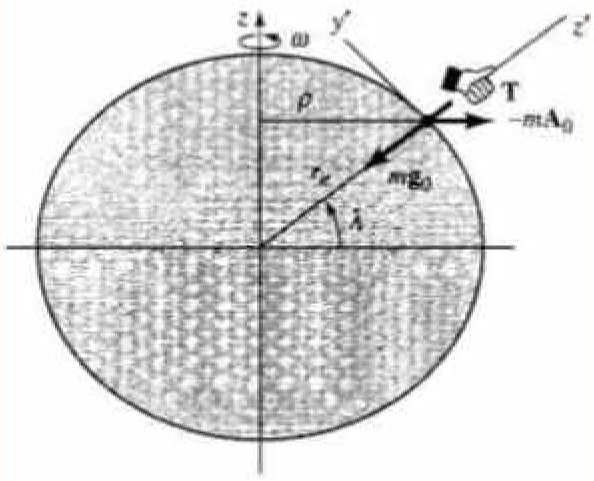


Figure (1) Gravitational force mg_0, inertial force $-\boldsymbol{mA0}$, and tension T acting on a plumb bob hanging just above the surface of the Earth at latitude A.

✓ The movement of this sphere with rotation of the earth does not indicate the center of the earth unless it is suspended either on the equator or above one of the poles.

Let us describe the motion of the plumb bob in a local frame of reference whose origin is at the position of the bob. The frame of reference is attached to the surface of the Earth. It is undergoing translation as well as rotation. The translation of the frame takes place along a circle whose radius is $(\boldsymbol{\rho} = \boldsymbol{r_e} \cos \boldsymbol{\lambda})$is the radius of the Earth and is the geocentric latitude of the plumb bob.

We now that the equation of motion is

$$\vec{F} - m\vec{A}_0 - 2m\vec{\omega} \times \vec{v}' - m\dot{\vec{\omega}} \times \vec{r}' - m\vec{\omega} \times (\vec{\omega} \times \vec{r}') = \vec{F}' \quad \ldots\ldots (4-31)$$

From our example, we can get the results and applied to the eq. (31)

1) $\vec{v}'$ and $\vec{a}'$ is **zero**, because the bob is stationary in the rotating frame of reference.

2) $\vec{r}' = \mathbf{0}$, because the bob sits at the origin of the frame or because the origin of the local coordinate system is centered on the bob.

3) $\dot{\vec{\omega}} = \mathbf{0}$, because the rotational speed of the Earth is constant.

Therefore, all the terms of the eq. (31) disappear for the static state, except for the real forces $\vec{F}$ and the inertial term $-m\vec{A}_0$, which arises because the local frame of reference is accelerating. Thus,

$$\vec{F} - m\vec{A}_0 = 0 \quad \ldots\ldots (4-36)$$

$\vec{F}$ is the vector sum of real gravitational force of the earth and the vertical tension (physical forces) of the plumb bob.

$$(m\vec{g}_0 + \vec{T}) - m\vec{A}_0 = 0 \quad \ldots\ldots (4-37)$$

$$\vec{T} = -m\vec{g} \quad \ldots\ldots (4-38)$$

$$m\vec{g}_0 - m\vec{g} - m\vec{A}_0 = 0 \quad \ldots\ldots (4-39)$$

$$\vec{g}_0 - \vec{g} - \vec{A}_0 = 0 \quad \ldots\ldots (4-40)$$

To evaluate the difference between $\vec{g}$ and $\vec{g}_0$, we use the law of *sines*:-

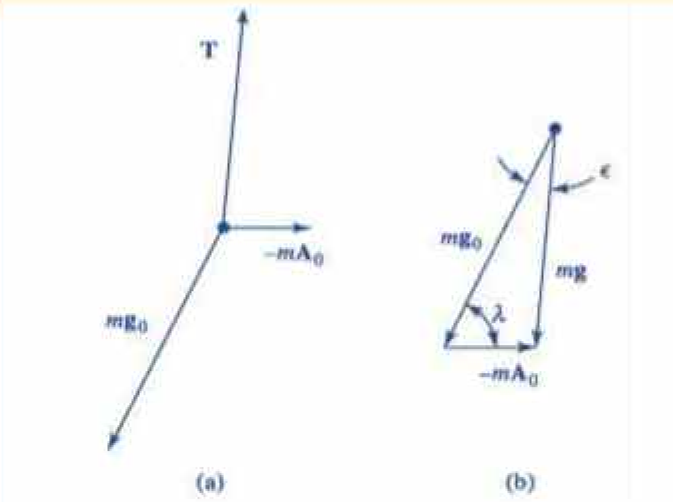
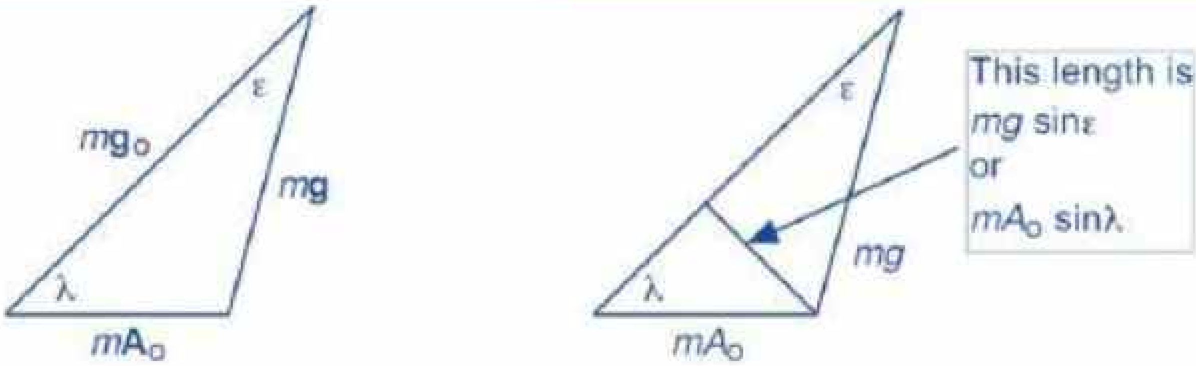


Figure (2) (a) Forces acting on a plumb bob at latitude (b) Forces defining the weight of the plumb bob, mg.



$$\sin\epsilon = \frac{m\vec{A}_0 \sin\lambda}{m\vec{g}} \quad \ldots\ldots\ldots\ldots\ldots(4-41)$$

$$\sin\epsilon = \frac{\vec{A}_0 \sin\lambda}{\vec{g}} \quad \ldots\ldots\ldots\ldots\ldots(4-42)$$

The centripetal acceleration $\vec{A}_0$ ($-\vec{A}_0$ is the centrifugal acceleration) given by

$$\vec{A}_0 = \omega^2\rho \quad \ldots\ldots\ldots\ldots\ldots(4-43)$$

$\because \rho = r_e \cos\lambda$

$$\therefore \vec{A}_0 = \omega^2 r_e \cos\lambda \quad \ldots\ldots\ldots\ldots\ldots(4-44)$$

Sub eq. (4-44) in eq. (4-42), we can get

$$\sin\epsilon = \frac{\omega^2 r_e \cos\lambda\ \sin\lambda}{\vec{g}} \quad \ldots\ldots\ldots\ldots\ldots(4-45)$$

Or, because ε is small

$$\epsilon = \frac{\omega^2 r_e}{\vec{g}} \cos\lambda\ \sin\lambda \quad \ldots\ldots\ldots\ldots\ldots(4-46)$$

$$\because \frac{1}{2}\sin 2\lambda = \cos\lambda\ \sin\lambda$$

$$\therefore \quad \epsilon = \frac{\omega^2 r_e}{2\vec{g}} \sin 2\lambda \quad \ldots\ldots\ldots\ldots\ldots(4-47)$$

Thus, vanishes at the equator $(\lambda = 0)$ and the poles $(\lambda = \pm 90^0)$ as we have already surmised. The maximum deviation of the direction of the plumb line from the center of the Earth occurs at $\lambda = 45^0$ where

$$\therefore \quad \epsilon_{max} = \frac{\omega^2 r_e}{2\vec{g}} \approx 1.7 \times 10^{-3}\ radian \approx 0.1^0 \quad \ldots\ldots\ldots(4-48)$$

Dynamic Effects: Motion of a Projectile

The equation of motion for a projectile near the Earth's surface (4-31) can be written

$$m\,\ddot{\vec{r}}' = \vec{F} + m\vec{g}_0 - m\vec{A}_0 - 2m\vec{\omega}\times\dot{\vec{r}}' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}') \;\ldots\ldots\ldots (4-49)$$

Where $\vec{F}$ represents any applied forces other than gravity. From eq. (4-39)

$$m\vec{g}_0 - m\vec{g} - m\vec{A}_0 = 0$$

$$m\vec{g}_0 - m\vec{A}_0 = m\vec{g} \;\ldots\ldots\ldots\ldots (4-50)$$

$$\therefore m\,\ddot{\vec{r}}' = \vec{F} + m\vec{g} - 2m\vec{\omega}\times\dot{\vec{r}}' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}') \;\ldots\ldots\ldots (4-51)$$

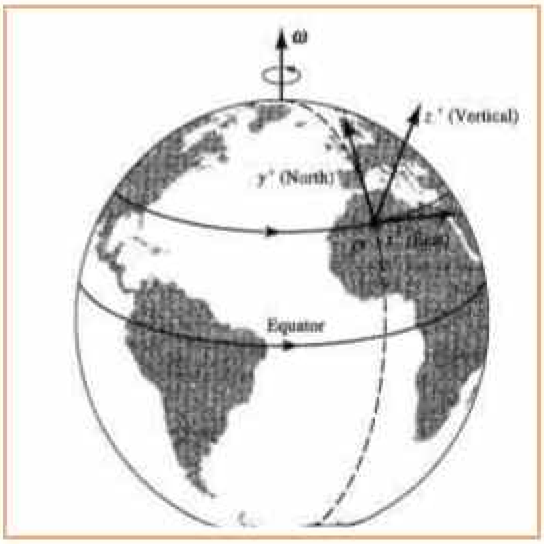
If we ignore air resistance, then $\vec{F} = 0$. Furthermore, the last term is very small compared with the other terms، so we can ignore it. Thus eq. (4-51) reduces to

$$m\,\ddot{\vec{r}}' = m\vec{g} - 2m\vec{\omega}\times\dot{\vec{r}}' \;\ldots\ldots\ldots (4-52)$$

In which the last term is the Coriolis force.

To solve the equation $(4-52)$, we will choose the directions of the coordinate axes $O'x'y'z'$ so that the $z'-axis$ is vertical (in the direction of the plumb line), the $x'-axis$ is to the east, and the $y'-axis$ pointing towards the north, as shown in fig (3). With this choice of axes, we have

$$\vec{g} = -\hat{k}'g \;\ldots\ldots\ldots\ldots\ldots\ldots (4-53)$$



The components of $\vec{\omega}$ in the primed system are

$$\vec{\omega} = \omega_{x'}\hat{i}' + \omega_{y'}\hat{j}' + \omega_{z'}\hat{k}' \;\ldots\ldots\ldots\ldots\ldots\ldots (4-54)$$

$$\omega_{x'} = 0 \quad , \qquad \omega_{y'} = \omega\ \cos\lambda \ , \qquad \omega_{z'} = \omega\ \sin\lambda \qquad \ldots\ldots (4-55)$$

$$\therefore \vec{\omega} = 0 + (\omega\ \cos\lambda)\hat{j}' + (\omega\ \sin\lambda)\hat{k}' \;\ldots\ldots\ldots\ldots\ldots\ldots (4-56)$$

$$\therefore \vec{\omega} \times \vec{\dot{r}}' = \begin{vmatrix} \hat{\imath}' & \hat{\jmath}' & \hat{k}' \\ \omega_{x'} & \omega_{y'} & \omega_{z'} \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots (4-57)$$

$$\vec{\omega} \times \vec{\dot{x}}' = (-1)^{1+1} \begin{vmatrix} \omega_{y'} & \omega_{z'} \\ \dot{y} & \dot{z} \end{vmatrix} = \left(\omega_{y'}\dot{z} - \omega_{z'}\dot{y}\right)\hat{\imath}' \quad \ldots\ldots\ldots\ldots (4-57a)$$

$$\vec{\omega} \times \vec{\dot{y}}' = (-1)^{1+2} \begin{vmatrix} \omega_{x'} & \omega_{z'} \\ \dot{x} & \dot{z} \end{vmatrix} = -\left(\omega_{x'}\dot{z} - \omega_{z'}\dot{x}\right)\hat{\jmath}' \quad \ldots\ldots\ldots\ldots (4-57b)$$

$$\vec{\omega} \times \vec{\dot{z}}' = (-1)^{1+3} \begin{vmatrix} \omega_{y'} & \omega_{z'} \\ \dot{y} & \dot{z} \end{vmatrix} = \left(\omega_{x'}\dot{y} - \omega_{y'}\dot{x}\right)\hat{k}' \quad \ldots\ldots\ldots\ldots (4-57c)$$

$$\therefore \vec{\omega} \times \vec{\dot{r}}' = \left(\omega_{y'}\dot{z} - \omega_{z'}\dot{y}\right)\hat{\imath}' - \left(\omega_{x'}\dot{z} - \omega_{z'}\dot{x}\right)\hat{\jmath}' + \left(\omega_{x'}\dot{y} - \omega_{y'}\dot{x}\right)\hat{k}'$$

$$\therefore \vec{\omega} \times \vec{\dot{r}}' = \left(\omega_{y'}\dot{z} - \omega_{z'}\dot{y}\right)\hat{\imath}' - \left(0 - \omega_{z'}\dot{x}\right)\hat{\jmath}' + \left(0 - \omega_{y'}\dot{x}\right)\hat{k}'$$

$$\therefore \vec{\omega} \times \vec{\dot{r}}' = \left(\omega_{y'}\dot{z} - \omega_{z'}\dot{y}\right)\hat{\imath}' + \left(\omega_{z'}\dot{x}\right)\hat{\jmath}' + \left(-\omega_{y'}\dot{x}\right)\hat{k}' \quad \ldots\ldots\ldots\ldots (4-58)$$

$$\therefore \vec{\omega} \times \vec{\dot{r}}' = \left(\omega\ \cos\lambda\,\dot{z} - \omega\ \sin\lambda\,\dot{y}\right)\hat{\imath}' + \left(\omega\ \sin\lambda\,\dot{x}\right)\hat{\jmath}' + \left(-\omega\ \cos\lambda\,\dot{x}\right)\hat{k}'$$

$$\therefore \vec{\omega} \times \vec{\dot{r}}' = \left(\omega\dot{z}\ \cos\lambda - \omega\dot{y}\ \sin\lambda\right)\hat{\imath}' + \left(\omega\dot{x}\ \sin\lambda\right)\hat{\jmath}' + \left(-\omega\,\dot{x}\cos\lambda\right)\hat{k}' \quad \ldots\ldots\ldots\ldots (4-59)$$

Rewrite eq. (4-52) with cancel m from each side, we get:-

$$\ddot{r}' = \vec{g} - 2\vec{\omega} \times \vec{\dot{r}}' \quad \ldots\ldots (4-60)$$

$$\ddot{x}' = 0 - 2\vec{\omega} \times \vec{\dot{x}}' \quad \ldots\ldots (4-60a)$$

$$\ddot{y}' = 0 - 2\vec{\omega} \times \vec{\dot{y}}' \quad \ldots\ldots (4-60b)$$

$$\ddot{z}' = -\vec{g} - 2\vec{\omega} \times \vec{\dot{z}}' \quad \ldots\ldots (4-60c)$$

Using eqs. (4-60 a,b,c) in eq. (4-59) we get:-

$$\ddot{x}' = -2\omega\left(\dot{z}'\ \cos\lambda - \dot{y}'\ \sin\lambda\right) \quad \ldots\ldots (4-61a)$$

$$\ddot{y}' = -2\omega\,\dot{x}'\ \sin\lambda \quad \ldots\ldots (4-61b)$$

$$\ddot{z}' = -\vec{g} + 2\omega\,\dot{x}'\cos\lambda \quad \ldots\ldots (4-61c)$$

These equations are not of the separated type, but we can integrate once with respect to t to obtain

$$\ddot{x}' = \frac{d\dot{x}'}{dt} \Longrightarrow \int d\dot{x}' = \int \ddot{x}'\,dt$$

$$\dot{x}' = \int d\dot{x}' = -2\omega\left(\int \frac{dz'}{dt}\cos\lambda\,dt - \int \frac{dy'}{dt}\sin\lambda\,dt\right)$$

$$\dot{x}' = -2\omega\left(\int dz'\ \cos\lambda - \int dy'\ \sin\lambda\right)$$

$$\dot{x}' = -2\omega(z'\ \cos\lambda - y'\ \sin\lambda) + \ \dot{x}'_0 \ \ldots\ldots\ldots\ldots\ldots (4-62a)$$

$$\ddot{y}' = \frac{d\dot{y}'}{dt} \Longrightarrow \int d\dot{y}' = \int \ddot{y}'\,dt$$

$$\dot{y}' = \int d\dot{y}' = -2\omega\int \frac{dy'}{dt}dt\ \sin\lambda$$

$$\dot{y}' = \int d\dot{y}' = -2\omega\int dx'\sin\lambda$$

$$\dot{y}' = -2\omega x'\sin\lambda + \dot{y}'_0 \ldots\ldots\ldots\ldots\ldots (4-62b)$$

$$\ddot{z}' = \frac{d\dot{z}'}{dt} \Longrightarrow \int d\dot{z}' = \int \ddot{z}'\,dt$$

$$\dot{z}' = \int d\dot{z}' = -\vec{g}\int dt + 2\omega\int \frac{dz'}{dt}dt\ \cos\lambda$$

$$\dot{z}' = -\vec{g}t + 2\omega\int dz'\cos\lambda$$

$$\dot{z}' = -\vec{g}t + 2\omega x'\cos\lambda + \dot{z}'_0 \ldots\ldots\ldots\ldots\ldots (4-62c)$$

Now sub eqs. (4-62 a, b, c) in eq. (4-61 a, b, c)

$$\ddot{x}' = -2\omega([-\vec{g}t + 2\omega x'\cos\lambda + \dot{z}'_0]\ \cos\lambda - [-2\omega x'\sin\lambda + \dot{y}'_0]\sin\lambda)$$

$$\ddot{x}' = 2\omega(\vec{g}t\cos\lambda - 2\omega x'\cos^2\lambda + \dot{z}'_0\cos\lambda + 2\omega x'\sin^2\lambda - \dot{y}'_0\sin\lambda)$$

$$\ddot{x}' = 2\omega\vec{g}t\cos\lambda - 4\omega^2x'\cos^2\lambda - 2\omega\dot{z}'_0\cos\lambda + 2\omega^2x'\sin^2\lambda - 2\omega\dot{y}'_0\sin\lambda$$

$$\ddot{x}' = \left(2\omega\vec{g}t\cos\lambda - 2\omega\dot{z}'_0\cos\lambda + 2\omega\dot{y}'_0\sin\lambda\right)$$

Where terms involving ω^2 have been ignored.

$$\ddot{x}' = 2\omega\vec{g}t\cos\lambda - 2\omega[\dot{z}'_0\cos\lambda - \dot{y}'_0\sin\lambda] \quad \ldots\ldots\ldots\ldots..(4-63)$$

By integrate this eq. (4-63), we get

$$\dot{x}' = \int d\ddot{x}'\,dt = -2\omega\vec{g}\left(\int t\,dt\ \cos\lambda - 2\omega[\dot{z}'_0\cos\lambda - \dot{y}'_0\sin\lambda]\int dt\right)$$

$$\dot{x}' = -2\omega\vec{g}\left(\frac{t^2}{2}\cos\lambda - 2\omega[\dot{z}'_0\cos\lambda - \dot{y}'_0\sin\lambda]t\right) + \dot{x}'_0$$

$$\dot{x}' = \omega\vec{g}t^2\cos\lambda - 2\omega t[\dot{z}'_0\cos\lambda - \dot{y}'_0\sin\lambda] + \dot{x}'_0 \ldots\ldots..(4-64)$$

And finally, by third integration to eq. (4-64)

$$\dot{x}' = \int\frac{dx'}{dt} \Longrightarrow \int dx' = \dot{x}'\int dt$$

$$x' = \int dx' = \omega\vec{g}\int t^2dt\cos\lambda + 2\omega[\dot{z}'_0\cos\lambda - \dot{y}'_0\sin\lambda]\int tdt + \int\dot{x}'_0\,dt$$

$$x'(t) = \frac{1}{3}\omega\vec{g}t^3\cos\lambda + \omega t^2[\dot{z}'_0\cos\lambda - \dot{y}'_0\sin\lambda] + \dot{x}'_0t + \dot{x}'_0 \ldots..(4-65)$$

Similarly,

$$\ddot{y}' = -2\omega[-2\omega(z'\ \cos\lambda - y'\ \sin\lambda) + \ \dot{x}'_0\ ]\sin\lambda$$

$$\ddot{y}' = [-4\omega^2(z'\ \cos\lambda - y'\ \sin\lambda)\sin\lambda - 2\omega\,\dot{x}'_0\ \sin\lambda]$$

Where terms involving ω^2 have been ignored.

$$\ddot{y}' = -2\omega\,\dot{x}'_0\ \sin\lambda \ \ldots\ldots\ldots\ldots..(4-66)$$

By integrate this eq. (4-66), we get

$$\dot{y}' = \int d\ddot{y}'\,dt = -2\omega\,\dot{x}'_0\ \sin\lambda\int dt$$

$$\dot{y}' = -2\omega t\, \dot{x}'_0 \, \sin\lambda + \dot{y}'_0 \quad \dots\dots\dots\dots(4-67)$$

And finally, by third integration to eq. (4-67)

$$y'(t) = -\omega t^2\, \dot{x}'_0 \, \sin\lambda + \dot{y}'_0 t + y'_0$$

$$y'(t) = \dot{y}'_0 t - \omega\, \dot{x}'_0 t^2 \, \sin\lambda + y'_0 \quad \dots\dots\dots\dots(4-68)$$

Similarly,

$$\ddot{z}' = -\vec{g} + 2\omega\left[-2\omega\left(z' \cos\lambda - y' \sin\lambda\right) + \dot{x}'_0\right]\cos\lambda$$

$$\ddot{z}' = -\vec{g} + 2\omega\, \dot{x}'_0\left[-4\omega^2\left(z' \cos\lambda - y' \sin\lambda\right) + 2\omega\, \dot{x}'_0\right]\cos\lambda$$

Where terms involving ω^2have been ignored

$$\ddot{z}' = -\vec{g} + 2\omega\, \dot{x}'_0 \cos\lambda \quad \dots\dots\dots\dots(4-69)$$

By integrate this eq. (4-69), we get

$$\dot{z}' = \int d\ddot{z}'\, dt = -\vec{g}\int dt + 2\omega\, \dot{x}'_0 \cos\lambda \int dt$$

$$\dot{z}' = -\vec{g}t + 2\omega\, \dot{x}'_0 t \cos\lambda + \dot{z}'_0 \quad \dots\dots\dots\dots(4-70)$$

And finally, by third integration to eq. (4-70)

$$z'(t) = -\frac{1}{2}\vec{g}t^2 + +\dot{z}'_0 t + 2\omega\, \dot{x}'_0 t^2 \cos\lambda + \dot{z}'_0 \quad \dots\dots\dots\dots(4-71)$$

4.5 The Foucault Pendulum:-

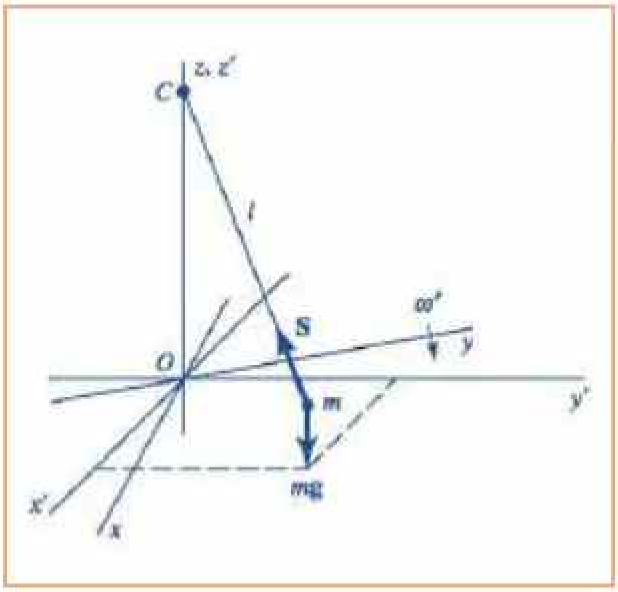
In this section we study the effect of Earth's rotation on the motion of a pendulum that is free to swing in any direction, the so-called **spherical pendulum**. As shown in Figure (4), the applied force acting on the pendulum bob is the vector sum of the weight mg and the tension S in the cord. The differential equation of motion is then

$$m\,\ddot{r}' = m\vec{g} + \mathbf{S} - 2m\vec{\boldsymbol{\omega}} \times \vec{\dot{r}}' \quad \dots\dots\dots\dots(4-72)$$

Now the $x'-$ and $y'-$components of the tension can be found simply by noting that the direction cosines of the vector **S** are $(-x'/l)$, $(-y'/l)$ and $-(l-z'/l)$, respectively. Consequently

$$S_x = \frac{-x'S}{l} \;,\; S_y = \frac{-y'S}{l} \;, S_y = \frac{(l-z')S}{l}$$

And the corresponding components of the differential equation of motion (4-72) are



$$m\,\ddot{x}' = 0 + \mathbf{S} - 2m\vec{\omega} \times \dot{\vec{x}}' \;\ldots\ldots\ldots\ldots\ldots(4-73)$$

$$m\,\ddot{x}' = \frac{-x'S}{l} - 2m\vec{\omega}(\dot{z}'\,\cos\lambda - \dot{y}'\,\sin\lambda) \;\ldots\ldots\ldots\ldots\ldots(4-74)$$

$\dot{z}' = 0$ *because there is no vertical motion,* *and* $S = mg$

$$m\,\ddot{x}' = \frac{-x'}{l}mg + 2m\vec{\omega}\,\dot{y}'\,\sin\lambda$$

$$\ddot{x}' = \frac{-x'}{l}g + 2\omega_{z'}\dot{y}' \;\ldots\ldots\ldots\ldots\ldots(4-75)$$

$$m\,\ddot{y}' = 0 + \mathbf{S} - 2m\vec{\omega} \times \dot{\vec{y}}' \;\ldots\ldots\ldots\ldots\ldots(4-76)$$

$$m\,\ddot{y}' = \frac{-y'S}{l} - 2m\vec{\omega}(\dot{x}'\,\sin\lambda) \;\ldots\ldots\ldots\ldots\ldots(4-77)$$

$$\ddot{y}' = \frac{-y'}{l}g - 2\omega_{y'}\dot{x}' \;\ldots\ldots\ldots\ldots\ldots(4-78)$$

The equations of transformation are

$$x' = x\cos\omega' t + y\sin\omega' t \;\ldots\ldots\ldots\ldots\ldots(4-79)$$

$$y' = -x\sin\omega' t + y\cos\omega' t \;\ldots\ldots\ldots\ldots\ldots(4-80)$$

$$x' = \left(\ddot{x}' + \frac{g}{l}x\right)\cos\omega' t + \left(\ddot{y}' + \frac{g}{l}y\right)\sin\omega' t = 0 \quad \ldots\ldots\ldots\ldots\ldots.(4-81)$$

$$\ddot{x}' + \frac{g}{l}x = 0 \quad \ldots\ldots\ldots\ldots\ldots.(4-81a)$$

$$\ddot{y}' + \frac{g}{l}y = 0 \quad \ldots\ldots\ldots\ldots\ldots.(4-81b)$$

ميكانيك تحليلي I

الفصل الخامس

المحاضرة الأولى

أ.د. رعد الحداد

Gravitation and Central Forces

5.1 Introduction

- ✓ The force whose line of influence passes at a fixed point or center of force called the **Central Force**.
- ✓ The central forces are important in physics because they include forces such as the earth's gravitational forces, electrostatic forces, and others.

Newton's Law of Universal Gravitation (for point particle)

The law can be stated as follows: −

> Every particle in the universe attracts every other particle with a force w
> magnitude is proportional to the product of the masses of the two particles an
> inversely proportional to the square of the distance between them. The dire
> of the force lies along the straight line connecting the two particles.

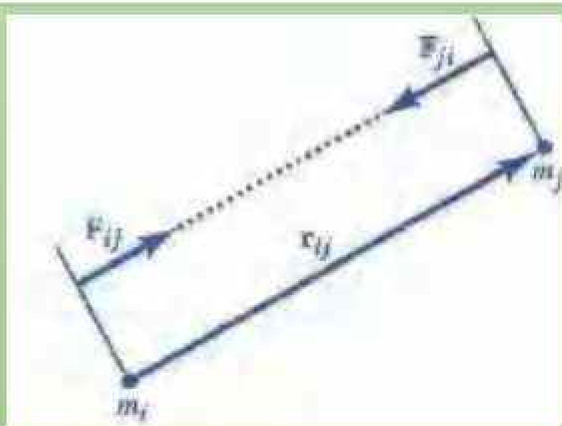
According to Newton's Law of Universal Gravitation, the force on a particle i of mass m_i exerted by a particle j of mass m_j is,

$$\vec{F}_{ij} = G\frac{m_i\, m_j}{r_{ij}^2}\left(\frac{\vec{r}_{ij}}{r_{ij}}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5-1)$$

Where $\vec{F}_{ij}$ is the force on particle i of mass m_i exerted by particle j of mass m_j.

- ✓ The vector $\vec{r}_{ij}$ is the directed line segme running from particle i to particle j, as shown in Fig (1).
- ✓ The law of action and reaction require that $\vec{F}_{ij} = -\vec{F}_{ji}$.
- ✓ The constant of proportionality G is known as the **universal constant of gravitation**.

$$G = 6.67259 \times 10^{-11}\ Nm^2\text{kg}^{-2}$$



Figure(1) Action and reaction in Newton's law of gravity.

✓ This is a central force because it only acts on the line connecting particle. If the magnitude of the force, as is the case with gravitation, is independent any direction (i.e. no angular or orientational dependence), the force is isotrop

5.2 Gravitational Force between a Uniform Sphere and a Particle

In the motion of free-falling particle, it has been emphasized that the force of the earth's attraction to a particle above its surface inversely proportional to the square distance between the particle and the center of the globe, meaning that the earth attracts all its mass together at one point. We will now demonstrate that this is true for any uniform spherical particle or symmetrical distribution of matter.

For any uniform spherical body or any spherically symmetric distribution matter, the gravitational force exerted by it on any external particle ca calculated by simply assuming that the entire mass of the distribution a though concentrated at its geometric center. Only an inverse square force law works this way.

Consider a thin uniform shell of mass **M** and radius **R**. Let **r** be the distance from the center **O** to a test particle **P** of mass **m** (Fig. 2). We assume that **r > R**. We shall divide the shell into circular rings of width $R\Delta\theta$ where the angle POQ is denoted by θ, **Q** being a point on the ring. Therefore, the circumference of the ring element is, $2\pi R\sin\theta$ and its mass ΔM is given by

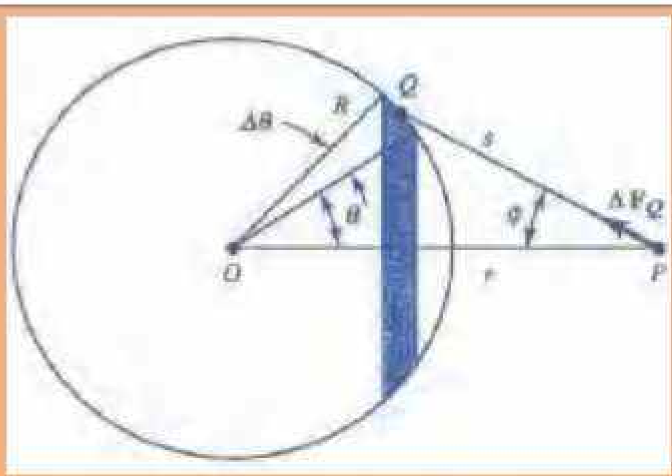


Fig. (2) Coordinates for calculating the gravitation field of a spherical shell.

$$\Delta M_{ring} \approx \rho 2\pi R^2 \sin\theta\, \Delta\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5-2)$$

Where ρ is the mass per unit area of the shell.

✓ Now what is the gravitational force exerted on a point mass $\boldsymbol{m}$ at $\boldsymbol{P}$ by a small subelement of the ring at **Q** (acting in the **PQ** direction)?

1) We resolve the force into components, one along **PO** of magnitude $\Delta F_Q \cos\phi$, the other perpendicular to **PO**, magnitude $\Delta F_Q\, sin\, \phi$.

2) From symmetry, we see that the vector sum of all of the perpendicular components exerted at $\boldsymbol{P}$ by the whole ring vanishes. Therefore, only one component along **PO** is remain ($\Delta F_Q \cos\phi$).

3) So, the force $\Delta \boldsymbol{F}$ exerted by the entire ring is in the $\boldsymbol{PO}$ direction with magnitude, is

$$\Delta\vec{F} = \mathrm{G}\frac{m\Delta M_{ring}\cos\phi}{s^2} = \mathrm{G}\frac{2\pi m\rho R^2\sin\theta\cos\phi}{s^2}\Delta\theta \;\ldots\ldots\ldots.(5-3)$$

Taking the limit that $\Delta\theta \rightarrow \theta$ and adding up all the rings, we obtain the total force exerted at **P** by the whole shell:

$$\vec{F} = \mathrm{G}m2\pi\rho R^2 \int_0^\pi \frac{\sin\theta\cos\phi\, d\theta}{s^2} \quad \ldots\ldots\ldots\ldots\ldots.(5-4)$$

To perform this integral, from the Ttriangle OPQ in figure (2), law of cosines for θ is:

$$s^2 = r^2 + R^2 - 2\pi R\cos\theta \;\ldots\ldots\ldots\ldots\ldots\ldots.(5-5)$$

From the law of cosines. But since **r** and **R** are constant,

$$sds = rR\, sin\,\theta\, d\theta \;\ldots\ldots\ldots\ldots\ldots\ldots.(5-6)$$

$$sin\,\theta\, d\theta = \frac{sds}{rR} \;\ldots\ldots\ldots\ldots\ldots\ldots(5-7)$$

Also, from the Ttriangle **OPQ** in figure (2), law of cosines for ϕ is:

$$R^2 = s^2 + r^2 - 2rs\cos\phi \;\ldots\ldots\ldots\ldots\ldots\ldots.(5-8)$$

$$\cos\phi = \frac{s^2 + r^2 - R^2}{2rs} \;\ldots\ldots\ldots\ldots\ldots\ldots(5-9)$$

$$\vec{F}_{shell} = \mathrm{G}m2\pi\rho R^2 \int_{r-R}^{r+R} \frac{s^2 + r^2 - R^2}{2rs} \frac{sds}{rR} \frac{1}{s^2}$$

$$\vec{F}_{shell} = \mathrm{G}m\pi\rho R \int_{r-R}^{r+R} \frac{s^2 + r^2 - R^2}{r^2 s^2} ds$$

$$\vec{F}_{shell} = \mathrm{G}\frac{m\pi\rho R}{r^2} \int_{r-R}^{r+R} \left(\frac{s^2}{s^2} ds + \frac{r^2 - R^2}{s^2} ds\right)$$

$$\vec{F}_{shell} = \mathrm{G}\frac{m\pi\rho R}{r^2} \int_{r-R}^{r+R} \left[1ds + \frac{r^2 - R^2}{s^2} ds\right]$$

$$\vec{F}_{shell} = \mathrm{G}\frac{m\pi\rho R}{r^2} \left[s + (r^2 - R^2)\left(\frac{s^{-1}}{-1}\right)\right]_{r-R}^{r+R}$$

$$\vec{F}_{shell} = \mathrm{G}\frac{m\pi\rho R}{r^2} \left[s - (r^2 - R^2)\frac{1}{s}\right]_{r-R}^{r+R}$$

$$\vec{F}_{shell} = \mathrm{G}\frac{m\pi\rho R}{r^2} \left[(r + R) - (r - R) - (r^2 - R^2)\left[\frac{1}{(r+R)} - \frac{1}{(r-R)}\right]\right]$$

$$\vec{F}_{shell} = \mathrm{G}\frac{m\pi\rho R}{r^2} \left(2R - (r^2 - R^2)\left[\frac{r - R - r - R}{(r+R)(r-R)}\right]\right)$$

$$\vec{F}_{shell} = \mathrm{G}\frac{m\pi\rho R}{r^2} \left(2R - (r^2 - R^2)\left[\frac{-2R}{r^2 - R^2}\right]\right)$$

$$\vec{F}_{shell} = \mathrm{G}\frac{m\pi\rho R}{r^2} (2R - [-2R])$$

$$\vec{F}_{shell} = \mathrm{G}\frac{m\pi\rho R}{r^2} (4R) \quad \Longrightarrow \quad \vec{F}_{shell} = \mathrm{G}\frac{4m\pi\rho R^2}{r^2}$$

$$\vec{F}_{shell} = \mathrm{G}\frac{mM_{shell}}{r^2} \quad \dots\dots\dots\dots\dots\dots (5-10)$$

✓ The result means that the uniformly spherical shell of the matter attracts an external particle as if the whole mass of the shell is concentrated in its center. This is true for:-

(1) Every concentric spherical portion of a solid uniform sphere.

(2) A non-uniform sphere if the density depends only on the radial distance $\boldsymbol{r}$.

We can write eq. (5-10) vectorially

$$\vec{F}_{shell} = -G\frac{mM_{shell}}{r^2}\,\hat{e}_r \dots\dots.\dots\dots\dots.(5-11)$$

Where $\hat{\boldsymbol{e}}_{\boldsymbol{r}}$ is the unit radial vector from the origin **O**.

5.3 Kepler's Laws of Planetary Motion

I. Law of Ellipses (1609)

The orbit of each planet is an ellipse, with the Sun located at one of its foci

II. Law of Equal Areas (1609)

A line drawn between the Sun and the planet sweeps out equal areas in equal times as the planet orbits the Sun.

III. Harmonic Law (1618)

The square of the sidereal period of a planet (the time it takes a planet to complete one revolution about the Sun relative to the stars) is directly proportional to the cube of the semi major axis of the planet's orbit.

5.4 Kepler's Second Law: Equal Areas

5.4 .1 Conservation of Angular Momentum

The angular momentum of a particle located a vector distance $\boldsymbol{r}$ from a given origin and moving with momentum $\boldsymbol{p}$ is defined to be the quantity

$$\vec{L} = r \times \vec{p} \dots\dots\dots\dots\dots\dots\dots\dots(5-12)$$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}\times\vec{p}\,) = \frac{d\vec{r}}{dt}\times\vec{p} + \vec{r}\times\frac{d\vec{p}}{dt} \dots\dots\dots\dots\dots\dots(5-13)$$

$$\frac{d\vec{L}}{dt} = \vec{v}\times\vec{p} + \vec{r}\times\frac{d\vec{p}}{dt} \dots\dots\dots\dots\dots\dots (5-14)$$

but $$\vec{v}\times\vec{p} = \vec{v}\times m\vec{v} = m\vec{v}\times\vec{v} = 0$$

$$\frac{d\vec{L}}{dt} = \vec{r}\times\frac{d\vec{p}}{dt} \dots\dots\dots\dots\dots\dots (5-15)$$

$$\therefore \quad \frac{d\vec{L}}{dt} = \vec{r}\times\vec{F} \dots\dots.\dots\dots\dots.. (5-16)$$

- ✓ The cross product $\vec{N} = \vec{r}\times\vec{F}$ is the moment of force, or torque, on the particle about the origin of the coordinate system.
- ✓ If $\vec{r}$ and $\vec{F}$ are collinear, this cross product vanishes and so does $\vec{L}$ ($d\vec{L}/dt = 0$). In such case the angular momentum $\vec{L}$, is a constant of the motion.

5.4.2 Angular Momentum and Area Velocity of a Particle Moving in a Central Field

To calculate the magnitude of the angular momentum, it is preferable to decompose the velocity vector $\vec{v}$ into its polar and transverse components in polar axes. The velocity of the particle is

$$\vec{v} = \hat{e}_r\dot{r} + \hat{e}_\theta r\dot{\theta} \dots\dots\dots\dots\dots\dots (5-17)$$

Where $\hat{e}_r$ is the unit radial vector and $\hat{e}_\theta$ is the unit transverse vector (Fig. 3a).

The magnitude of the angular momentum is

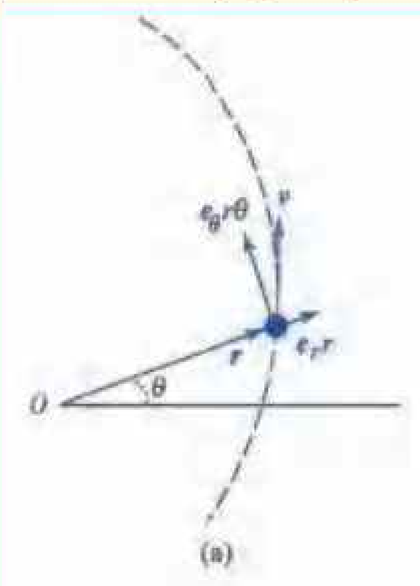
$$\vec{L} = |\vec{r}\times m\vec{v}| = \left|r\hat{e}_r\times m\left(\hat{e}_r\dot{r} + \hat{e}_\theta r\dot{\theta}\right)\right|. \dots (5-18)$$

$$\vec{L} = mr\dot{r}\left|\hat{e}_r\times\hat{e}_r\right| + mr^2\dot{\theta}\left|\hat{e}_r\times\hat{e}_\theta\right|$$

$$\vec{L} = mr^2\dot{\theta}\left|\hat{e}_r\times\hat{e}_\theta\right| \dots\dots\dots\dots\dots\dots\dots.\dots (5-19)$$

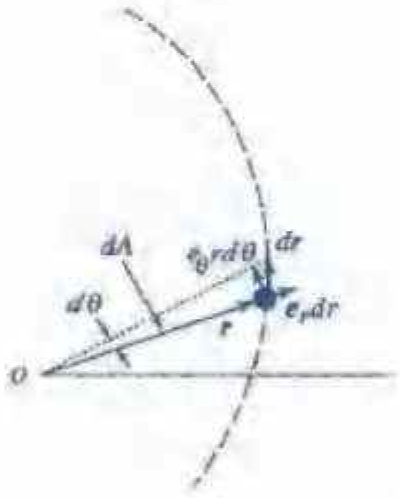
Note $\left|\hat{e}_r\times\hat{e}_r\right| = 0$ *and* $\left|\hat{e}_r\times\hat{e}_\theta\right| = 1$

$$\therefore\ \vec{L} = mr^2\dot{\theta} = constant \ \dots.\dots\dots..(5-20)$$



The angular momentum of a particle is related to time rate of the area traveled by the position vecto[r]. To illustrate this, suppose the fig. (3b) showing two consecutive position vectors, $\vec{r} + d\vec{r}$ representing the motion of a particle in a time interval of dt. The area of the triangle between the vectors is:

$$dA = \frac{1}{2}|\vec{r} \times d\vec{r}| \quad \dots\dots\dots\dots\dots (5-21)$$



(b)

$$dA = \frac{1}{2}\left|r\hat{e}_r \times (\hat{e}_r d\vec{r} + \hat{e}_\theta r\, d\theta)\right| \quad \dots\dots\dots\dots\dots (5-22)$$

$$dA = \frac{1}{2}(r d\vec{r})|\hat{e}_r \times \hat{e}_r| + \frac{1}{2}r(r\, d\theta)|\hat{e}_r \times \hat{e}_\theta| \quad \dots\dots\dots\dots\dots (5-23)$$

$$dA = \frac{1}{2}r^2 d\theta \quad \dots\dots\dots\dots\dots (5-24)$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} \dots\dots\dots\dots\dots (5-25)$$

$$\dot{A} = \frac{1}{2}r^2\dot{\theta} \dots\dots\dots\dots\dots (5-26)$$

Rewrite eq. (5-20)

$$\dot{\theta} = \frac{\vec{L}}{mr^2} = constant \quad \dots\dots\dots\dots (5-27) \quad sub\ in\ eq.(5-26)$$

$$\dot{A} = \frac{1}{2}r^2\frac{\vec{L}}{mr^2}$$

$$\dot{A} = \frac{\vec{L}}{2m} = constant \dots\dots\dots\dots\dots (5-28)$$

$$dA = \frac{1}{2}|\vec{r} \times d\vec{r}| = \frac{1}{2}|\vec{r} \times \vec{v}dt| = \frac{\vec{L}}{2m} = constant. \dots (5-29)$$

EXAMPLE 6.4.1

Let a particle be subject to an attractive central force of the form $f(r)$ where r is the distance between the particle and the center of the force. Find $f(r)$ if all circular orbits are to have identical areal velocities, $\dot{A}$.

Sol/

$$\ddot{\vec{r}} = \frac{d}{dt}(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta) = \ddot{r}\hat{e}_r + \dot{r}(\dot{\hat{e}}_r) + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}(\dot{\hat{e}}_\theta)$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$$

In this case

$$a_r = (\ddot{r} - r\dot{\theta}^2) = -r\dot{\theta}^2 = a_c$$

Now

$$f(r) = -mr\dot{\theta}^2$$

So we use

$$\dot{A} = \frac{1}{2}r^2\dot{\theta} \implies \boxed{f(r) = -\frac{4m}{r^3}\dot{A}^2} \implies \boxed{f(r) \propto \frac{1}{r^3}}$$

Therefore, the attractive force for which all circular orbits have identical areal velocities (and angular momenta) is the inverse cube.

5.5 Kepler's First Law: The Law of Ellipses

To derive Kepler's 1st law, we start with Newton's second law expressed in polar coordinates:

$$m\ddot{\vec{r}} = f(r)\hat{r} \dots\dots\dots\dots\dots\dots (5-30)$$

Where $f(r)$ is the central, isotropic force that acts on the particle of mass m. Recall from lecture 3 that we derived the radial component of the acceleration to be:

$$\vec{a}_r = \ddot{\vec{r}}_r = \ddot{r} - r\dot{\theta}^2 \dots\dots\dots\dots\dots\dots (5-31)$$

$$\therefore m(\ddot{r} - r\dot{\theta}^2) = f(r) \dots\dots\dots\dots\dots\dots (5-32)$$

And the θ component of the acceleration was given as:

$$\vec{a}_\theta = \ddot{\vec{r}}_\theta = 2\dot{r}\dot{\theta} - r\ddot{\theta} \dots\dots\dots\dots\dots\dots (5-33)$$

$$\therefore m(2\dot{r}\dot{\theta} - r\ddot{\theta}) = 0 \dots\dots\dots\dots\dots\dots (5-34)$$

From the last expression (eq.5-32), we see that:

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (5-35)$$

$$r^2\dot{\theta} = Constant = \ell = \frac{L}{m} \ldots\ldots\ldots\ldots\ldots\ldots (5-36)$$

Where ℓ is the angular momentum per unit mass. This just confirms the fact that the angular momentum is constant when a particle is moving under the action of a central force.

$$\dot{\theta} = \ell/r^2 \ldots\ldots\ldots\ldots\ldots\ldots (5-37)$$

We can theoretically solve the differential equations (5-32) and (5-36) for a known polar function $f(r)$ to get r and θ as a function of time t [In most case, we are concerned only with the path in space (the orbit) without regard to the time t]. To achieve this, consider the variable $u = 1/r$

$$r = \frac{1}{u} \ldots\ldots\ldots\ldots\ldots (5-38)$$

$$\dot{\theta} = \ell u^2 \Longrightarrow \ell = \frac{\dot{\theta}}{u^2} \ldots\ldots\ldots\ldots\ldots\ldots (5-39)$$

$$\dot{r} = -\frac{\dot{u}}{u^2} = -\frac{\dot{\theta}}{u^2}\frac{du}{d\theta} = -\ell\frac{du}{d\theta} \ldots\ldots\ldots\ldots\ldots (5-40)$$

$$\ddot{r} = -\ell\frac{d}{dt}\frac{du}{d\theta} = -\ell\frac{d\theta}{dt}\frac{d}{d\theta}\frac{du}{d\theta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5-41)$$

$$\ddot{r} = -\ell\dot{\theta}\frac{d^2u}{d\theta^2} = -\ell^2u^2\frac{d^2u}{d\theta^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5-42)$$

Substituting the values found for r, $\dot{\theta}$, and $\ddot{r}$ into eq.(5-32), we obtain

$$\ddot{r} - r\dot{\theta}^2 = \frac{f(r)}{m} \Longrightarrow -\ell^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}\left(\ell^2u^4\right) = \frac{f(1/u)}{m}$$

$$-\ell^2u^2\frac{d^2u}{d\theta^2} - \left(\ell^2u^3\right) = \frac{f\left(u^{-1}\right)}{m}$$

$$\frac{d^2u}{d\theta^2}+u=-\frac{f(u^{-1})}{m\ell^2u^2}\ \dots\dots\dots\dots\dots\dots\ (5-43)$$

EXAMPLE 6.5.1

A particle in a central field moves in the spiral orbit

$$r=c\theta^2$$

Determine the force function.

Solution:

We have

$$u=\frac{1}{c\theta^2}$$

and

$$\frac{du}{d\theta}=\frac{-2}{c}\theta^{-3}\qquad\qquad \frac{d^2u}{d\theta^2}=\frac{6}{c}\theta^{-4}=6cu^2$$

Then from Equation 6.5.10b

$$6cu^2+u=-\frac{1}{ml^2u^2}f(u^{-1})$$

Hence,

$$f(u^{-1})=-ml^2(6cu^4+u^3)$$

and

$$f(r)=-ml^2\left(\frac{6c}{r^4}+\frac{1}{r^3}\right)$$

Thus, the force is a combination of an inverse cube and inverse–fourth power law.

EXAMPLE 6.5.2

In Example 6.5.1 determine how the angle θ varies with time.

Solution:

Here we use the fact that $l=r^2\dot{\theta}$ is constant. Thus,

$$\dot{\theta}=lu^2=l\frac{1}{c^2\theta^4}$$

or

$$\theta^4 d\theta = \frac{l}{c^2} dt$$

and so, by integrating, we find

$$\frac{\theta^5}{5} = lc^{-2}t$$

where the constant of integration is taken to be zero, so that $\theta = 0$ at $t = 0$. Then we can write

$$\theta = \alpha t^{1/5}$$

where α = constant = $(5lc^{-2})^{1/5}$.

ميكانيك تحليلي I

الفصل الخامس

المحاضرة الثانيه

أ.د. رعد الحداد

5.5.1 Inverse-Square

One of the most important central fields is one in which the force changes inversely with the square of the polar distance. Consider

$$f(r) = \frac{-k}{r^2} \dots\dots\dots\dots\dots\dots\dots\dots (5-44)$$

Where$(k = GmM)$, with m the mass of the Earth and M the mass of the sun. For the following treatment, we assume $M \gg m$, and take the sun as being stationary (but later we will generalize). As we now that the orbit equation is

$$\frac{d^2u}{d\theta^2} + u = -\frac{f(u^{-1})}{m\ell^2u^2} \dots\dots\dots\dots\dots\dots\dots (5-43)$$

Now Substituting eq. (5-44) into the orbit equation, we find,

$$\frac{d^2u}{d\theta^2} + u = -\frac{(-k/r^2)}{m\ell^2(1/r^2)}$$

$$\frac{d^2u}{d\theta^2} + u = \left(\frac{k}{r^2}\right)\frac{(r^2)}{m\ell^2}$$

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{m\ell^2} \dots\dots\dots\dots\dots\dots\dots (5-45)$$

Equation (5-45) has the same form as the one that describes the simple harmonic oscillator, but with an additive constant. If we solve this equation without the constant term $k/m\ell^2$, the general solution is

$$u(\theta) = A\cos(\theta - \theta_0) + k/m\ell^2 \dots\dots\dots\dots\dots\dots\dots (5-46)$$

$$1/r = A\cos(\theta - \theta_0) + k/m\ell^2 \dots\dots\dots\dots\dots\dots\dots (5-47)$$

$$r = \frac{1}{k/m\ell^2 + A\cos(\theta - \theta_0)} \dots\dots\dots\dots\dots\dots\dots (5-48)$$

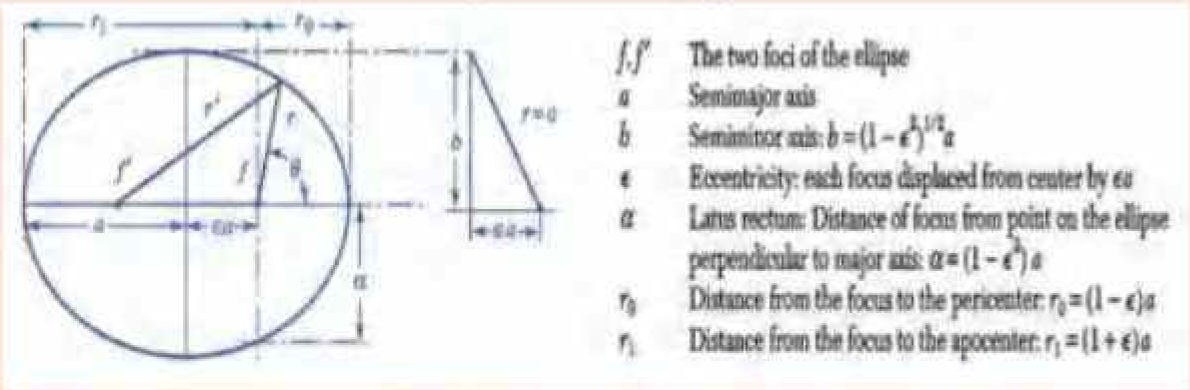
The constants of integration A and θ_0 can be obtain from the initial conditions. Note through the equation that the value of r approaches the minimum values when $\theta = \theta_0$. Since the value of θ_0 specifies the inclination of the orbit relative to the coordinate axes, so we can without losing the generality of the equation choose $\theta_0 = 0$ when looking at the shape of the orbit.

$$r = \frac{1}{k/m\ell^2 + A\cos\theta}$$

$$r = \frac{1}{k/m\ell^2\left[1 + \left(\frac{A}{k/m\ell^2}\right)\cos\theta\right]}$$

$$\therefore r = \frac{m\ell^2/k}{1 + (Am\ell^2/k)\cos\theta} \quad \dots\dots\dots.\dots.(5-49)$$

Figure 4 The ellipse

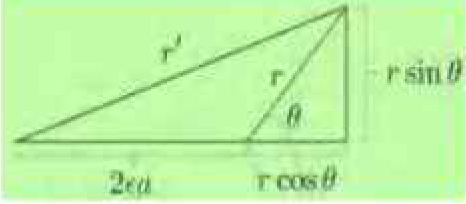


An ellipse is defined to be the locus of points whose sum of distance from two foci, f and f', is a constant, that is,

$$r + r' = constant = (1-a)a + (1+a)a = 2a \dots\dots\dots.\dots.(5-50)$$

Where a the semimajor axis of the ellipse, and the two foci is are offset from its center, each by an amount ϵa. ϵ is called the eccentricity of the ellipse. From the figure above, we may use the **Pythagorean Theorem** to find the geometric relationship between r and r'. Let's show that this property implies $r(\theta)$ like above

Pythagoras Theorem:-



$$r'^2 = (r\sin\theta)^2 + (2\epsilon a + r\cos\theta)^2 \dots(5-51)$$

Substitute ($r' = 2a - r$) in eq. (5-51)

$$(2a - r)^2 = (r\sin\theta)^2 + (2\epsilon a + r\cos\theta)^2$$

$$4a^2 - 4ar + r^2 = r^2\sin^2\theta + 4\epsilon^2 a^2 + 4\epsilon ar\cos\theta + r^2\cos^2\theta$$

$$4a^2 - 4ar + r^2 = r^2\sin^2\theta + r^2\cos^2\theta + 4\epsilon^2 a^2 + 4\epsilon ar\cos\theta$$

$$4a^2 - 4ar + r^2 = r^2(\sin^2\theta + \cos^2\theta) + 4\epsilon a(\epsilon a + r\cos\theta)$$

$$4a^2 - 4ar + r^2 = r^2 + 4\epsilon a(\epsilon a + r\cos\theta)$$

$$4a^2 - 4ar = 4\epsilon a(\epsilon a + r\cos\theta)$$

$$a - r = \epsilon(\epsilon a + r\cos\theta)$$

$$a - r = \epsilon^2 a + \epsilon r\cos\theta$$

$$a - \epsilon^2 a = r + \epsilon r\cos\theta$$

$$a(1-\epsilon^2) = r(1+\epsilon\cos\theta)$$

$$\therefore \quad r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5-52a)$$

Not that at $= \pi/2$, $\cos\pi/2 = 0$, then $r = a(1-\epsilon^2) = \alpha$

$$\therefore \quad r = \frac{\alpha}{1+\epsilon\cos\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5-52b)$$

When we comparing this equation with equation (5-49), we find that

$$\alpha = \frac{m\ell^2}{k} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5-53)$$

$$\epsilon = \frac{Am\ell^2}{k} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5-54)$$

Note:

$$r_0 = (\theta = 0) = \frac{\alpha}{1+\epsilon} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5-55)$$

$$r_1 = (\theta = \pi) = \frac{\alpha}{1-\epsilon} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5-56)$$

Actually, the equation (5-52b) does not describe any ellipse! The orbit can be:

$a)\ \epsilon = 0 \quad Circle$ $\qquad b)\ 0 < \epsilon < 1 \quad ellipse$

$c)\ \epsilon = 1 \quad parabola$ $\qquad d)\ \epsilon > 1 \quad hyperbola$

5.6 LawKepler's Third Law: The Harmonic Law

Kepler's third law relates the orbital period to distance from the Sun. He tried to connect the planetary motions with all fields of abstraction and harmony: geometrical figures, numbers, and musical harmonies. So, called the harmonic law. In this section, we will show how the third law can be derived from Newton's laws of motion and the inverse-square law of gravity. Starting with Equation (5-28), Kepler's second law:-

$$\dot{A} = \frac{\vec{L}}{2m} = \frac{\ell}{2} \ldots\ldots\ldots\ldots\ldots (5-57)$$

Integrating this equation over some period , we obtain,

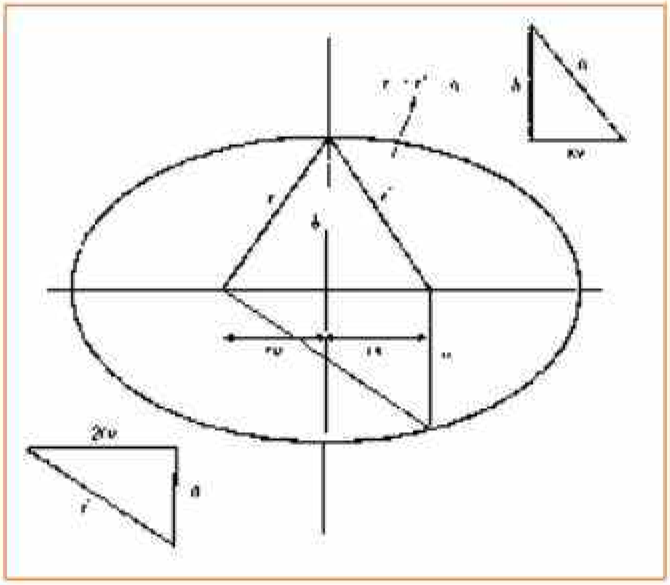
$$\int_0^\tau \dot{A}\, dt = A = \frac{\ell}{2}\tau \quad \Rightarrow \quad \tau = \frac{2A}{\ell} \ldots\ldots\ldots\ldots\ldots (5-58)$$

The area of an ellipse is

$$A = \pi ab \ldots\ldots\ldots\ldots (5-59)$$

$$\therefore \tau = \frac{2\pi ab}{\ell} \ldots\ldots\ldots (5-60)$$

To find the value of b, we used the Pythagorean Theorem, from figure (6)



$$b^2 + \epsilon^2 a^2 = a^2 \ldots\ldots (5-61)$$

$$b^2 = a^2 - \epsilon^2 a^2 \Longrightarrow b^2 = a^2(1-\epsilon^2)$$

$$b = a\sqrt{(1-\epsilon^2)} \ldots\ldots\ldots\ldots (5-62)$$

$$\therefore \tau = \frac{2\pi a^2}{\ell}\sqrt{(1-\epsilon^2)} \ldots\ldots\ldots (5-63)$$

$$\tau^2 = \frac{4\pi^2 a^4}{\ell^2}(1-\epsilon^2) \ldots\ldots\ldots (5-64)$$

Or in term of $a(1-\epsilon^2) = \alpha$

$$\left(1-\epsilon^2\right)=\frac{\alpha}{a}\ \ldots\ldots\ldots\ldots(5-65)$$

$$\therefore \tau^2=\frac{4\pi^2 a^4}{\ell^2}\frac{\alpha}{a}\ \ldots\ldots\ldots\ldots(5-66)$$

$$\therefore \tau^2=4\pi^2\left(\frac{\alpha}{\ell^2}\right)a^3\ \ldots\ldots\ldots\ldots(5-67)$$

Sub eq. (5-53) in eq. (5-67)

$k=\mathrm{G}mM$

$$\tau^2=4\pi^2\left(\frac{m\ell^2/k}{\ell^2}\right)a^3 \Longrightarrow \tau^2=\left(\frac{4\pi^2 m}{\mathrm{G}mM}\right)a^3\ldots\ldots\ldots\ldots\ldots(5-68)$$

$$\therefore \tau^2=\left(\frac{4\pi^2}{\mathrm{G}M}\right)a^3\ldots\ldots\ldots\ldots(5-69)$$

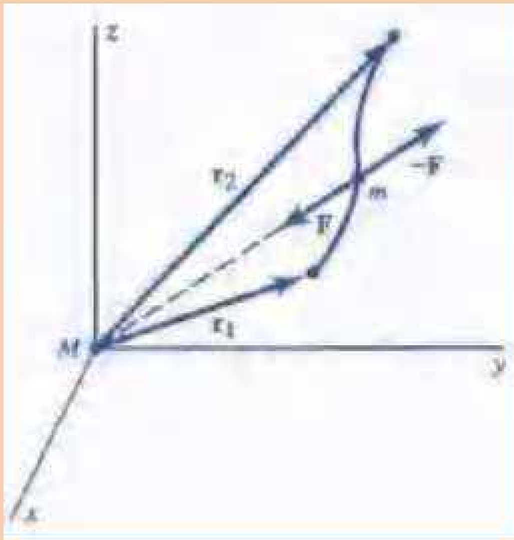
The square of a planet's orbital period is proportional to the cube of its "distance" from the Sun.

5.7 Potential Energy in a Gravitational Field: Gravitational Potential

Let us consider the work W required to move a test particle of mass m along some prescribed path in the gravitational field of another particle of mass M placed at the origin of our coordinate system in Figure (7a). Because the force $\vec{F}$ on the test particle is given by

$$\vec{F}=-\mathrm{G}\frac{mM}{r^2}\,\hat{e}_r\ \ldots\ldots\ldots\ldots(5-70)$$

Then to overcome this force an external force $-\vec{F}$ must be applied.

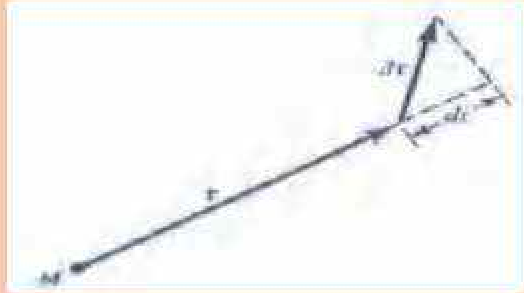


Thus, the work dW done in moving the test particle through a displacement dr is:

$$dW=-\vec{F}\cdot dr=\mathrm{G}\frac{mM}{r^2}\,\hat{e}_r\cdot d\vec{r}\ \ldots\ldots\ldots\ldots(5-71)$$

Now we can resolve $d\vec{r}$ into two components: $\hat{e}_r\, dr$ parallel to $\hat{e}_r$ (the radial component) and the other at right angles to $\hat{e}_r$. (Figure 7b). Clearly,

$$\hat{e}_r \cdot d\vec{r} = dr \; \dots\dots\dots\dots (5-72)$$



And so W is given by

$$dW = GmM \int_{r_1}^{r_2} \frac{dr}{r^2} = -GmM\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \dots\dots\dots\dots (5-73)$$

Where r_1 and r_2 are the radial distances of the particle at the beginning and end of the path. Thus, the work is **independent** of the particular path taken; it depends only on the endpoints. This verifies a fact we already knew, namely that a **central force** described by an inverse-square law is **conservative**.

Potential energy: - The work done in moving the test particle from some arbitrary reference position r_1 to the position r_2. We take the reference position to be $r_1 = \infty$. This assignment is usually a convenient one, because the gravitational force between two particles vanishes when they are separated by ∞. Thus, putting $r_1 = \infty$. and $r_2 = r$ in Equation (5-73), we have

$$V(r) = GmM \int_{\infty}^{r} \frac{dr}{r^2} = -\frac{GmM}{r} \dots\dots\dots\dots (5-74)$$

Forces and potential energies can be thought of as being generated not by actions at a distance but by local actions of matter with an existing field. To do this, we introduce the quantity Φ, called the **gravitational potential**

$$\Phi = \lim_{m\to 0}(V/m) \dots\dots\dots\dots (5-75)$$

The **gravitational potential** at a distance r from a particle of mass M is

$$\Phi = -GM/r \dots\dots\dots\dots (5-76)$$

The limit $m \to 0$ to ensure that the presence of the test particle does not affect the distribution of the other matter and change the thing we are trying to define.

If we have a number of particles $M_1, M_2, \dots\dots, M_i \dots$ located at positions $r_1, r_2, \dots . r_i, \dots .$, then the gravitational potential at the point $\vec{r}(x, y, z)$ is the sum of the gravitational potentials of all the particles, that is,

$$\Phi(x, y, z) = \sum \Phi_i = -G \sum_i \frac{M_i}{|\vec{r} - \vec{r}_i|} \dots\dots\dots\dots\dots (5-77)$$

For a continuous distribution

$$\Phi = -G \int \frac{dM}{|\vec{r} - \vec{r}'|} \dots\dots\dots\dots\dots (5-78)$$

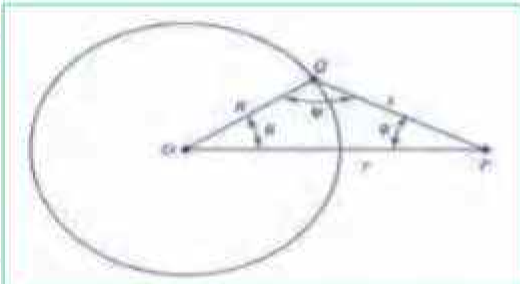
$$\Phi = -G \int \frac{\rho dV'}{|\vec{r} - \vec{r}'|} \dots\dots\dots\dots\dots (5-79)$$

$$\Phi = -G \int \frac{\rho d^3 r'}{|\vec{r} - \vec{r}'|} \dots\dots\dots\dots\dots (5-80)$$

Since $\quad \vec{F} = -\vec{\nabla} V \Rightarrow mg = -\vec{\nabla}\Phi$

$$\therefore \vec{g} = -\vec{\nabla}\Phi \quad \dots\dots\dots (5-81) \quad \textit{gravitational field intensity.}$$

H.W (6.7.2):- Find the potential function and the gravitational field intensity in the plane of a thin circular ring.



5.8 Potential Energy in a General Central Field

We showed previously that a central field of the inverse-square type is conservative. Now are all central forces ($\vec{F}$) conservative?

$$\vec{F} = \hat{r} f(r) \quad \dots\dots\dots\dots\dots (5-82) \quad \textit{A general isotropic central field}$$

$$\vec{\nabla} \times \vec{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} e_r & e_\theta r & e_\phi r \sin\theta \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ F_r & rF_\theta & rF_\phi \sin\theta \end{vmatrix} \dots\dots\dots\dots\dots (5-83)$$

For our central force $F_r = f(r)$, $F_\theta = 0$, and $F_\phi = 0$. The curl then reduces to

$$\vec{\nabla} \times \vec{F} = \frac{e_\theta}{r \sin\theta} \frac{\partial f}{\partial \phi} - \frac{e_\phi}{r} \frac{\partial f}{\partial \theta} = 0 \quad \dots\dots\dots\dots\dots (5-84) \quad \therefore \ \vec{F} \ conservative$$

5.9 Energy Equation of an Orbit in a Central Field

The square of the speed is given in polar coordinates from eq. (1-52, lect 3)

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \quad \dots\dots\dots\dots\dots\dots (1-52)$$

$$\vec{v} \cdot \vec{v} = v^2 = \dot{r}^2 + r^2\dot{\theta}^2 \quad \dots\dots\dots\dots\dots\dots (5-85)$$

Because a central force is conservative, the total energy $T + V$ is constant and is given by

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) = E = constant \quad \dots\dots\dots\dots\dots\dots (5-86)$$

$$u = \frac{1}{r}, \qquad \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = \frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} \quad \Longrightarrow \quad \dot{r} = -r^2\,\dot{\theta}\,\frac{du}{d\theta}$$

$$-r^2\,\dot{\theta} = \ell \quad \Longrightarrow \quad \dot{r} = \ell\,\frac{du}{d\theta}, \quad \ell^2 = r^4\dot{\theta}^2 \Longrightarrow u^2\ell^2 = \frac{r^4}{r^2}\dot{\theta}^2 = r^2\dot{\theta}^2$$

$$\frac{1}{2}m\left[\ell^2\left(\frac{du}{d\theta}\right)^2 + u^2\ell^2\right] + V\left(u^{-1}\right) = E \quad \dots\dots\dots\dots\dots\dots (5-87)$$

$$\frac{1}{2}m\ell^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] + V\left(u^{-1}\right) = E \quad \dots\dots\dots\dots\dots\dots (5-88)$$

The preceding equation is called the energy equation of the orbit.

EXAMPLE 6.9.1

In Example 6.5.1 we had for the spiral orbit $r = c\theta^2$

$$\frac{du}{d\theta} = \frac{-2}{c}\theta^{-3} = -2c^{1/2}u^{3/2}$$

so the energy equation of the orbit is

$$\frac{1}{2}ml^2(4cu^3 + u^2) + V = E$$

$$V(r) = E - \frac{1}{2}ml^2\left(\frac{4c}{r^3} + \frac{1}{r^2}\right)$$

Force function

$$f(r) = -\frac{dV}{dr}$$

Same as obtained from differential equation of the orbital

5.10 Orbital Energies in an Inverse-Square Field

The potential energy function for an inverse-square force field is

$$V(r) = \frac{-k}{r^2} = -ku \dots\dots\dots\dots\dots\dots\dots\dots (5-89)$$

So the energy equation of the orbit (Equation 5-88) becomes

$$\frac{1}{2}m\ell^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] - ku = E \dots\dots\dots\dots\dots\dots\dots\dots (5-90)$$

$$\frac{1}{2}m\ell^2\left(\frac{du}{d\theta}\right)^2 + \frac{1}{2}m\ell^2u^2 - ku = E \implies \left(\frac{du}{d\theta}\right)^2 + u^2 - \frac{2ku}{m\ell^2} = \frac{2E}{m\ell^2}$$

$$\left(\frac{du}{d\theta}\right)^2 = -u^2 + \left(\frac{2k}{m\ell^2}\right)u + \left(\frac{2E}{m\ell^2}\right) \dots\dots\dots\dots\dots\dots\dots\dots (5-91)$$

$$\frac{du}{d\theta} = \sqrt{-u^2 + \left(\frac{2k}{m\ell^2}\right)u + \left(\frac{2E}{m\ell^2}\right)} \dots\dots\dots\dots\dots\dots\dots\dots (5-92)$$

$$d\theta = \frac{du}{\sqrt{-u^2 + \left(\frac{2k}{m\ell^2}\right)u + \left(\frac{2E}{m\ell^2}\right)}} \dots\dots\dots\dots\dots\dots\dots\dots (5-93)$$

define, $a = -1$, $b = 2k/m\ell^2$ and $c = 2E/m\ell^2$, with take the integral

$$\int_{\theta_0}^{\theta} d\theta = \frac{du}{\sqrt{au^2 + bu + c}} \dots\dots\dots\dots (5-94) \quad use\ a\ table\ of\ integrals$$

$$\implies (\theta - \theta_0) = \frac{1}{\sqrt{-a}}\cos^{-1}\left(-\frac{b + 2au}{\sqrt{b^2 - 4ac}}\right) \quad \dots (5-95a)$$

$$\implies \cos\left[\sqrt{-a}(\theta - \theta_0)\right] = -\frac{b - 2au}{\sqrt{b^2 - 4ac}} \quad \dots (5-95b)$$

$$\implies u = \frac{\sqrt{b^2 - 4ac}}{-2a}\cos\left[\sqrt{-a}(\theta - \theta_0)\right] + \frac{b}{-2a} \quad \dots (5-95c)$$

Now we replace u with $1/r$ and insert the values for the constants a, b, and c into Equation (5-95c).

$$\frac{1}{r}=\frac{\sqrt{\left(\frac{2k}{m\ell^2}\right)^2-4\,(-1)\,2E/m\ell^2}}{2(-1)}\cos\left[\sqrt{-(-1)}(\theta-\theta_0)\right]+\frac{2k/m\ell^2}{-2(-1)}$$

$$\frac{1}{r}=\frac{1}{2}\sqrt{4\frac{k^2}{m^2\ell^4}+4\frac{2E}{m\ell^2}}\cos(\theta-\theta_0)+\frac{k}{m\ell^2}\ \dots\dots.\dots\dots.(5-96)$$

$$\frac{1}{r}=\frac{1}{2}\frac{2k}{m\ell^2}\left(\sqrt{1+\frac{2m\ell^2E}{k^2}}\cos(\theta-\theta_0)\right)+\frac{k}{m\ell^2}\ \dots\dots\dots.(5-97)$$

$$\frac{1}{r}=\frac{k}{m\ell^2}\left(\sqrt{1+\frac{2Em\ell^2}{k^2}}\cos(\theta-\theta_0)\right)+\frac{k}{m\ell^2}\ \dots\dots\dots\dots(5-98)$$

$$\frac{1}{r}=\frac{k}{m\ell^2}\left[\sqrt{1+\frac{2Em\ell^2}{k^2}}\cos(\theta-\theta_0)+1\right]\ \dots\dots\dots.\dots\dots(5-99)$$

$$r=\frac{m\ell^2/k}{1+\sqrt{1+\frac{2Em\ell^2}{k^2}}\cos(\theta-\theta_0)}\ \dots\dots.\dots\dots(5-100)$$

If we set $\theta_0=0$ and comparing with eq. (5-52a), we see that again it represents a conic section whose eccentricity is

$$\epsilon=\sqrt{1+\frac{2E}{k}\frac{m\ell^2}{k}}\ \dots\dots.\dots\dots(5-101)$$

We know that $a(1-\epsilon^2)=\alpha=m\ell^2/k^2$, and so

$$\epsilon^2=1+\frac{2E}{k}(1-\epsilon^2)a\ \dots\dots.\dots\dots(5-102)$$

$$0 = (1 - \epsilon^2) + \frac{2E}{k}(1 - \epsilon^2)a \ldots\ldots\ldots.\ldots\ldots\ldots (5-103)$$

$$0 = 1 + \frac{2E}{k}a \ldots\ldots\ldots.\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots (5-104)$$

$$\frac{2E}{k}a = -1 \ldots\ldots\ldots.\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots (5-105)$$

$$E = -\frac{k}{2a} \ldots\ldots\ldots.\ldots\ldots\ldots (5-106)$$

$E < 0$	$\Longrightarrow$	$\epsilon < 1$	Elliptic
$E = 0$	$\Longrightarrow$	$\epsilon = 1$	Parabola
$E > 0$	$\Longrightarrow$	$\epsilon > 1$	Hyperbola

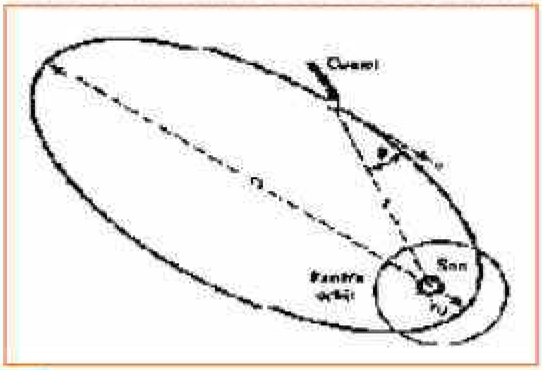
Because $E = T + V$ and is constant, the closed orbits are those for which $T < |V|$, and the open orbits are those for which $T \geq |V|$.

In the Sun's gravitational field the force constant $k = GM_\odot m$, where $M_\odot$ is the mass of the Sun and m is the mass of the body. The total energy is then

$$\frac{mv^2}{2} - \frac{GM_\odot m}{r} = E = \text{constant} \qquad (6.10.11)$$

so the orbit is an ellipse, a parabola, or a hyperbola depending on whether v^2 is less than, equal to, or greater than the quantity $2GM_\odot/r$, respectively.

H.W (6.10.1): A comet is observed to have a speed v when it is a distance r from the Sun, and its direction of motion makes an angle the radius vector from the Sun. Find the eccentricity of the comet's orbit.



H.W (6.10.2): see book.

H.W $\Rightarrow$ 5.2, 5.3, 5.10, 5.11 $\Leftarrow$

ميكانيك تحليلي I

الفصل الخامس

المحاضرة الثالثه

أ.د. رعد الحداد

5.11 Limits of the Radial Motion: Effective Potential

$$E = \frac{1}{2}mv^2 + V(r) \dots\dots.\dots\dots (5-107)$$

$$v^2 = \dot{r}^2 + \left(r\dot{\theta}\right)^2 \dots\dots\dots\dots\dots\dots (5-108)$$

$$l = |\vec{r} \times \vec{v}| = rr\dot{\theta} = r^2\dot{\theta} \dots\dots\dots\dots\dots\dots (5-109)$$

The expression for the energy of central-force motion was:-

$$E = \frac{1}{2}m\left[\dot{r}^2 + \frac{l^2}{r^2}\right] + V(r) \dots\dots.\dots\dots (5-110)$$

$$E = \frac{1}{2}m\dot{r}^2 + \left[\frac{ml^2}{2r^2} + V(r)\right] \dots\dots.\dots\dots (5-111)$$

The last term inside the square brackets is called the **effective potential** ($U(r)$). Therefore we can write the last term as

$$U(r) = \frac{ml^2}{2r^2} + V(r) \dots\dots.\dots\dots (5-112)$$

The first term is called **the centrifugal potential**.

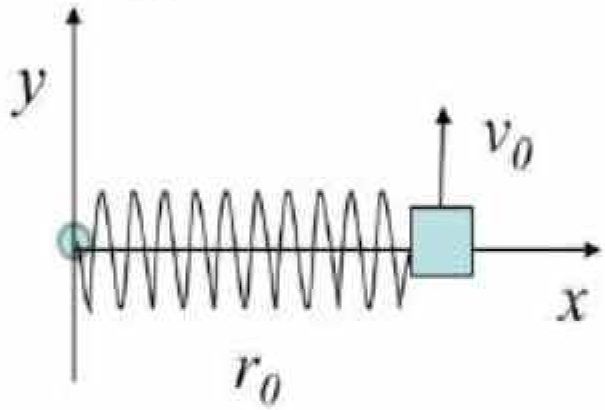
$$\therefore \quad E = \frac{1}{2}m\dot{r}^2 + U(r) \dots\dots.\dots\dots (5-113)$$

In this equation, we see that, as far as the radial motion is concerned, the particle behaves in exactly the same way as a particle of mass m moving in one-dimensional motion under a potential energy function U(r).

Now we discuss the harmonic motion (ignore the natural length of the spring).

- ✓ For circular motion $\dot{r} = 0$ and the energy become E_{min} at

$$\left.\frac{\partial U(r)}{\partial r}\right|_{r_0} = 0$$



Therefore eq. (5-113) becomes

$$U(r) - E = 0 \ldots\ldots.\ldots\ldots(5-114)$$

Or

$$\frac{ml^2}{2r^2} + V(r) - E = 0 \ldots\ldots.\ldots\ldots(5-115)$$

The allowed values of r are those for which $U(r) \leq E$, because $\dot{r}^2$ is necessarily positive or zero. Thus, it is possible to determine the range of the radial motion without knowing anything about the orbit.

Example: - for inverse square law, suppose $V(r) = -k/r$, then eq. (5-112) is

$$U(r) = \frac{ml^2}{2r^2} - \frac{k}{r} \ldots\ldots.\ldots\ldots(5-116)$$

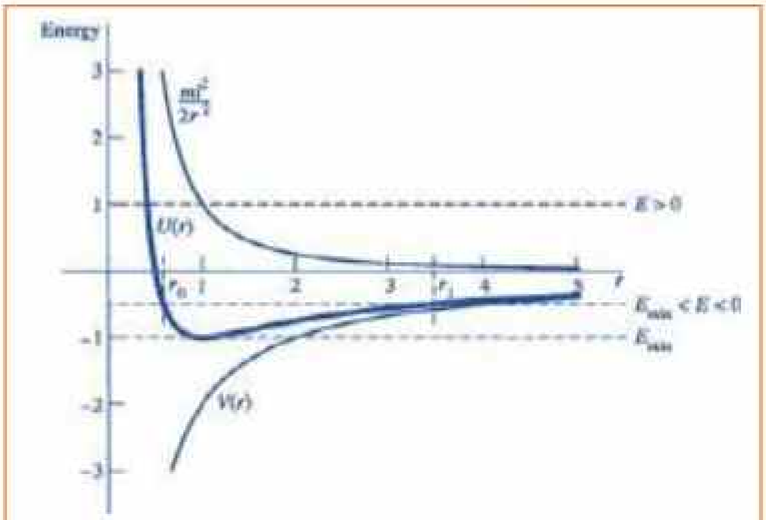


Figure show the effective potential for the inverse-square law of force and limits of the radial motion.

$$U(r) = E < 0$$

$$E = \frac{ml^2}{2r^2} - \frac{k}{r}$$

$$Er^2 + kr - \frac{ml^2}{2} = 0$$

$$ax^2 + bx + c = 0$$

And by using Quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = r, \quad a = E \ , \quad b = k \quad , \quad c = -\frac{ml^2}{2}$$

$$r_{\pm} = \frac{-k \pm \sqrt{k^2 - 4(E)\left(-\frac{ml^2}{2}\right)}}{2E} = \frac{k \pm \sqrt{k^2 + 2Eml^2}}{-2E}$$

$$r_{1,0} = \frac{k \pm \sqrt{k^2 - 2|E|ml^2}}{2|E|} \ \ldots.\ldots\ldots(5-117)$$

If $E < 0$ the orbits are bound, the two roots are both positive, and the resulting orbit is an **ellipse** in which r_0 and r_1 are the pericenter and apocenter, respectively. When the energy is equal to its minimum possible value. That value can be getting by make the argument under the square root equal to zero

$$k^2 + 2Eml^2 = 0 \quad \Longrightarrow \quad k^2 = -2Eml^2$$

$$E = E_{min} = -\frac{k^2}{2ml^2} \quad \dots\dots\dots(5-118)$$

$$r_0 = \frac{k \pm \sqrt{k^2 - 2\frac{k^2}{2ml^2}ml^2}}{-2E_{min}} \quad \dots\dots\dots\dots\dots(5-119)$$

$$r_0 = \frac{k}{2|E_{min}|} = \frac{ml^2}{k} \quad \dots\dots\dots(5-120) \quad \textit{The orbit is a circle}$$

For $E \geq 0$, Equation $(5-117)$ has only a single, positive real root corresponding to a parabola $(E = 0)$ or a hyperbola $(E > 0)$.

5.12 Nearly Circular Orbits in Central Fields: Stability

- ✓ A circular orbit is possible under any attractive central force.
- ✓ Not all central forces result in stable circular orbits.

Let us discuss the following question: If a particle traveling in a circular orbit suffers a slight disturbance, does the ensuing orbit remain close to the original circular path? To answer the query, we refer to the radial differential equation of motion (Equation 5-32):-

$$m(\ddot{r} - r\dot{\theta}^2) = f(r) \dots\dots\dots\dots\dots\dots(5-32)$$

because $\dot{\theta}^2 = l/r^2 =$, we can write the radial equation as follows:

$$m\ddot{r} = \frac{ml^2}{r^3} + f(r) \dots\dots\dots\dots\dots\dots(5-121)$$

$$m\ddot{r} = \frac{ml^2}{r^3} + \frac{dV(r)}{dr} \dots\dots\dots\dots\dots\dots(5-122)$$

$$m\ddot{r} = -\frac{d}{dr}\left(\frac{ml^2}{2r^2}\right) + \frac{dV(r)}{dr} \dots\dots\dots\dots\dots\dots(5-123)$$

$$m\ddot{r} = -\frac{d}{dr}\left(\frac{ml^2}{2r^2} + V(r)\right) \dots\dots\dots\dots\dots\dots(5-124)$$

$$\because U(r) = \frac{ml^2}{2r^2} + V(r) \dots\dots\dots\dots(5-112)$$

$$\therefore m\ddot{r} = -\frac{dU(r)}{dr} \dots\dots\dots\dots\dots\dots(5-125)$$

Now for a circular orbit, r is constant, and $\dot{r} = \ddot{r} = 0$. Thus, calling a the radius of the circular orbit($r = a$), we have

$$f(a) = -\frac{ml^2}{a^3} \dots\dots\dots\dots\dots\dots(5-126)$$

We can expended r about a: $x = r + a$.

$$m\ddot{r} = \frac{ml^2}{(x+a)^3} + f(x+a) \dots\dots\dots\dots\dots\dots(5-127)$$

$$m\ddot{r} = ml^2(x+a)^{-3} + f(x+a) \dots\dots\dots\dots\dots\dots(5-128)$$

$$m\ddot{r} = ml^2a^{-3}\left(1 - 3\frac{x}{a} + ..\right)^{-3} + [f(a) + f'(a)x + \cdots] \dots\dots\dots(5-129)$$

$$m\ddot{r} + \left[-3\frac{x}{a}f(a) + f'(a)x + \cdots\right] \dots\dots\dots(5-130)$$

$$m\ddot{r} + \left[-\frac{3}{a}f(a) + f'(a) + \cdots\right]x \dots\dots\dots(5-131) \quad eq.(6.12.6)$$

if we ignore terms involving x^2 and higher powers of x. Now, if the coefficient of x (the quantity in brackets) in Equation 6.12.6 is positive, then the equation is the same as that of the simple harmonic oscillator. In this case the particle, if perturbed, oscillates harmonically about the circle $r = a$, so the circular orbit is a stable one. On the other hand, if the coefficient of x is negative, the motion is nonoscillatory, and the result is that x eventually increases exponentially with time; the orbit is unstable. (If the coefficient of x is zero, then higher terms in the expansion must be included to determine the stability.) Hence, we can state that a circular orbit of radius a is stable if the force function $f(r)$ satisfies the inequality

$$f(a)+\frac{a}{3}f'(a)<0$$

For example, if the radial force function is a power law, namely,

$$f(r)=-cr^n$$

then the condition for stability reads

$$-ca^n-\frac{a}{3}cna^{n-1}<0$$

which reduces to

$$n>-3$$

Thus, the inverse-square law ($n=-2$) gives stable circular orbits,

5.13 Apsides and Apsidal Angles for Nearly Circular Orbits

- ✓ An apsis, or apse, is a point in an orbit at which the radius vector assumes an extreme value (maximum or minimum).
- ✓ The perihelion and aphelion points are the apsides of planetary orbits.
- ✓ The angle swept out by the radius vector between two consecutive apsides is called the apsidal angle. Thus, the apsidal angle is π for elliptic orbits under the inverse-square law of force.
- ✓ we have seen that r oscillates about the circle $r = a$ (if the orbit is stable). We have from Equation (5-131)

$$\tau_r = 2\pi\sqrt{\frac{m}{[-(3/a)f(a)-f'(a)]}} \quad \text{.........} (5-132)$$

The apsidal angle ψ in this case is just the amount by which the polar angle θ increases during the time that r oscillates from a minimum value ($t=0$) the succeeding maximum value ($t=\tau_r/2$)($half\ a\ period$). Now since

$$\dot{\theta} = \frac{1}{l^2} \approx \frac{1}{a^2} = \frac{\sqrt{-\frac{a^3 f(a)}{m}}}{a^2} = \sqrt{-\frac{a^3 f(a)}{m a^4}}$$

$$\dot{\theta} = \sqrt{-\frac{f(a)}{ma}} \quad \ldots\ldots\ldots\ldots (5-133)$$

$$\frac{d\theta}{dt} = \sqrt{-\frac{f(a)}{ma}} \Longrightarrow \int_0^{\psi} d\theta = \sqrt{-\frac{f(a)}{ma}} \int_0^{\tau_r/2} dt$$

$$\psi = \frac{\tau_r}{2}\sqrt{-\frac{f(a)}{ma}} \quad \ldots\ldots\ldots\ldots (5-134)$$

by sub eq. (5-132) in eq. (5-134), we get

$$\psi = \frac{2\pi}{2}\sqrt{-\frac{f(a)}{ma}} \sqrt{\frac{m}{[-(3/a)f(a) - f'(a)]}}$$

$$\psi = \pi \sqrt{\frac{-f(a)}{[-3f(a) - af'(a)]}} = \pi \sqrt{\frac{f(a)}{3f(a) + af'(a)}}$$

$$\psi = \pi \sqrt{\frac{f(a)}{3f(a)} + \frac{f(a)}{af'(a)}} = \pi \sqrt{\frac{1}{3} + \frac{f(a)}{af'(a)}}$$

$$\psi = \pi \left[3 + \frac{af'(a)}{f(a)}\right]^{-1/2} \quad \ldots\ldots\ldots\ldots (5-135)$$

Thus, for the power law of force $f(r) = -cr^n$, we obtain

$$f(a) = -ca^n \Longrightarrow f'(a) = -cna^{n-1}$$

$$\therefore \psi = \pi \left[3 + \frac{a(-cn)a^{n-1}}{-ca^n}\right]^{-1/2} \quad \ldots\ldots\ldots (5-136)$$

$$\psi = \pi \left[3 + \frac{na^n}{a^n}\right]^{-1/2} \quad \ldots\ldots\ldots (5-137)$$

$$\psi = \pi \ [3 + n]^{-1/2} \quad \ldots\ldots\ldots (5-138)$$

$$\psi = \frac{\pi}{\sqrt{[3+n]}} \quad \ldots\ldots\ldots\ldots\ldots (5-139)$$

The apsidal angle is independent of the size of the orbit in this case. The orbit is *reentrant*, or repetitive, in the case of the inverse-square law ($n = -2$) for which $\psi = \pi$ and in the case of the linear law ($n = 1$) for which $\psi = \pi/2$. If, however, say $n = 2$, then $\psi = \pi/\sqrt{5}$, which is an irrational multiple of π, and so the motion does not repeat itself.

If the law of force departs slightly from the inverse-square law, then the apsides either advances or regresses steadily, depending on whether the apsidal angle is slightly greater or slightly less than π. (See Figure 6.13.1.)

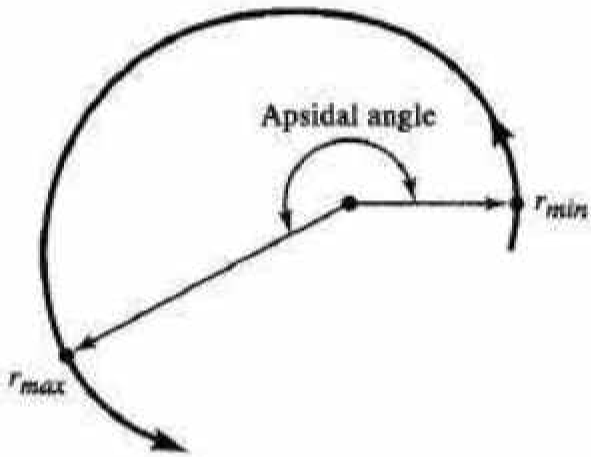


Figure 6.13.1 Illustrating the apsidal angle.

EXAMPLE 6.13.1

Let us assume that the gravitational force field acting on the planet Mercury takes the form

$$f(r) = -\frac{k}{r^2} + \epsilon r$$

where ϵ is very small. The first term is the gravitational field due to the Sun, and the second term is a repulsive perturbation due to a surrounding ring of matter. We assume this matter distribution as a simple model to represent the gravitational effects of all the other planets, primarily Jupiter. The perturbation is linear for points near the Sun and in the plane of the surrounding ring, as previously explained in Example 6.7.2. The apsidal angle, from Equation 5-135, is

$$\Psi = \pi\left(3 + a\frac{2ka^{-3} + \epsilon}{-ka^{-2} + \epsilon a}\right)^{-1/2}$$
$$= \pi\left(\frac{1 - 4k^{-1}\epsilon a^3}{1 - k^{-1}\epsilon a^3}\right)^{-1/2} = \pi\left(1 - \frac{\epsilon}{k}a^3\right)^{1/2}\left(1 - 4\frac{\epsilon}{k}a^3\right)^{-1/2}$$
$$\approx \pi\left(1 - \tfrac{1}{2}\frac{\epsilon}{k}a^3\right)\left(1 + 2\frac{\epsilon}{k}a^3\right)$$
$$\approx \pi\left(1 + \tfrac{3}{2}\frac{\epsilon}{k}a^3\right)$$

In the last step, we used the binomial expansion theorem to expand the terms in brackets in powers of ϵ/k and kept only the first-order term. The apsidal angle advances if ϵ is positive and regresses if it is negative.

H.W ⇒ 6.21 6.22 6.24 6.25 6.27 ⇐

ميكانيك تحليلي I

الفصل السادس

المحاضرة الأولى

أ.د. رعد الحداد

Dynamics of Systems of Particles

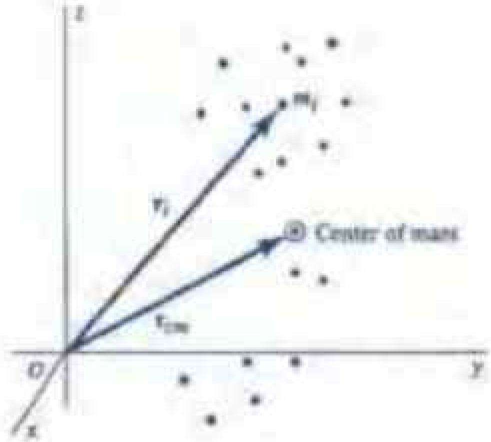
To study a system of a large group of free particles, we will focus our attention primarily on general appearance of the movement of that group.

6.1 Introduction: Center of Mass and Linear Momentum of a System

System consists of n particles of masses $m_1, m_2, \dots, m_n$ whose position vectors are $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$. The center of mass of the system as the point whose position vector $\vec{r}_{cm}$ (Fig.1) is given by:-

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n}{m_1 + m_2 + \cdots + m_n} \quad \dots\dots(6.1)$$

$$\vec{r}_{cm} = \frac{\sum_i m_i \vec{r}_i}{m} \quad \dots\dots(6.2)$$



$m = \sum_i m_i$ *is the total mass of the system.*

$$x_{cm} = \frac{\sum_i m_i x_i}{m} \qquad y_{cm} = \frac{\sum_i m_i y_i}{m} \qquad z_{cm} = \frac{\sum_i m_i z_i}{m} \quad \dots\dots(6.3)$$

Linear Momentum

The linear momentum $\vec{p}$ of the system define as the vector sum of the linear momenta of the individual particles, namely,

$$\vec{p} = \sum_i \vec{p}_i = \sum_i m_i v_i \quad \dots\dots(6.4)$$

$$\vec{p} = m\vec{v}_{cm} \quad \dots\dots(6.5)$$

That is mean, the linear momentum of a system of particles is equal to the velocity of the center of mass multiplied by the total mass of the system.

Suppose now that there are external forces $\vec{F}_1, \vec{F}_2, \dots, \vec{F}_i, \dots . \vec{F}_n$ acting on the respective particles (that is $\vec{F}_n$ $act\ on\ m_n$). In addition, there may be internal forces of interaction between any two particles of the system. We denote these internal forces by $\vec{F}_{ij}$ meaning the **force exerted on particle i** by particle j, with the understanding that $\vec{F}_{ij} = 0$. The equation of motion of particle i is then or the **Newton's 2nd law for each particle i**

$$\vec{F}_i + \sum_i \vec{F}_{ij} = m_i \ddot{\vec{r}}_i = \dot{\vec{p}}_i \dots \dots \dots \dots \dots \dots \dots \dots \dots (6.6)$$

Where $\vec{F}_i$ the external is force acting on particle i, and $\vec{F}_{ij}$ is the inter-particle interactions. If we sum over all the i (adding n particles):

$$\sum_i \vec{F}_i + \sum_{i,j} \vec{F}_{ij} = \sum_i m_i \ddot{\vec{r}}_i = \sum_i \dot{\vec{p}}_i = \dot{\vec{p}}_{cm} \dots \dots \dots \dots \dots (6.7)$$

By Newton's 3rd law, $\vec{F}_{ij} = -\vec{F}_{ji}$ and so

$$\sum_{i,j} \vec{F}_{ij} = 0 \dots \dots \dots \dots \dots \dots \dots \dots (6.8)$$

$$\sum_i \vec{F}_i = \dot{\vec{p}}_i = m\vec{a}_{cm} \dots \dots \dots \dots \dots \dots (6.9)$$

In words: The acceleration of the center of mass of a system of particles is the same as that of a single particle having a mass equal to the total mass of the system and acted on by the sum of the external forces.

EXAMPLE 7.1 1

At some point in its trajectory a ballistic missile of mass m breaks into three fragments of mass $m/3$ each. One of the fragments continues on with an initial velocity of one-half the velocity $\mathbf{v}_0$ of the missile just before breakup. The other two pieces go off at right angles to each other with equal speeds. Find the initial speeds of the latter two fragments in terms of v_0.

Solution:

At the point of breakup, conservation of linear momentum is expressed as

$$m\mathbf{v}_{cm} = m\mathbf{v}_0 = \frac{m}{3}\mathbf{v}_1 + \frac{m}{3}\mathbf{v}_2 + \frac{m}{3}\mathbf{v}_3$$

The given conditions are: $\mathbf{v}_1 = \mathbf{v}_0/2$, $\mathbf{v}_2 \cdot \mathbf{v}_3 = 0$, and $v_2 = v_3$. From the first we get, on cancellation of the m's, $3\mathbf{v}_0 = (\mathbf{v}_0/2) + \mathbf{v}_2 + \mathbf{v}_3$, or

$$\tfrac{5}{2}\mathbf{v}_0 = \mathbf{v}_2 + \mathbf{v}_3$$

Taking the dot product of each side with itself, we have

$$\tfrac{25}{4}v_0^2 = (\mathbf{v}_2 + \mathbf{v}_3) \cdot (\mathbf{v}_2 + \mathbf{v}_3) = v_2^2 + 2\mathbf{v}_2 \cdot \mathbf{v}_3 + v_3^2 = 2v_2^2$$

Therefore,

$$v_2 = v_3 = \frac{5}{2\sqrt{2}}v_0 = 1.77v_0$$

6.2 Angular Momentum and Kinetic Energy of a System

The angular momentum $\vec{L}$ of a system of particles is defined accordingly، as the vector sum of the individual angular momenta, namely,

$$\vec{L} = \sum_{i=1}^{n}(r_i \times \vec{p}_i) \dots\dots\dots\dots\dots\dots (6.10)$$

$$\vec{L} = \sum_{i=1}^{n}(r_i \times m_i v_i) \dots\dots\dots\dots\dots\dots (6.11)$$

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n}(\vec{v}_i \times m_i \vec{v}_i) + \sum_{i=1}^{n}(\vec{r}_i \times m_i \vec{a}_i) \dots\dots\dots\dots\dots\dots (6.12)$$

The first term on the right vanishes, because, $\vec{v}_i \times \vec{v}_i = 0$,

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n}(\vec{r}_i \times m_i \vec{a}_i) \dots\dots\dots\dots\dots\dots (6.13)$$

The term $m_i \vec{a}_i$ is equal to the total force acting on particle i, we can write

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n}\left(\vec{r}_i \times \left[\vec{F}_i + \sum_{j=1}^{n}\vec{F}_{ij}\right]\right) \text{……….………………} (6.14)$$

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n}\left(\vec{r}_i \times \vec{F}_i + \sum_{j=1}^{n}\vec{r}_i \times \vec{F}_{ij}\right) \text{……….………………} (6.15)$$

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n}\vec{r}_i \times \vec{F}_i + \sum_{i=1}^{n}\sum_{j=1}^{n}\vec{r}_i \times \vec{F}_{ij} \text{……….………………} (6.16)$$

$\vec{F}_i$ denotes the total external force on particle i, and $\vec{F}_{ij}$ denotes the (internal) force exerted on particle i by any other particle j.

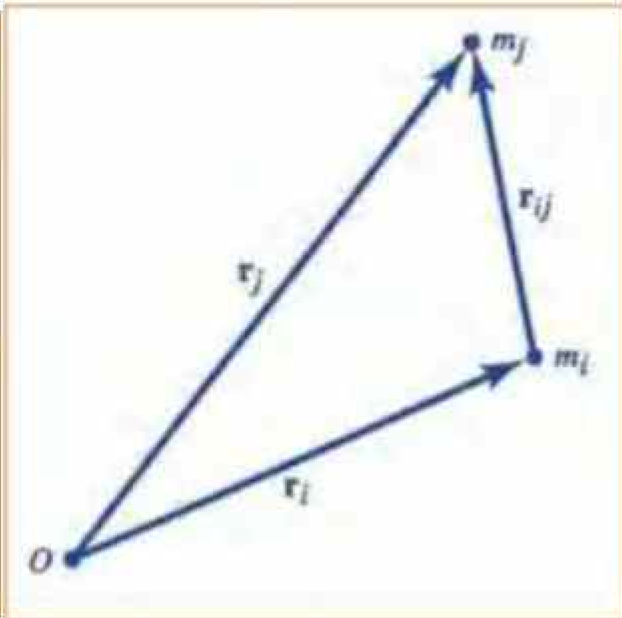
$$\sum_{i,j=1}^{n}\vec{r}_i \times \vec{F}_{ij} = \sum_{i<j}^{n}\left(\vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \vec{F}_{ji}\right) \text{…………………… …………} (6.17)$$

The vector displacement of particle j relative to particle i by $\vec{r}_{ij}$ we see from the triangle shown in Figure 2 that

$$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j \text{…………} (6.18)$$

Therefore, because $\vec{F}_{ij} = -\vec{F}_{ji}$ expression (6.17) reduces to

$$-\vec{r}_{ij} \times \vec{F}_{ij} \text{…………} (6.19)$$



$$\sum_{i,j=1}^{n}\vec{r}_i \times \vec{F}_{ij} = \sum_{i<j}^{n}\left(\vec{r}_i \times \vec{F}_{ij} - \vec{r}_j \times \vec{F}_{ij}\right) = \sum_{i<j}^{n}\left(\vec{r}_i - \vec{r}_j\right) \times \vec{F}_{ij} \text{…………} (6.20)$$

$$\therefore \frac{d\vec{L}}{dt} = \sum_{i=1}^{n}\vec{r}_i \times \vec{F}_i + \sum_{i<j}^{n}\left(\vec{r}_i - \vec{r}_j\right) \times \vec{F}_{ij} \text{………………………………} (6.21)$$

If the forces are central between particles, then $(\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij} = 0$

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i = N \; \dots\dots\dots\dots (6.22) \quad (Torque)$$

$$\frac{d\vec{L}}{dt} = N \; \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (6.23) \quad (Torque)$$

That is mean, **the time rate of change of the angular momentum of a system** is equal to **the total moment of all the external forces acting on the system**.

- ✓ If a system is **isolated**, then $\boldsymbol{N = 0}$, and the angular momentum remains constant in both magnitude and direction:

$$\vec{L} = \sum_{i=1}^{n} (\vec{r}_i \times m_i \vec{v}_i) = constant\ vector \; \dots\dots\dots\dots\dots\dots (6.24)$$

This is a statement of the principle of conservation of angular momentum.

- ✓ The angular momentum of an isolated system is also constant in the case of a system of moving charges when the angular momentum of the electromagnetic field is considered.

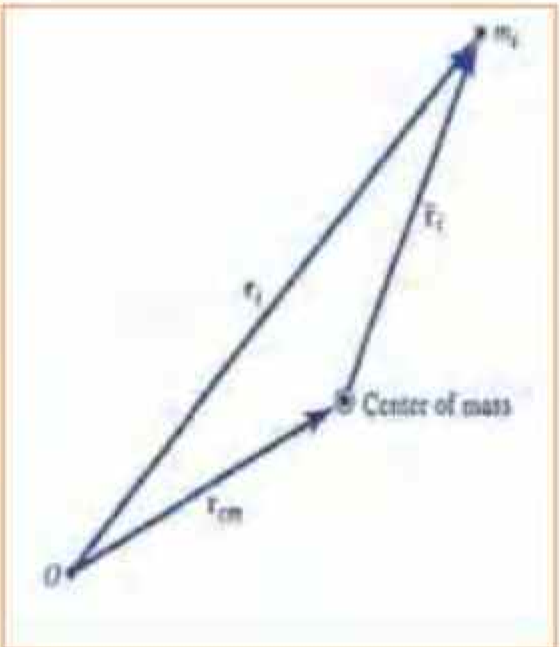
The angular momentum will now expressed in terms of the motion of the center of mass. As shown in Figure 3, we can express each position vector in the form

$$\vec{\bar{r}}_i = \vec{r}_{cm} + \vec{r}_i \; \dots\dots\dots\dots\dots\dots (6.25)$$

Where $\vec{r}_i$ is the position of particle i relative to the center of mass. Taking the derivative with respect to t, we have

$$\vec{\bar{v}}_i = \vec{v}_{cm} + \vec{v}_i \; \dots\dots\dots\dots\dots\dots (6.26)$$

Where $\vec{v}_{cm}$ is the velocity of the center of mass and $\vec{v}_i$ is the velocity of particle i relative to the center of mass.



The expression for $\vec{L}$ can, therefore eq. (6.24) can be written as

$$\vec{L} = \sum_{i=1}^{n}\left[\left(\vec{r}_{cm} + \vec{\vec{r}}_i\right) \times m_i\left(\vec{v}_{cm} + \vec{\vec{v}}_i\right)\right] \quad \ldots\ldots.\ldots\ldots\ldots\ldots\ldots(6.27)$$

$$= \sum_{i=1}^{n}(\vec{r}_{cm} \times m_i\vec{v}_{cm}) + \sum_{i=1}^{n}(\vec{r}_{cm} \times m_i\vec{\vec{v}}_i) + \sum_{i=1}^{n}(\vec{\vec{r}}_i \times m_i\vec{v}_{cm}) + \sum_{i=1}^{n}(\vec{\vec{r}}_i \times m_i\vec{v}_i)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(6.28)$$

$$= \vec{r}_{cm} \times \left(\sum_{i=1}^{n} m_i\right)\vec{v}_{cm} + \vec{r}_{cm} \times \underline{\sum_{i=1}^{n} m_i\vec{\vec{v}}_i} + \underline{\left(\sum_{i=1}^{n} m_i\vec{\vec{r}}_i\right)} \times \vec{v}_{cm} + \sum_{i=1}^{n}(\vec{\vec{r}}_i + m_i\vec{v}_i)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(6.29)$$

Now, from Equation 6.24, we have

$$\vec{\vec{r}}_i = \vec{r}_i - \vec{r}_{cm} \ldots\ldots.\ldots\ldots\ldots\ldots\ldots(6.30)$$

$$\sum_{i=1}^{n} m_i\vec{\vec{r}}_i = \sum_{i=1}^{n} m_i\,(r_i - \vec{r}_{cm}) = \sum_{i=1}^{n} m_i r_i - m_i\vec{r}_{cm} = 0 \ldots\ldots.\ldots\ldots\ldots\ldots(6.31)$$

Similarly, from Equation 6.26, we have

$$\vec{\vec{v}}_i = \vec{v}_i - \vec{v}_{cm} \ldots\ldots.\ldots\ldots\ldots\ldots\ldots(6.32)$$

$$\sum_{i=1}^{n} m_i\vec{\vec{v}}_i = \sum_{i=1}^{n} m_i\,(v_i - \vec{v}_{cm}) = \sum_{i=1}^{n} m_i v_i - m_i\vec{v}_{cm} = 0 \ \ldots\ldots\ldots\ldots\ldots(6.33)$$

$$\therefore \vec{L} = \vec{r}_{cm} \times \left(\sum_{i=1}^{n} m_i\right)\vec{v}_{cm} + \sum_{i=1}^{n}(\vec{r}_i + m_i\vec{v}_i) \quad \ldots\ldots.\ldots\ldots\ldots\ldots(6.34)$$

By differentiation with respect to t, all summations in the expansion of $\vec{L}$ vanish, and we can

$$\vec{L} = \vec{r}_{cm} \times m_i\vec{v}_{cm} + \sum_{i=1}^{n} \vec{r}_i + m_i\vec{v}_i \qquad \ldots\ldots.\ldots\ldots\ldots\ldots(6.35)$$

Expressing the angular momentum of a system in terms of an "orbital" part (motion of the center of mass) and a "spin" part (motion about the center of mass).

H.W (7.2.1) A long, thin rod of length $\boldsymbol{l}$ and mass $\boldsymbol{m}$ hangs from a pivot point about which it is free to swing in a vertical plane like a simple pendulum. Calculate the total angular momentum of the rod as a function of its instantaneous angular velocity $\boldsymbol{\omega}$.

Kinetic Energy of a System

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{n} \frac{1}{2} m_i (v_i \cdot v_i) \quad \dots\dots.\dots\dots\dots (6.36)$$

Now, sub eq. 6.25, in eq. (6.35), we have

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i (\vec{v}_{cm} + \vec{\bar{v}}_i) \cdot (\vec{v}_{cm} + \vec{\bar{v}}_i) \quad \dots\dots.\dots\dots\dots (6.37)$$

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i \vec{v}_{cm}^2 + \sum_{i=1}^{n} \frac{1}{2} m_i (\vec{v}_{cm} \cdot \vec{\bar{v}}_i) + \sum_{i=1}^{n} \frac{1}{2} m_i \vec{\bar{v}}_i^2 + \sum_{i=1}^{n} \frac{1}{2} m_i (\vec{\bar{v}}_i \cdot \vec{v}_{cm})$$

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i \vec{v}_{cm}^2 + \sum_{i=1}^{n} m_i (\vec{v}_{cm} \cdot \vec{\bar{v}}_i) + \sum_{i=1}^{n} \frac{1}{2} m_i \vec{\bar{v}}_i^2 \quad \dots\dots.\dots\dots\dots (6.38)$$

$$T = \frac{1}{2} \vec{v}_{cm}^2 \left(\sum_{i=1}^{n} m_i \right) + \vec{v}_{cm} \cdot \sum_{i=1}^{n} m_i \vec{\bar{v}}_i + \sum_{i=1}^{n} \frac{1}{2} m_i \vec{\bar{v}}_i^2 \quad \dots\dots.\dots\dots\dots (6.39)$$

As before, the second summation $\sum_{i=1}^{n} m_i \vec{\bar{v}}_i$ *vanishes.*

$$\therefore T = \sum_{i=1}^{n} \frac{1}{2} m_i \vec{v}_{cm}^2 + \sum_{i=1}^{n} \frac{1}{2} m_i \vec{\bar{v}}_i^2 \quad \dots\dots..\dots.\dots\dots\dots (6.40)$$

The first term is the kinetic energy of translation of the whole system, and the second is the kinetic energy of motion relative to the mass center.

$$\therefore T = T_{cm} + T_{relative} \quad \dots\dots..\dots.\dots\dots\dots (6.41)$$

H.W (7.2.2) Calculate the total kinetic energy of the rod length $\boldsymbol{l}$ and mass $\boldsymbol{m}$ hangs from a pivot point about which it is free to swing in a vertical plane like a simple pendulum. Show that the total energy obtained for the rod according to this theorem is equivalent to that obtained by direct calculation.

6.3 Motion of Two Interacting Bodies: The Reduced Mass

Let us consider the motion of a system consisting of two bodies, treated here as particles, that interact with each other by a central force. We assume the system is isolated, and, hence, the center of mass moves with constant velocity. For simplicity, we take the center of mass as the origin. We have then

$$m_1\bar{r}_1 + m_2\bar{r}_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.42)$$

$$\bar{r}_2 = -\frac{m_1}{m_2}\bar{r}_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.43)$$

Where, as shown in Figure (4), the vectors $\vec{r}_1$ and $\vec{r}_2$ represent the positions of the particles m_2 and m_2, respectively, relative to the center of mass. Now, if **n** is the position vector of particle **1** relative to particle **2**, then

$$\vec{R} = \bar{r}_1 - \bar{r}_2 = \bar{r}_1\left(1 + \frac{m_1}{m_2}\right) \quad \ldots\ldots (6.44)$$

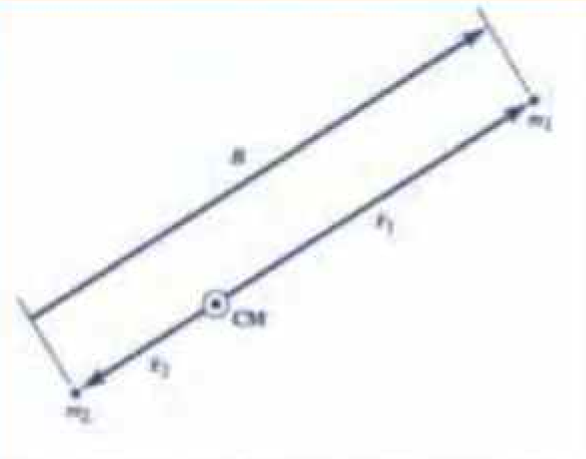


Figure (4) the relative position vector **B** for the two-body problem.

$$\vec{R} = \bar{r}_1\left(\frac{m_1 + m_2}{m_2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.45)$$

$$\bar{r}_1 = \left(\frac{m_2}{m_1 + m_2}\right)\vec{R} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.46)$$

The differential equation of motion of particle 1 relative to the center of mass is

$$m_1\frac{d^2\bar{r}_1}{dt^2} = \vec{F}_1 = f(R)\frac{\vec{R}}{R} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.47)$$

$$m_1\left(\frac{m_2}{m_1+m_2}\right)\frac{d^2\vec{R}}{dt^2}=f(R)\frac{\vec{R}}{R} \quad \ldots (6.48)$$

$$\left(\frac{m_1 m_2}{m_1+m_2}\right)\frac{d^2\vec{R}}{dt^2}=f(R)\frac{\vec{R}}{R} \quad \ldots (6.49)$$

$$\mu\frac{d^2\bar{r}_1}{dt^2}=f(R)\frac{\vec{R}}{R} \quad \ldots (6.50)$$

$$\therefore \mu=\left(\frac{m_1 m_2}{m_1+m_2}\right) \quad \ldots (6.51)$$

The quantity μ is called the reduced mass.

$$f(R)=-\frac{\mathrm{G}\, m_1 m_2}{R^2} \quad \ldots (6.52)$$

$$\therefore \mu\ddot{\vec{R}}=-\frac{\mathrm{G}\, m_1 m_2}{R^2}\hat{R} \quad \ldots (6.53)$$

6.4 Collisions

Whenever two bodies undergo a collision, the force that either exerts on the other during the contact is an internal force, if the bodies are regarded together as a single system. The total linear momentum is unchanged. We can, therefore, write

Conservation of Momentum:

$$\vec{p}_1+\vec{p}_2=\vec{p}'_1+\vec{p}'_2 \quad \ldots (6.54)$$

$$m_1\vec{v}_1+m_2\vec{v}_2=m_1\vec{v}'_1+m_2\vec{v}'_2 \quad \ldots (6.55)$$

Conservation of Energy:

$$\frac{\vec{p}_1^2}{2m_1}+\frac{\vec{p}_2^2}{2m_2}=\frac{\vec{p}'^2_1}{2m_1}+\frac{\vec{p}'^2_2}{2m_2}+Q \quad \ldots (6.56)$$

$$\frac{1}{2}m_1\vec{v}_1^2+\frac{1}{2}m_2\vec{v}_2^2=\frac{1}{2}m_1\vec{v}'^2_1+\frac{1}{2}m_2\vec{v}'^2_2+Q \quad \ldots (6.57)$$

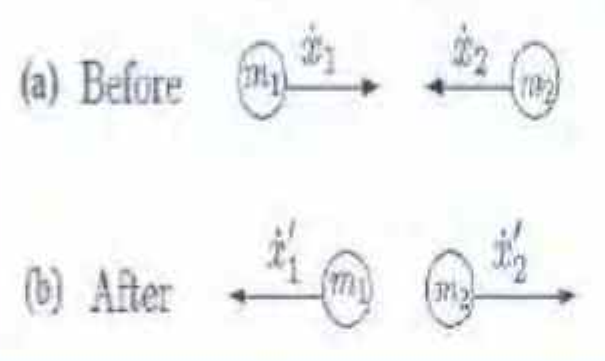
Here the quantity **Q** is introduced to indicate the net loss or gain in kinetic energy that occurs as a result of the collision.

In the case of an elastic collision, no change takes place in the total kinetic energy, so that **Q = 0**.

1. If an **energy loss** does occur, then **Q** is **positive**. This is called an **exoergic collision**.
2. If there is **gain in energy**, then **Q** is **negative**. This collision is called **endoergic**.

Direct Collisions

Let us consider the special case of a head-on collision of two bodies, or particles, in which the motion takes place entirely on a single straight line, the x-axis, as shown in Figure (5a). In this case the momentum balance equation (Equation 6.55) can be written



$$m_1\dot{x}_1 + m_2\dot{x}_2 = m_1\dot{x}'_1 + m_2\dot{x}'_2 \quad \dots\dots\dots\dots\dots\dots\dots\dots (6.58)$$

$$\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 = \frac{1}{2}m_1\dot{x}'^2_1 + \frac{1}{2}m_2\dot{x}'^2_2 + Q \quad \dots\dots\dots\dots\dots\dots (6.59)$$

It is often convenient in this kind of problem to introduce another parameter called the **coefficient of restitution** (ϵ), Instead of introducing Q.

$$\epsilon = \frac{|\dot{x}'_1 - \dot{x}'_2|}{|\dot{x}_1 - \dot{x}_2|} = \frac{v'}{v} \quad \dots\dots\dots\dots\dots\dots (6.60)$$

1. This quantity (ϵ) is defined as the ratio of the speed of separation v' to the speed of approach v.
2. The numerical value of (ϵ) depends primarily on the composition and physical makeup of the two bodies.
3. $\epsilon = 1$ for elastic collision.
4. In the case of a totally inelastic collision, the two bodies stick together after colliding, so that $\epsilon = 0$.

Impulse in Collisions

Forces of extremely short duration in time, such as those exerted by bodies undergoing collisions, are called **impulsive forces**. If we confine our attention to one body, or particle, the differential equation of motion is:-

$$\vec{F} = \frac{d(m\vec{v})}{dt} \dots\dots\dots.\dots\dots\dots\dots (6.61)$$

$$\vec{F}dt = d(m\vec{v}) \dots\dots\dots.\dots\dots\dots\dots (6.62)$$

Let us take the time integral over the interval $t = t_1$ to $t = t_2$. This is the time during which the force is considered to act. Then we have

$$\int_{t_1}^{t_2} \vec{F}dt = \Delta(m\vec{v}) \dots\dots\dots.\dots\dots\dots\dots (6.63)$$

The time integral of the force is the impulse. It is customarily denoted by the symbol $\vec{P}$. Equation 6.63 is, accordingly, expressed as

$$\vec{P} = \Delta(m\vec{v}) \dots\dots\dots.\dots\dots\dots\dots (6.63) \quad (Impulse)$$

H.W ⇒ 7.1 7.3 7.4 7.5 7.6 7.8 7.11 ⇐

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الفصل السادس

المحاضرة الثانيه

أ.د. رعد الحداد

6.5 Oblique Collisions and Scattering: Comparison of Laboratory and Center of Mass Coordinates

We now turn our attention to the most general case of collisions in which the motion is not confined to a single straight line. Where the momentum equations must be used in the form of vector algebra, i.e. the equations (6.54) and (6.55). Let us study the special case of a particle of mass m_1 with initial velocity $\vec{v}_1$ (the incident particle) that strikes a particle of mass m_2 that is initially at rest (the target particle). The momentum equations in this case are

Conservation of Momentum:

$$\vec{p}_1 = \vec{p}'_1 + \vec{p}'_2 \quad \text{(6.64)}$$

$$m_1\vec{v}_1 = m_1\vec{v}'_1 + m_2\vec{v}'_2 \quad \text{(6.65)}$$

Conservation of Energy:

$$\frac{\vec{p}_1^2}{2m_1} = \frac{\vec{p}'^2_1}{2m_1} + \frac{\vec{p}'^2_2}{2m_2} + Q \quad \text{(6.66)}$$

$$\frac{1}{2}m_1\vec{v}_1^2 = \frac{1}{2}m_1\vec{v}'^2_1 + \frac{1}{2}m_2\vec{v}'^2_2 + Q \quad \text{(6.67)}$$

Consider the particular case in which the masses of the incident and target particles are the same ($m_1 = m_2 = m$). Then the energy balance equation (eq. 6.66) can be written

$$\frac{\vec{p}_1^2}{2m} = \frac{\vec{p}'^2_1}{2m} + \frac{\vec{p}'^2_2}{2m} + Q \quad \text{(6.68)}$$

$$\vec{p}_1^2 = \vec{p}'^2_1 + \vec{p}'^2_2 + 2mQ \quad \text{(6.69)}$$

Now if we take the dot product of each side of the momentum (eq. 6.64) with itself, we get

$$\vec{p}_1 \cdot \vec{p}_1 = (\vec{p}'_1 + \vec{p}'_2) \cdot (\vec{p}'_1 + \vec{p}'_2) = \vec{p}'^2_1 + \vec{p}'^2_2 + 2\vec{p}'_1 \cdot \vec{p}'_2 \quad \text{(6.70)}$$

Comparing Equations 6.69 and 6.70, we see that

$$\vec{p}'_1 \cdot \vec{p}'_2 = mQ \quad \text{(6.71)}$$

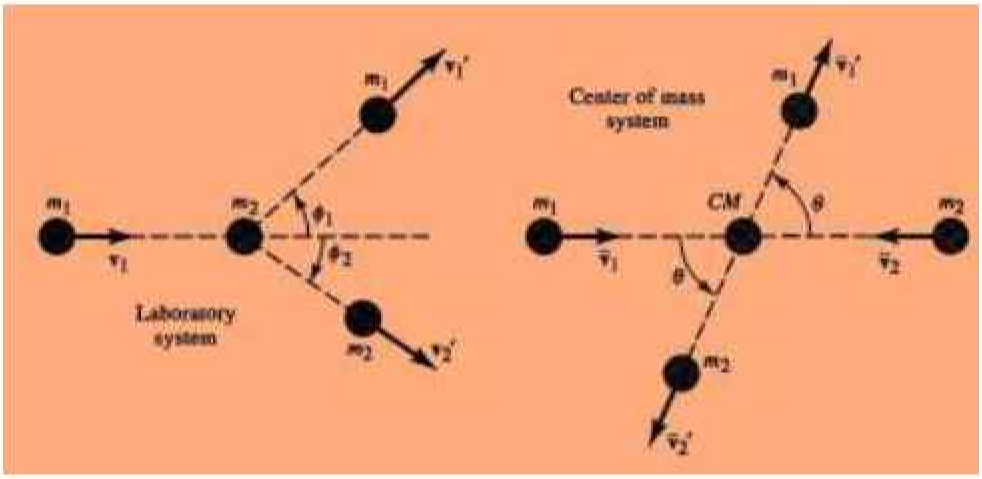
For an elastic collision ($Q = 0$) we have, therefore,

$$\vec{p}'_1 \cdot \vec{p}'_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (6.72)$$

So the two particles emerge from the collision at right angles to each other.

Center of Mass Coordinates

Figure (6) shows a plot of velocity vectors in the laboratory system and the center of mass system. Where ϕ_1 represent the angle of deflection of the incident particle after it strikes the target particle, and ϕ_2 represent the angle that the line of motion of the target particle makes with the line of motion of the incident particle. In the center of mass system, because the center of mass must lie on the line joining the two particles at all times, both particles approach the center of mass, collide, and recede from the center of mass in opposite directions. The angle θ denotes the angle deflection of the incident particle in the center of mass system as indicated.



From the definition of the center of mass, the linear momentum in the center of mass system is zero both before and after the collision. Hence, we can write

$$\bar{p}_1 \cdot \bar{p}_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (6.73)$$

$$\bar{p}'_1 \cdot \bar{p}'_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (6.74)$$

$$\frac{\bar{p}_1^2}{2m_1} + \frac{\bar{p}_2^2}{2m_2} = \frac{\bar{p}_1'^2}{2m_1} + \frac{\bar{p}_2'^2}{2m_2} + Q \ldots\ldots\ldots\ldots\ldots\ldots (6.75)$$

If $\bar{p}_1^2 = \bar{p}_2^2$ and $\bar{p}_1'^2 = \bar{p}_2'^2$

$$\frac{\bar{p}_1^2}{2m_1} + \frac{\bar{p}_1^2}{2m_2} = \frac{\bar{p}_1'^2}{2m_1} + \frac{\bar{p}_1'^2}{2m_2} + Q \ldots\ldots\ldots\ldots\ldots\ldots (6.76)$$

$$\frac{\bar{p}_1^2}{2}\left(\frac{m_1+m_2}{m_1 m_2}\right)=\frac{\bar{p}'^2_1}{2}\left(\frac{m_1+m_2}{m_1 m_2}\right)+Q.\ldots\ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots(6.77)$$

$$\frac{\bar{p}_1^2}{2\mu}=\frac{\bar{p}'^2_1}{2\mu}+Q.\ldots\ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots(6.78)$$

The momentum relations, Equations 6.73 and 6.74 expressed in terms of velocities,

$$m_1\bar{v}_1+m_2\bar{v}_2=0\quad\ldots\ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots(6.79)$$

$$m_1\bar{v}'_1+m_2\bar{v}'_2=0\quad\ldots\ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots(6.80)$$

$$\vec{r}_{cm}=\frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_1+m_2}\quad\ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots(6.81)$$

$$\vec{v}_{cm}=\frac{m_1\vec{v}_1+m_2\vec{v}_2}{m_1+m_2}\quad\ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots(6.82)$$

$\vec{v}_2=\mathbf{0}$ at the rest.

$$\therefore\ \vec{v}_{cm}=\frac{m_1\vec{v}_1}{m_1+m_2}\quad\ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots(6.83)$$

From lecture (12) we have

$$\vec{\bar{v}}_i=\vec{v}_i-\vec{v}_{cm}\ldots\ldots.\ldots\ldots\ldots\ldots...(6.32)$$

$$\vec{\bar{v}}_1=\vec{v}_1-\vec{v}_{cm}\ldots\ldots.\ldots\ldots\ldots\ldots...(6.84)$$

$$\vec{\bar{v}}_1=\vec{v}_1-\left(\frac{m_1}{m_1+m_2}\right)\vec{v}_1\quad\ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots(6.85)$$

$$\vec{\bar{v}}_1=\left(1-\frac{m_1}{m_1+m_2}\right)\vec{v}_1\quad\ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots(6.86)$$

$$\vec{\bar{v}}_1=\left(\frac{m_1+m_2-m_1}{m_1+m_2}\right)\vec{v}_1\quad\ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots(6.87)$$

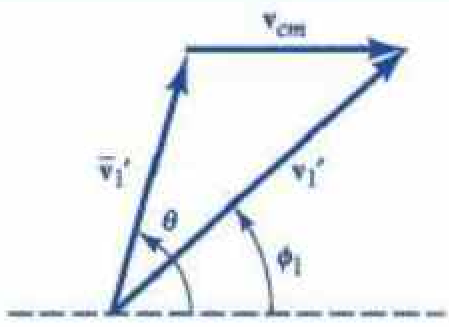
$$\vec{\bar{v}}_1=\left(\frac{m_2\vec{v}_1}{m_1+m_2}\right)\quad\ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots(6.88)$$

Velocity vectors in the laboratory system and the center of mass system.

The relationships among the velocity $\vec{v}_{cm}$, v'_1 and $\vec{v}'_1$ are shown in Figure (7). From the figure, we see that

$$v'_1 \sin\phi_1 = \vec{v}'_1 \sin\theta \qquad \ldots\ldots\ldots(6.89)$$

$$\vec{v}'_1 \cos\phi_1 = \vec{v}'_1 \cos\theta + \vec{v}_{cm} \qquad \ldots\ldots\ldots(6.90)$$



We divided eq. (6.89) on eq. (6.90), we get

$$\tan\phi_1 = \frac{\sin\theta}{\left(\frac{\vec{v}_{cm}}{\vec{v}'_1}\right) + \cos\theta} = \frac{\sin\theta}{\gamma + \cos\theta} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.91)$$

Where γ is a numerical parameter whose value is given by

$$\gamma = \frac{\vec{v}_{cm}}{\vec{v}'_1} = \frac{m_1}{m_1 + m_2}\frac{\vec{v}_1}{\vec{v}'_1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.92)$$

Sub eq. (6.88) in eq. (6.92), we get

$$\gamma = \frac{m_1}{m_1 + m_2}\frac{\vec{v}_1}{\vec{v}'_1}\frac{m_1 + m_2}{m_2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.93)$$

$$\gamma = \frac{m_1}{m_2}\frac{\vec{v}_1}{\vec{v}'_1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.94)$$

For an Elastic collision

$$Q = 0 \quad \Rightarrow \quad \vec{p}_1 = \vec{p}'_1 \quad \Rightarrow \vec{v}_1/\vec{v}'_1 = 1, \;\; So$$

$$\tan\phi_1 = \frac{\sin\theta}{\frac{m_1}{m_2} + \cos\theta} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.95)$$

When $m_1 \neq m_2$

$$\tan\phi_2 = \frac{\sin\theta}{1 - \cos\theta} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.96)$$

$$tan\,\phi_2 = cot(\theta/2) \qquad \dots\dots\dots\dots\dots\dots\dots\dots (6.97)$$

$$\phi_2 = \left(\frac{\pi}{2} - \frac{\theta}{2}\right) \qquad \dots\dots\dots\dots\dots\dots\dots\dots (6.98)$$

Two Cases:

$$m_1 \gg m_2 \quad \Rightarrow tan\,\phi_1 \approx tan\,\theta \quad \Rightarrow \phi_1 = \theta$$

$$m_1 \gg m_2 \quad \Rightarrow tan\,\phi_1 = \frac{sin\,\theta}{1 + cos\,\theta} = \frac{2\,sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)}{2\,cos^2\left(\frac{\theta}{2}\right)} = tan\left(\frac{\theta}{2}\right) \Rightarrow \phi_1 = \frac{\theta}{2}$$

In the general case of nonelastic collisions, it is left as a problem to show that γ is expressible as

$$\gamma = \frac{m_1}{m_2}\left[1 - \frac{Q}{T}\left(1 + \frac{m_1}{m_2}\right)\right]^{-1/2} \dots\dots\dots\dots\dots\dots\dots\dots (6.99)$$

in which T is the kinetic energy of the incident particles as measured in the laboratory system.

EXAMPLE 7.6.1

In a nuclear scattering experiment a beam of 4-MeV alpha particles (helium nuclei) strikes a target consisting of helium gas, so that the incident and the target particles have equal mass. If a certain incident alpha particle is scattered through an angle of 30° in the laboratory system, find its kinetic energy and the kinetic energy of recoil of the target particle, as a fraction of the initial kinetic energy T of the incident alpha particle. (Assume that the target particle is at rest and that the collision is elastic.)

Ans/

For elastic collisions with particles of equal mass, we know from Equation 6.72 that $\phi_1 + \phi_2 = 90°$ (see Figure 6). Hence, if we take components parallel to and perpendicular to the momentum of the incident particle, the momentum balance equation (Equation 6.64) becomes

$$p_1 = p_1' \cos\phi_1 + p_2' \sin\phi_1$$
$$0 = p_1' \sin\phi_1 - p_2' \cos\phi_1$$

in which $\phi_1 = 30°$. Solving the preceding pair of equations for the primed components, we find

$$p_1' = p_1 \cos\phi_1 = p_1 \cos 30^\circ = \frac{\sqrt{3}}{2} p_1$$
$$p_2' = p_1 \sin\phi_1 = p_1 \sin 30^\circ = \tfrac{1}{2} p_1$$

Therefore, the kinetic energies after impact are

$$T_1' = \frac{p_1'^2}{2m_1} = \frac{3}{4}\frac{p_1^2}{2m_1} = \frac{3}{4}T = 3 \text{ MeV}$$
$$T_2' = \frac{p_2'^2}{2m_2} = \frac{1}{4}\frac{p_1^2}{2m_1} = \frac{1}{4}T = 1 \text{ MeV}$$

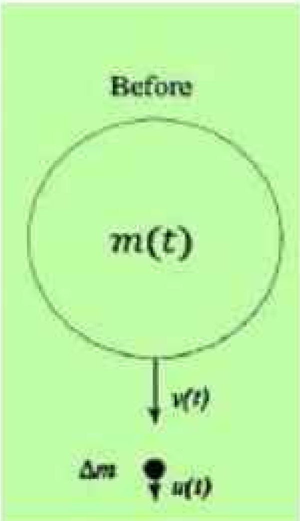
6.6 Motion of a Body with Variable Mass: Rocket Motion

Rockets propel themselves by burning fuel explosively and ejecting the resultant gasses at high exhaust velocities. Thus, they lose mass as they accelerate. In each case, mass is continually being added to or removed from the body in question, and this change in mass affects its motion. Here we derive the general differential equation that describes the motion of such objects.

- ✓ We have to be careful when establishing the differential equations of motion for the state of object whose mass changes over time.
- ✓ The equation of motion also applies to rockets, but in that case the rate of change of mass is a negative quantity.
- ✓ We now take the general state of an object whose mass changes during motion. Examine Figure (8).

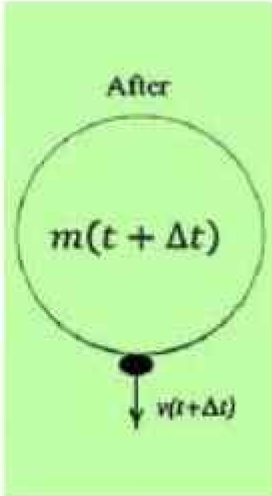
1. Before

A large mass is moving through some medium that is infested with small particles that stick to the mass as it strikes them. Thus, the larger body is continually gathering up mass as it moves through the medium. At some time t, its mass is $m(t)$ and its velocity is $v(t)$. The small particles are, in general, not at rest but are moving through the medium also with a velocity that we assume to be $u(t)$.



$$\left(\vec{p}_{total}\right)_t = m\vec{v} + \Delta m\vec{u} \quad \dots\dots\dots\dots\dots\dots\dots\dots (6.100)$$

After



At time $t + \Delta t$, the large moving object has collided with some of these smaller particles and accumulated an additional small amount of mass Δm. Thus, its mass is now $m(t + \Delta t) = m(t) + \Delta m$ and its velocity has changed to $v(t + \Delta t)$.

$$\left(\vec{p}_{total}\right)_{t+\Delta t} = (m + \Delta m)(\vec{v} + \Delta\vec{v}) \quad \ldots\ldots\ldots\ldots (6.101)$$

Suppose that $\vec{F}_{ext}$ represents the external force acting on the body at a certain time (t) and Δm represents increase in the mass of the body that occurs over a small-time interval Δt. The total linear momentum of the system is

$$\vec{F}_{ext}\Delta t = \Delta\vec{p} = \left(\vec{p}_{total}\right)_{t+\Delta t} - \left(\vec{p}_{total}\right)_{t} \quad \ldots\ldots\ldots\ldots (6.102)$$

$$\Delta\vec{p} = (m + \Delta m)(\vec{v} + \Delta\vec{v}) - (m\vec{v} + \Delta m\vec{u}) \quad \ldots\ldots\ldots\ldots (6.103)$$

$$\Delta\vec{p} = m\vec{v} + m\Delta\vec{v} + \Delta m\vec{v} + \Delta m\Delta\vec{v} - m\vec{v} - \Delta m\vec{u} \quad \ldots\ldots\ldots\ldots (6.104)$$

$$\Delta\vec{p} = m\Delta\vec{v} + \Delta m\vec{v} + \Delta m\Delta\vec{v} - \Delta m\vec{u} \quad \ldots\ldots\ldots\ldots (6.105)$$

$$\Delta\vec{p} = m\Delta\vec{v} + \Delta m\Delta\vec{v} - \Delta m\vec{u} + \Delta m\vec{v} \quad \ldots\ldots\ldots\ldots (6.106)$$

$$\Delta\vec{p} = (m + \Delta m)\Delta\vec{v} - (\vec{u} - \vec{v})\Delta m \quad \ldots\ldots\ldots\ldots (6.107)$$

We let $\vec{V} = \vec{u} - \vec{v}$ be the velocity of Δm relative to m. Hence

$$\Delta\vec{p} = (m + \Delta m)\Delta\vec{v} - \vec{V}\Delta m \quad \ldots\ldots\ldots\ldots (6.108)$$

And on dividing by Δt we obtain

$$\frac{\Delta\vec{p}}{\Delta t} = (m + \Delta m)\frac{\Delta\vec{v}}{\Delta t} - \vec{V}\frac{\Delta m}{\Delta t} \quad \ldots\ldots\ldots\ldots (6.109)$$

In the limit as $\Delta t \to 0$, we have

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} - \vec{V}\frac{dm}{dt} = \vec{F}_{ext} \quad \ldots\ldots\ldots\ldots (6.110)$$

$$\vec{F}_{ext} = \dot{\vec{p}} = m\dot{\vec{v}} - \vec{V}\dot{m} \quad \ldots\ldots\ldots\ldots (6.111)$$

- ✓ The term $\vec{V}\dot{m}$ in Equation 6.111 is called the thrust of the rocket, and its direction is opposite the direction of $\vec{V}$, the relative velocity of the exhaust products.
- ✓ The term $\Delta m \Delta \vec{v}$ is a higher order term, which is neglected when taking the limit.

Case 1: falling of body through a fog or mist: $\vec{u} = 0$ (mist is at rest), so $\vec{V} = -\vec{v}$:

$$\vec{F}_{ext} = m\dot{\vec{v}} + \vec{v}\dot{m} = \frac{d\vec{p}}{dt}(m\vec{v}) \quad \ldots\ldots\ldots\ldots\ldots\ldots (6.112)$$

Note that this equation is not general, it applies only if the initial velocity of the matter that is being swept us is zero ($\vec{u} = 0$)

Case 2: Consider the motion of a rocket in a space (no any force of gravity, air resistance, and so on) in which the external force on it is zero. Thus, in Equation 6.111, $\vec{F}_{ext} = 0$, and we have

$$m\dot{\vec{v}} = \vec{V}\dot{m} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6.113)$$

$$d\vec{v} = \vec{V}\frac{dm}{m} = -\vec{V}\left|\frac{dm}{m}\right| \ldots\ldots\ldots\ldots\ldots\ldots (6.114)$$

We can now integrate to find $\vec{v}$ as follows:

$$\int d\vec{v} = \vec{V}\int \frac{dm}{m} \ldots\ldots\ldots\ldots\ldots\ldots (6.115)$$

If we assume that $\vec{V}$ is constant, then we can integrate between limits to find the speed as function of m:

$$\int_{v_0}^{v} d\vec{v} = -\vec{V}\int_{m_0}^{m} \frac{dm}{m} \ldots\ldots\ldots\ldots\ldots\ldots (6.116)$$

$$\vec{v} - \vec{v}_0 = -\vec{V}\ln\frac{m}{m_0} \ldots\ldots\ldots\ldots\ldots\ldots (6.117)$$

$$\vec{v} - \vec{v}_0 = \vec{V}\ln\frac{m_0}{m} \ldots\ldots\ldots\ldots\ldots\ldots (6.118)$$

$$\vec{v} = \vec{v}_0 + \vec{V} \ln \frac{m_0}{m} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.119)$$

Multi-Stage Rockets

The equation of motion of the rocket with gravity acting is given by Equation 6.111

$$m \frac{d\vec{v}}{dt} - \vec{V} \frac{dm}{dt} = m\vec{g} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.120)$$

Note that $\vec{F}_{ext} = -m\vec{g}\vec{j}$ and $\vec{V} = -V\vec{j}$ (the fuels velocity is downwards). Hence

$$m \frac{d\vec{v}}{dt} + V \frac{dm}{dt} = -mg \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.121)$$

Rearranging this equation, we get

$$\frac{1}{V} \frac{d\vec{v}}{dt} + \frac{1}{m} \frac{dm}{dt} = -\frac{g}{V} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.122)$$

$$\frac{d\vec{v}}{V} + \frac{dm}{m} = -\frac{g}{V}\, dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.123)$$

$$\frac{d\vec{v}}{V} = -\frac{dm}{m} - \frac{g}{V}\, dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.124)$$

$$\tau_s = \frac{V}{g} \quad (specic\ impulse) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.125)$$

$$\therefore \frac{d\vec{v}}{V} = -\frac{dm}{m} - \frac{1}{\tau_s}\, dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.126)$$

$$\frac{d\vec{v}}{V} = -\frac{dm}{m} - \frac{g}{V}\, dt \ > 0 \ \text{(Lift Off)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.127)$$

This occurs only when

$$\frac{|dm|}{m} > \frac{g}{V}\, dt \quad \Rightarrow \left|\frac{dm}{dt}\right| > \frac{m_0}{\tau_s} \ldots\ldots\ldots\ldots (6.128)$$

$$\int_{0}^{v_f} \frac{d\vec{v}}{V} = -\int_{m_0}^{m_f} \frac{dm}{m} - \frac{1}{\tau_s}\int_{0}^{\tau_B} dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.129)$$

whereτ_B is "burn time".

$$\frac{v_f}{V} = -\ln[m]_{m_0}^{m_f} - \frac{\tau_B}{\tau_s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.130)$$

$$\frac{v_f}{V} = -\ln\left(\frac{m_f}{m_0}\right) - \frac{\tau_B}{\tau_s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.131)$$

$$\ln\left(\frac{m_f}{m_0}\right) = -\frac{v_f}{V} - \frac{\tau_B}{\tau_s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.132)$$

$$\ln\left(\frac{m_f}{m_0}\right) = -\left(\frac{v_f}{V} + \frac{\tau_B}{\tau_s}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.133)$$

$$\left(\frac{m_f}{m_0}\right) = e^{-\left(\frac{v_f}{V}+\frac{\tau_B}{\tau_s}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.134)$$

$$\frac{m_f}{m_0} = \frac{m_{Rocket} + m_{Payload}}{m_{Rocket} + m_{Payload} + m_{Ful}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.135)$$

$$\frac{m_f}{m_0} = \frac{m_R + m_P}{m_R + m_P + m_F} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.136)$$

Sub eq. (6.137) in eq. (6.131), we get

$$\frac{v_f}{V} = -\ln\left[\frac{m_R + m_P}{m_R + m_P + m_F}\right] - \frac{\tau_B}{\tau_s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.137)$$

$$\therefore \frac{v_f}{V} = \ln\left[\frac{m_R + m_P + m_F}{m_R + m_P}\right] - \frac{\tau_B}{\tau_s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.138)$$

$$\frac{v_f}{V} + \frac{\tau_B}{\tau_s} = \ln\left[\frac{m_R + m_P + m_F}{m_R + m_P}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.139)$$

$$\therefore \left[\frac{m_R + m_P + m_F}{m_R + m_P}\right] = e^{\left(\frac{v_f}{V}+\frac{\tau_B}{\tau_s}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.140)$$

$\mathbf{H.W \Rightarrow 7.22 \quad 7.23 \quad 7.24 \quad 7.25 \quad \Leftarrow}$

$\mathbf{H.W \Rightarrow 7.26 \quad 7.27 \quad 7.28 \quad \Leftarrow}$