جامعة بغداد / كلية العلوم

قسم الرياضيات

المادة: نظرية البيان

المرحلة الثانية / الكورس الثاني

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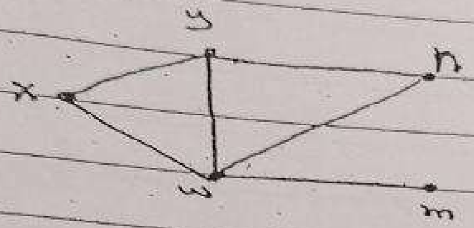
Definition: A graph is a non empty finite set of vertices and a finite set of edges we denote a graph in the form $G = (V, E)$ where V is the vertices and E is the edges.

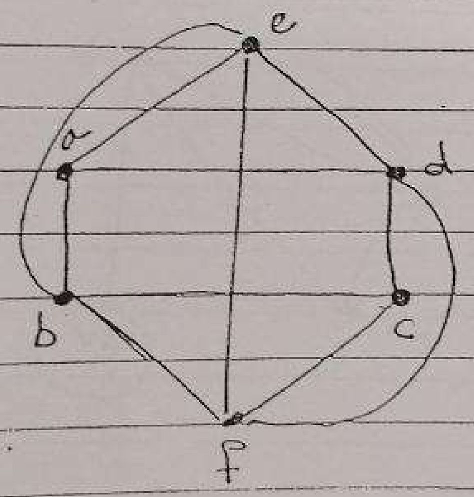
Example:



$V(G) = \{x, y, w, m, n\}$

$E(G) = \{ [x,y], [x,w], [w,m], [w,n], [y,w]$
$[y,n] \}$

Example:



$V(G) = \{a, b, c, d, e, f\}$

$E(G) = \{ [a,e], [a,b], [b,c], [b,f], [c,d], [c,f]$
$[b,e], [d,f], [a,d], [b,c], [e,f] \}$

Definitions:

Let $G=(V,E)$ be a graph.

1. The order of G is the number of vertices denoted by $|V(G)|$
2. The size of G is the number of edges denoted by $|E(G)|$

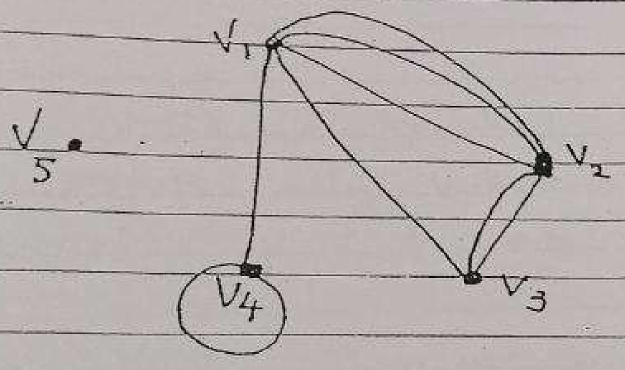
Example:

Let $V(G)=\{v_1, v_2, v_3, v_4, v_5\}$ and

$E(G)=\{[v_1,v_2], [v_1,v_3], [v_1,v_2], [v_1,v_4], [v_2,v_3], [v_2,v_1], [v_4,v_4], \{v_3,v_2\}\}$

$|V(G)| = 5$

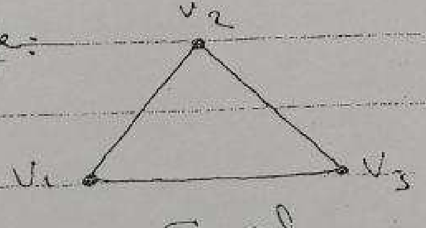
$|E(G)| = 8$



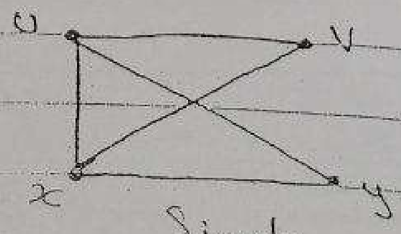
Definition:

A graph G is said to be simple if it has distincit edges and distincit vertices

Example:



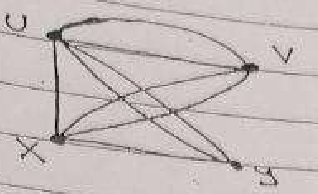
Simple



Simple

2. A graph G is said to be multiple if ∃ more than one edge of vertices

Example:



multiple graph

3. A graph G is said to be finite if ∃ a finite edges otherwise G is said to be infinite

Example:

Let $G = (V, E)$ such that

$V = \{ V_1, V_2, V_3, V_4, \ldots \}$

$E = \{ [V_1, V_2], [V_1, V_3], [V_2, V_3], [V_2, V_3] \}$.

G is finite.

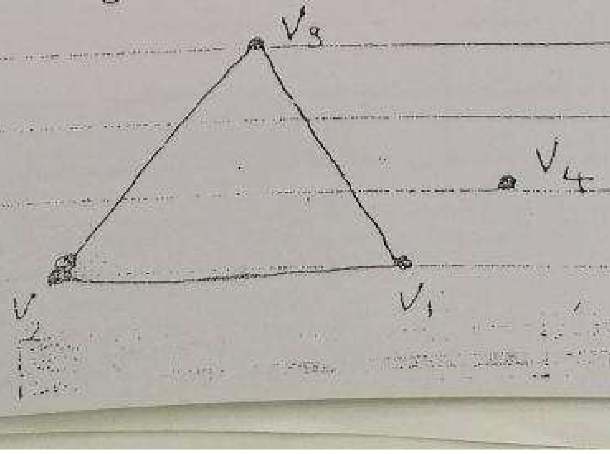
Definition:

Let $G = (V, E)$ be a graph.

1. We denote the edge between any two vertices for exactly, V_1, V_2 by $e = [V_1, V_2]$

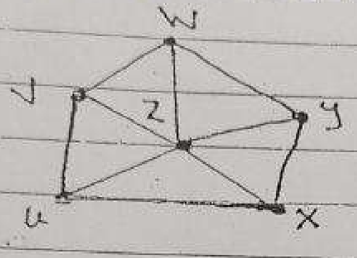
2. If $\exists e = [V_1, V_1]$, then we say that e is loop

3. Let V be a vertices if there is no any edge with, we say V is isolated (A vertex of degree zero)

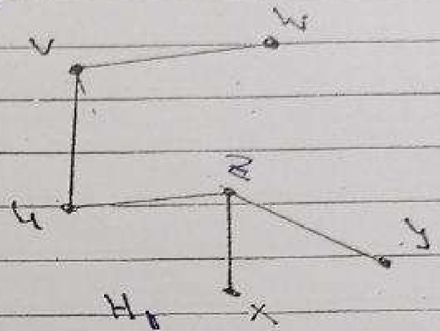


Definition: subgraph تعريف البيان الجزئي

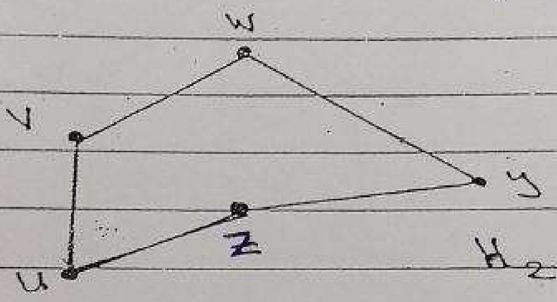
Let G be a graph, a graph H is called a subgraph of G, denoted by ($H \subseteq G$), if the vertices of H are vertices of G and the edges of H ~~has~~ has edges of G. (i.e a graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$)



G



H_1



H_2

Definition: spanning subgraph تعريف البيان الموسع

Let $G = (V, E)$, be a graph. A subgraph H of G is spanning if $V(H) = V(G)$ and $E(H) \subseteq E(G)$

~~denoted by $G = \langle H \rangle$~~ subgraph

Example:

Let $H = H_1 \cup H_2$

$G = \langle H \rangle$, $G = \langle H_1 \rangle$?, $G = \langle H_2 \rangle$?

✓ ✗

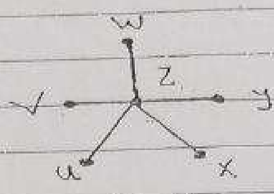
Example:

$E = \{ e_{uv}, e_{vw}, e_{xy}, e_{yw}, e_{ux} \}$

$S = [w, x]$

then, ~~G-~~ F : →

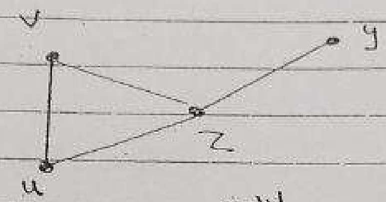
البيان الجزئي F من G



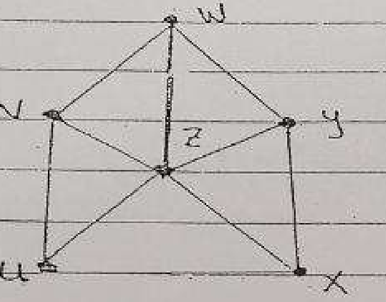
G - F

~~G-~~ S : →

البيان الجزئي S من G



G : →

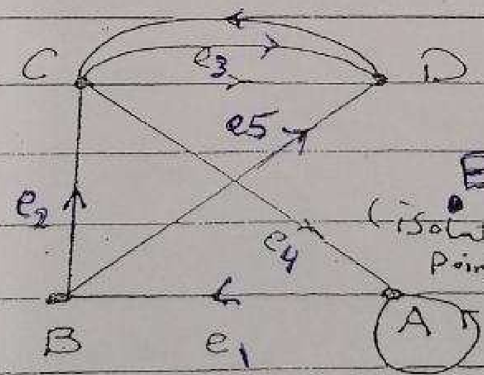


Type of Graph:

Graph is of two types:

1. Directed graph
2. ~~U~~ Indirected graph

Example :

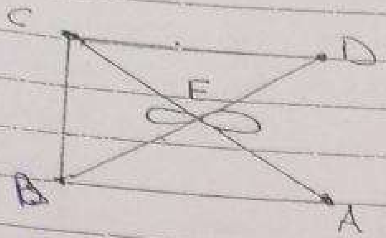


Directed graph

$\deg(A) = 4$

(لان اللوب يمثل بطريقتين)

Example.



(indirected graph)

$deg(E) = 8$

$deg(C) = 3$

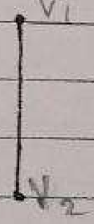
$deg(A) = 2$

Adjacent vertex الرؤوس المتجاورة

Definition: Let G be a graph with $G = (V, E)$

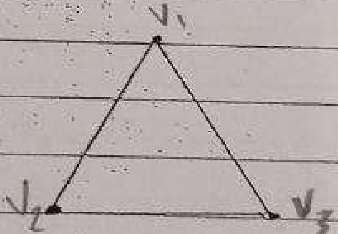
① Two vertices u and v are said to be "adjacent" if there exists an edge between them
ضلع

② Regular graph: A simple graph is said to be "regular" if all vertices of G are of equal degree



one regular graph

(a)



2-regular graph

(b)

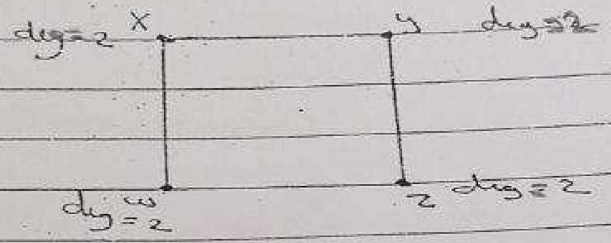
③. Degree of vertex:

The degree of vertex V of a graph G is the number of edges of G, which are incident with the V, i.e.:

$\deg(V)$ = number of edges incident with vertices v

عدد الأضلاع المشتركة مع الرأس v هو عدد الأضلاع للرأس v

Remark: If e is loop, then we have (2) in degree



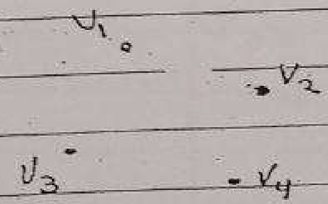
2-regular graph

Definition:

Let $G = (V, E)$ be a graph

1. We say that G is null graph if: (empty graph) (رسم خالي)

$E(G) = \{\phi\}$



2. We say that G is "trivial graph" if

$V(G) = \{\text{one vertex}\}$ and $E(G) = \{\phi\}$

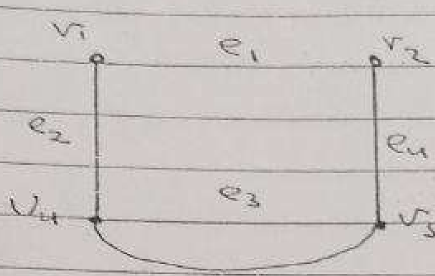
G:

(رأس واحد)

Pseudo graph (9)

Pseudo graph:

We say that G is pseudo graph if G has no parallel edges.



non-pseudograph

Hand Shaking lemma:

The sum of the degree of the all vertices of graph G is even

i.e/ $\left(\sum_{v \in G} \deg(v) = \text{even} \right)$

ليكن $G(V,E)$ بيان بحيث
$|V| = p$ and $|E| = q$
$\sum_{i=1}^{p} \rho(v_i) = 2q$ فإن

Proof: Suppose that the G is a graph with

$V(G) = \{v_1, v_2, \ldots, v_n\}$

Let $d_j = \deg_G(v_j)$

ملخص البرهان: كل ضلع يحسب مرتين
عمراً بين مختلفين او متساويين فهو
يدرك بدرجتين في مجموع درجات
رؤوس البيان G، نستنتج ان مجموع درجات
الرؤوس = ضعف عدد الأضلاع

Now, we prove that $\sum_{v_j \in G} d_j = 2|E|$

where $|E|$ is the number of edges

Consider the set,

$S = \{(x, e) : x \in V(G),\ e \in E(G),\ x \text{ incident with } e\}$

choose a vertex $v_1 \in V$.

This can be done in n ways.

Since $\boxed{d_i = \deg(v_i)}$ there are d edges

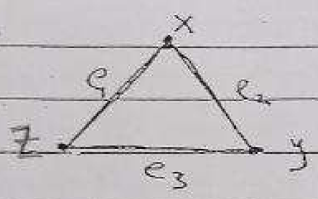
incident with this vertices

$$|S| = \sum_{i=1}^{v} d_i \quad \text{(1)}$$

Example.

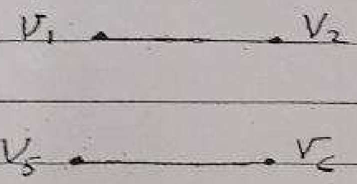
$E(u) = \{[v_1, v_2], [v_5, v_6]\}$

$|E| = 2$



$S = \{(x, e_1), (x, e_2), (y, e_3), (y, e_2), (z, e_1), (z, e_3)\}$

Example:

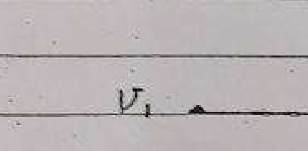


deg = 1

Now choose an edge e in

$E(G)$ this can be done in $|E|$ ways

this edge has two end vertices



$(v_1, e), (v_2, e)$ has two element in S

Class

$\therefore \quad |S| = 2|E| \quad \ldots (2)$

From (1) and (2) we get.

$$\sum_{i=1}^{n} d_j = 2|E|$$

Corollary: If $K = \delta(G)$ is the minimum degree of all the vertices of a graph G.

Then

$$K|V| \leq \sum_{V \in G} \deg(V) = 2|E|$$

u is the set of vertices of even degree,

inparticular, if G is a K-regular graph

then

$$K|V| = \sum_{V \in G} \deg(V) = 2|E|$$

Corollary②: Any graph has even number of odd vertices.

أي أن
(عدد الرؤوس الفردية الدرجة يكون عدد زوجيا)

Proof:
Let W be the set of vertices of odd degree and U be the set of vertices of even degree, then

لتكن W مجموعة الرؤوس ذات درجة فردية و U مجموعة رؤوس ذات الدرجة الزوجية فإن
حسب المبرهنة السابقة

$$\sum_{V\in G} \deg(V) = 2|E|$$

$\deg v_1 + \deg v_2 = 2|E|$

$\sum \deg(V) = 2|E|$

$$\sum_{V\in W} \deg(V) + \underbrace{\sum_{V\in U} \deg(V)}_{\text{even}} = 2.|E|$$

نجزئها الى فردية وزوجية

$$\sum_{V\in W} + \underbrace{\sum_{V\in U}}_{\text{even}} = 2|E|$$

هنا

$$\Rightarrow \sum_{V\in W} \deg(V) \text{ is even}$$

$\sum_{V\in W}$ يكون زوجي

Example: Suppose G is a ~~non~~ in directied graph with 12 degree, if G has 6 vertices of degree 3 and the rest have less than 3, then determinet the minimum number of vertices G can have.

عدد الحافات

$6 = $ عدد

Sol: From first Theorem of graph Theory

$$\sum_{i=1}^{n} \deg(V_i) = 2|E|$$

$$= 2 * 12 = 24$$

where n = The number of vertices in graph G

We have 6 vertices with degree 3 and remaining $n-6$ vertices of degree less than 3 [they are of degree 1 or 2]

$$\sum_{i=1}^{6} d(v_i) + \sum_{j=7}^{n} d(v_j) = 24$$

$$\underset{(3+3+3+3+3+3)}{\underset{3(6)}{18}} + \sum_{j=7}^{n} d(v_j) = 24$$

$$\sum_{j=7}^{n} d(v_j) = 24 - 18 = 6$$

$$\therefore \sum_{j=7}^{n} d(v_j) = 6$$

① If the remaining vertices are of degree 1, then

$$1(n-6) = 6 \Rightarrow \boxed{n=12}$$

② If the remaining vertices are of degree 2, then

$$2(n-6) = 6 \Rightarrow \boxed{n=9}$$

Hence the given graph can have minimum of 9 vertices and maximum of 12 vertices.

Example:
Let G be a simple graph all of whose vertices have degree 3 and $|E| = 2|V| - 3$. What can be said about G?

Sol)

$$\sum_{i=1}^{n} d(V_i) = 2|E| \quad ; \quad n = |V| \quad -(1)$$

عدد الرؤوس

$$= 2\left(2\,|V| - 3\right)$$

$$= 2(2n - 3)$$

$$\Rightarrow \sum_{i=1}^{n} d(v_i) = 4n - 6$$

مجموع

$$\sum_{i=1}^{6} 3 = 3 + 3 + \dots + 3$$

$$\boxed{* \quad \sum_{i=1}^{n} 3 = 3n \quad ; \quad \text{مجموع الدرجات}}$$

$$3n = 2(2n - 3)$$

$$3n = 4n - 6$$

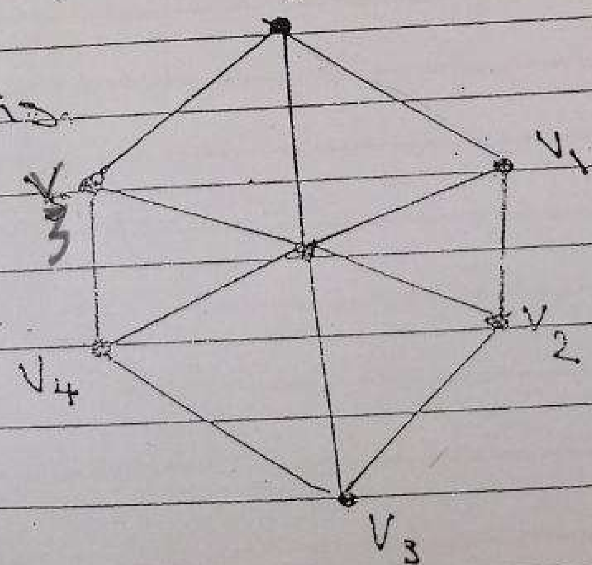
$$\therefore 4n - 6 = 3n \Rightarrow \boxed{n = 6} \rightarrow$$ عدد الرؤوس (no. of vertices)

$$2|V| - 3 = 2n - 3$$

$$|E| = 2 \times 6 - 3 = 9 \rightarrow$$ عدد الاضلاع edges

$\therefore$ G has 6 vertices and 9 edges

The graph is given by:



(15)

Example:

Construct a graph on 12 vertices with 2 of them having degree 1, three having degree 3, and the remaining having degree 10

Sol)

$$\sum_{i=1}^{12} d(v_i) = 1+1+3+3+3+10+10+10+10+10+10+10$$

$\sum d(v_i) = 2|E|$

$$= 81 \text{, which is odd}$$

So we can not have a graph with the given vertices.

Example:

Find the number of vertices in a graph G, if has 21 edges, 3 vertices of degree 4, and the other vertices of degree 3

Sol) (H.W)

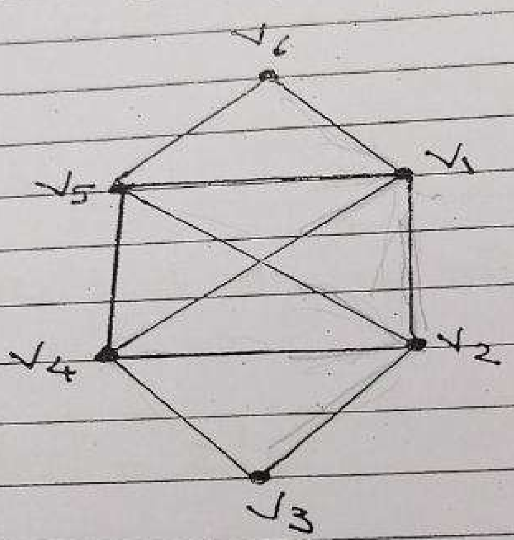
Definition:

If $v_1, v_2, \ldots, v_n$ are the vertices of G, then the sequence $\{d_1, d_2, \ldots, d_n\}$, where $d_i = d(v_i)$ is called the degree sequence of G.

In general.

$$\underbrace{\delta}_{\text{The minimum degree}}(G) = d_1 \le d_2 \le \ldots \le d_n = \underbrace{\Delta}_{\text{The maximum degree}}(G)$$

Example:



$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

$E = \{8\}$

$d_1 = d(v_1) = 4$ $\qquad$ $d_2 = d(v_2) = 4$

$d_3 = d(v_3) = 2$ $\qquad$ $d_4 = d(v_4) = 4$

$d_5 = d(v_5) = 4$

$d_6 = d(v_6) = 2$

The degree sequences of $\{2, 2, 4, 4, 4, 4\}$

$\delta(G) = 2$, $\Delta(G) = 4$

اصغر   اكبر

(H.W)

Ex: Is there any graph with degree sequences $\{1, 3, 3, 5, 6, 6\}$? (هل توجد رسم بيانات)

Sol> No,

From the first theorem of graph theory we have

$$\sum_{i=1}^{n} d(v_i) = 2|E|$$

Theorem:
For a non-directed graph G we have

$$\delta(G) \leq \frac{2|E|}{|V|} \leq \Delta(G)$$

Proof: Suppose G is a graph with n vertices $\{v_1, v_2, \ldots, v_n\}$ and $|E|$ edges

Form the first theorem $\sum_{i=1}^{n} d(v_i) = 2|E|$

Since $\delta(G) = \min\{d_1, d_2, \ldots, d_n\}$

$\delta(G) \leq d_1,\ \delta(G) \leq d_2, \ldots, \delta(G) \leq d_n$

(i.e.) $\delta(G) \leq d_i$ for all $1 \leq i \leq n$

and

Since $\Delta(G) = \max\{d_1, \ldots, d_n\}$

$\Delta G \geq d_1$
$\Delta G \geq d_2$

$\Rightarrow \Delta(G) \geq d_i$ for all $1 \leq i \leq n$ $\quad \Delta G \geq d_i$

$$\left.\begin{array}{l} \delta(G) \leq d_1 \\ \delta(G) \leq d_2 \\ \vdots \\ \delta(G) \leq d_n \end{array}\right\} \Rightarrow n\,\delta(G) \leq \underbrace{d_1 + d_2 + \cdots + d_n}_{\sum d(v_i)}$$

$n\,\Delta(G) \geq \sum d_i = 2|E|$

$n\,\delta(G) \leq 2|E|$, Similarly

$n\,\delta(G) \geq d_1 + d_2 + d_3 + \cdots + d_n$

$\Rightarrow \Delta(G)$

$\delta(G) \leq 2|E|$ and $2.|E| \leq n\,\Delta(G)$

$2|E|$

$\Rightarrow n\,\delta(G) \leq 2|E| \leq n\,\Delta(G)$

$\Rightarrow \delta(G) \leq \dfrac{2|E|}{n} \leq \Delta(G)$

$\therefore \delta(G) \leq \dfrac{2|E|}{|V|} \leq \Delta(G)$

Example:
Show that there is no simple graph with 12 vertices and 28 edges in the following cases:-

① The degree of each vertices there is either 3 or 4

② The degree of each vertices there is either 3 or 6

Sol/
① We assume that there is a graph with 12 vertices and 28 edges
and there is k vertices of degree 3
So there is $n-k$ vertices of degree 4
where is the number of all vertices in graph,
From the first theorem of graph theory we have.

$$\sum_{i=1}^{12} d(v_i) = 2|E|$$

[illegible]

$$\sum_{i=1}^{k} d(v_i) + \sum_{i=k+1}^{12} d(v_i) = 2 \times 28$$

$3k + 48 - 4k = 56$

$$3k + 4(12-k) = 56$$

$\therefore k = -8$ C!

Which is a contradiction

Since k is a positive integer. Hence, There is no simple graph with 12 vertices and 28 edges.

(2)

Here $3k + 6(12-k) = 56$

$3k + 72 - 6k = 56$

$-3k = -16$

$k = \frac{16}{3}$

$\Longrightarrow k = 16/3$

which cannot hold since k has to be integer.

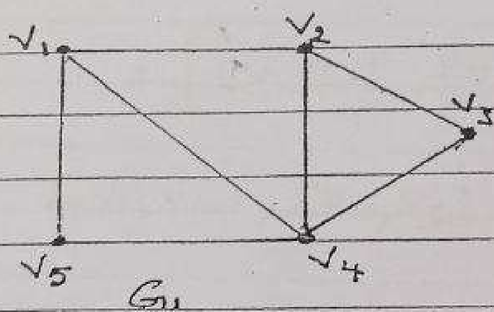
Operations on the graph

Generally their are two operation on the graph

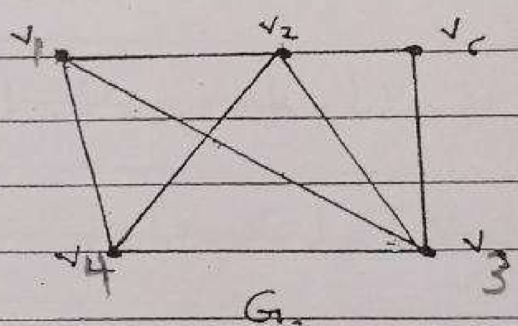
1. Intersection of graphs. تقاطع البيانات
2. Union of graphs. اتحاد البيانات

Definition 1:
Let G_1 and G_2 be two graph. The intersection of G_1 and G_2, denoted by $G_1 \cap G_2$, is the graph whose vertex set is $V(G_1) \cap V(G_2)$ and whose edge set is $E(G_1) \cap E(G_2)$

Example:



G_1



G_2

$G_1 \cap G_2$: ? (H.w)

$V(G_1 \cap G_2) = V(G_1) \cap V(G_2) = \{v_1, v_2, v_3, v_4\}$

$[v_1, v_4]$

$E(G_1 \cap G_2) = E(G_1) \cap E(G_2) = \{[v_1, v_2], [v_2, v_3], [v_3, v_4], [v_2, v_4]\}$

H.w الرسم

Definition 2 : Let G_1 and G_2 be two graphs. The union of G_1 and G_2, denoted by $G_1 \cup G_2$ is the graph whose vertex set is $V(G_1) \cup V(G_2)$ and whose edge set is $E(G_1) \cup E(G_2)$

Example:

In the same graphs in the previous example, we have.

$G_1 \cup G_2$: ? (H.w)

$$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$$

$$= \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

$$E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$$

$$= \{[v_1, v_2], [v_2, v_3], [v_3, v_4], [v_2, v_4],$$

$$[v_1, v_4], [v_4, v_5], [v_5, v_1], [v_2, v_6], [v_3, v_6]$$

$$[v_1, v_3]\}$$

H.w الرسم

Remark: Let $G_1, G_2, \ldots, G_n$ be graphs, then:

① $G = \bigcap_{i=1}^{n} G_i$: $V(G) = \bigcap_{i=1}^{n} V(G_i)$, $E(G) = \bigcap_{i=1}^{n} E(G_i)$

② $H = \bigcup_{i=1}^{n} G_i$: $V(H) = \bigcup_{i=1}^{n} V(G_i)$, $E(H) = \bigcup_{i=1}^{n} E(G_i)$

Definition:

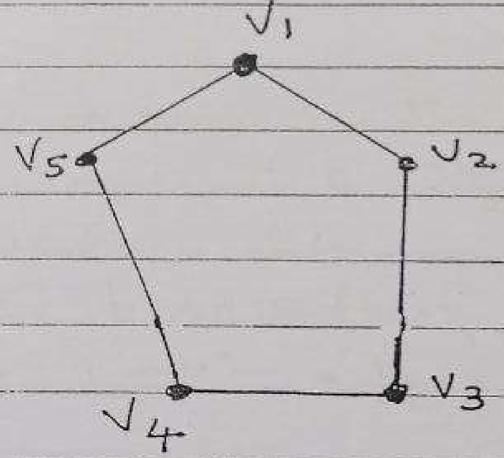
The complement G^c of a graph $G=(V,E)$ is the graph with vertices set V such that two vertices are adjacent in G^c iff they are non-adjacent in G

i.e (G^c is a graph with)

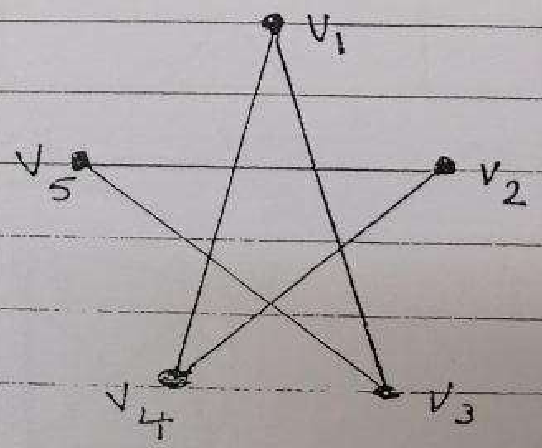
$V(G^c)=V(G)$ and $E(G^c)=\{[u,v] : [u,v] \notin E(G)\}$

Example:

G :

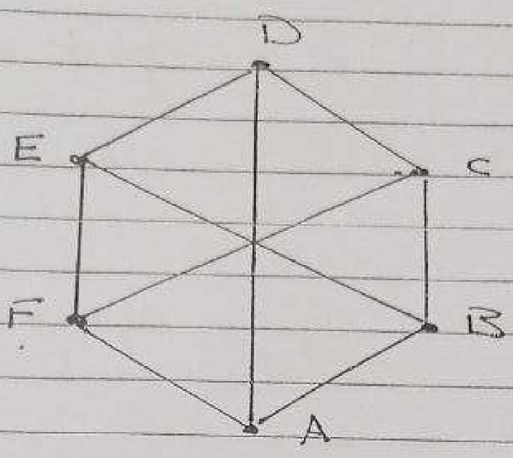


G^c :



(H.w)

G :



G^c : ?

For any simple graph G, the number of edges of G is less than or equal to $\frac{n(n-1)}{2}$ where n is the number of vertices of G.

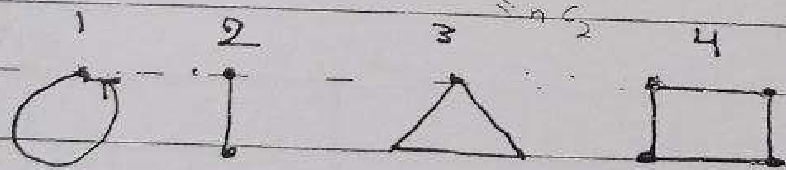
<u>Remark</u>: If G has n vertices, then G can has

$$|E| = \frac{n(n-1)}{2} \text{ edges.}$$

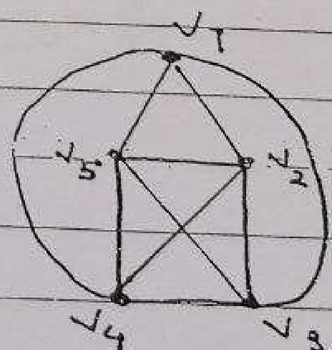
$= {}^nC_2$

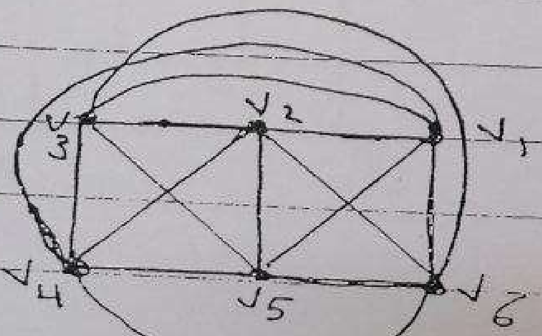
1 2 3 4

loop not edges



$n=1$ $n=2$ $n=3$ $n=4$





$= {}^{n}C_2 - m$

$$|E(G^c)| = \frac{n(n-1)}{2} - m,$$

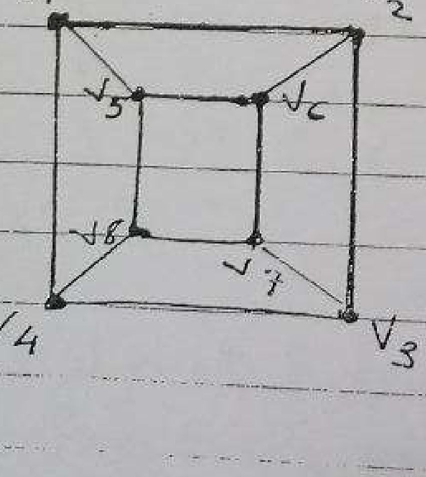
where m is the number of edges of G.

(H.W):

1. If G is a simple graph with 15 edges and G^c has 13 edges, how many vertices dose G have?

2.

2. Find the complement of the grap.

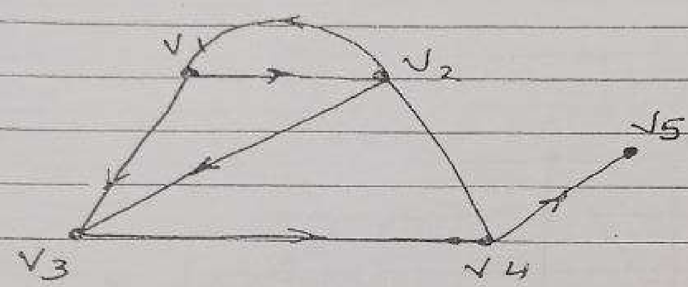


Path:

Definition: A path in the graph is a sequences $v_1, v_2, \ldots, v_k$ of vertices each edjacent of the next, and a choice of an edge between each "v_i" to "v_{i+1}" so that no edge is chosen more than once

Example:

Consider the graph G as:



Path: $v_1 v_2 v_1 v_3 v_4 v_5$

Definition:

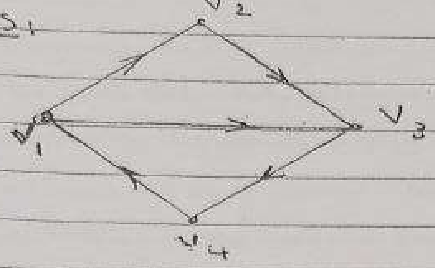
A graph $G=(V,E)$ is said to be connected if for every pair of distinict vertices "a" and "b" in G there ~~exist~~ is a path

A directed graph is said to be strougly connected if for every pair of disitincit vertices "u" and and "v" in G, there is a directed path from "u" to "v"

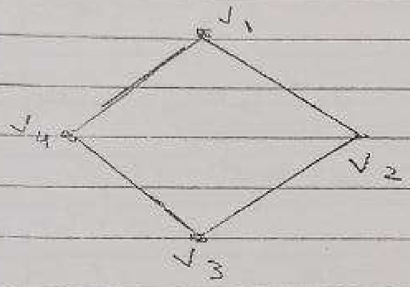
A directed graph is said to be weakly connected if for every pair of distinict vertices, there is a path without ~~taking~~ taking the direction

Examples:

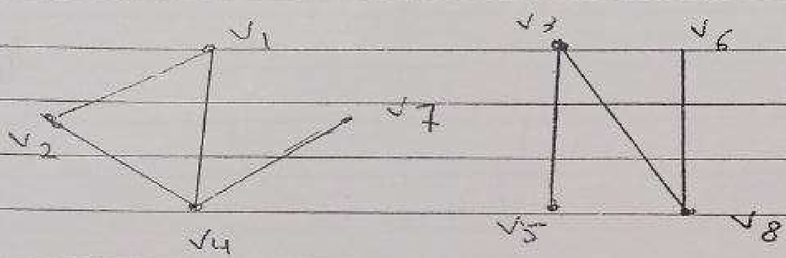
G_1:



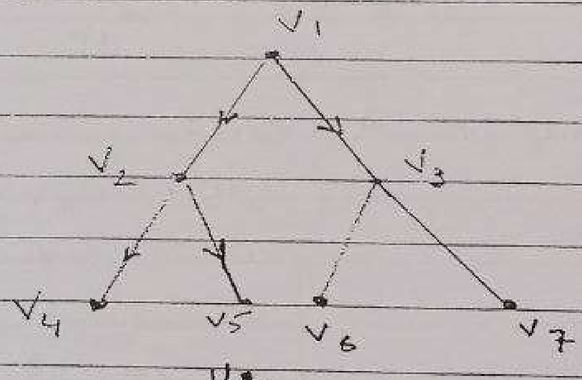
G_2:



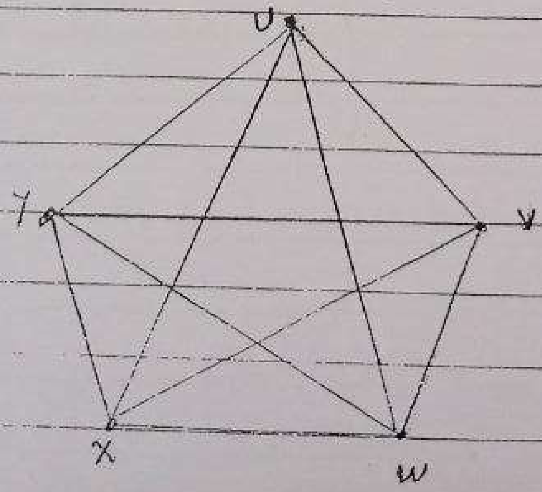
G_3:



G_4:



G_5



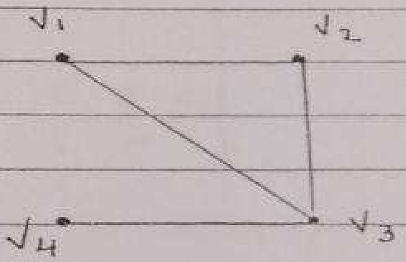
Definition:

Let $G=(V,E)$ be a graph with $V=\{v_1,v_2,\dots,v_n\}$. The adjacency matrix is given by:

$$A=\begin{array}{c|c} & \begin{array}{cccc} v_1 & v_2 & & v_n \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ \\ v_n \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \end{array}$$

$$a_{ij}=\begin{cases} 1 & \text{if there is an edge from } v_i \text{ to } v_j \\ 0 & \text{o.w} \end{cases}$$

Example:



The adjacency matrix A is given by:

$$A=\begin{array}{c|c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Theorem:

Let G be a graph with n-vertices $v_1, v_2, \dots, v_n$ and A be the adjcency matrix of G. Let $G = [b_{ij}]$ such that

$$B = A + A^2 + \dots + A^{n-1}$$

If $b_{ij} \neq 0$ for all $i \neq j$, then G is connected

Example:

If G is the graph of perivious example, show that G is connected

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Sol:

$$B = A + A^2 + A^3$$

$$A^2 = A \cdot A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

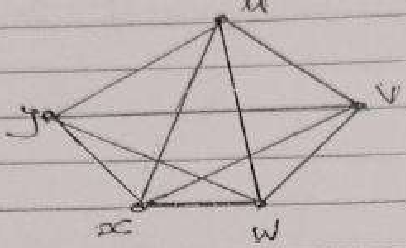
$$A^3 = A \cdot A \cdot A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \\ 4 & 4 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 5 & 6 & 2 \\ 5 & 4 & 6 & 2 \\ 6 & 6 & 5 & 4 \\ 2 & 2 & 4 & 1 \end{bmatrix}$$

Since $b_{ij} \neq 0$ for all $i \neq j$, then G is connected.

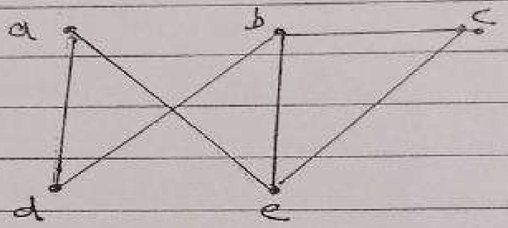
(H.W)

1. Let G_1 be a graph.



Find B = ? Is G_1 connected ?

2. If G_2:



Find B ? Is G connected.

المرئية ابيان 5

Isomorphism of graphs

Definition:

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graph. A function $f: G \to H$ is called isomorphism if:

(1) f is one to one.
(2) f is onto
(3) For each pair of vertices x and y of G
$\{x, y\} \in E(G)$ iff $\{f(x), f(y)\} \in E(H)$

$$x \bullet\!\!-\!\!\!-\!\!\bullet y \equiv f(x) \bullet\!\!-\!\!\!-\!\!\bullet f(y)$$

i.e, two vertices x and y are adjacent in G if and only if $f(x)$ and $f(y)$ are adjacent in H.

Note: (1) If the graph G is isomorphic to H then we write $G \simeq H$

(2) If two graphs are isomorphic then they must have:
1. the same number of vertices.
2. the same number of edges.
3. An equal number of vertices with a given degree.

(3) These conditions are necessary but not sufficient. i.e,
two graphs for which these conditions hold need not be isomorphic.

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Result: The isomorphism of simple graphs is an equivalent relation

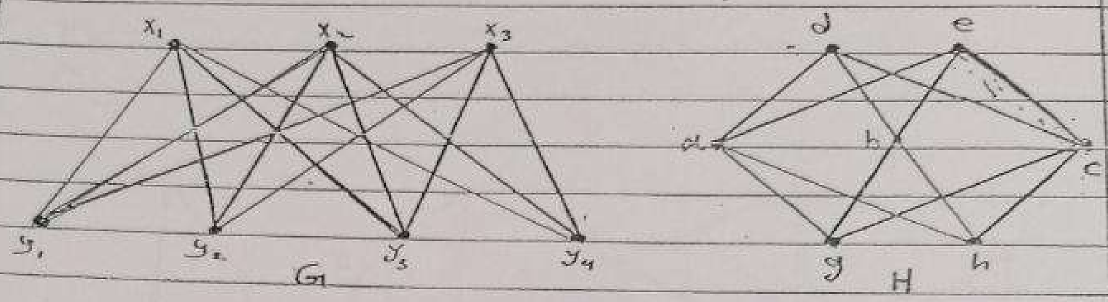
Proof,

1. A graph G is isomorphic to itself by the identity function
   i.e $I: G \to G$
   So isomorphism is reflexive.

2. Suppose that the graph G_1 is isomorphic to G_2
   Then there exists a function
   $f: G_1 \to G_2$
   which is one to one and onto and that preserves) adjacency
   Hence f is bijection.
   So there exists $f^{-1}: G_2 \to G_1$ which is one to one and onto and preserves adjacency and non adjacency.
   So G_2 is isomorphic to G_1
   Hence isomorphism is symmetric.

3. Suppose $G_1 \simeq G_2$ and $G_2 \simeq G_3$
   Then there are one to one correspondences f and g from G_1 to G_2 and from G_2 to G_3 that preserve adjacency and non adjacency
   It follows that $g \circ f : G_1 \to G_3$ is one to one, onto that preserve adjacency and non adjacency.
   $\therefore$ G_1 is isomorphic to G_3
   Hence isomorphism is transitive.
   Thus isomorphism is an equivalence relation

Example: Consider the following two graphs



G is a graph have 7 vertices and 12 edges and

{3,3,3,4,4,4} Given their degree sequence.

Defin a map

$$f : G \longrightarrow H \text{ by}$$

$f(x_1) = a$, $f(x_2) = b$, $f(x_3) = c$

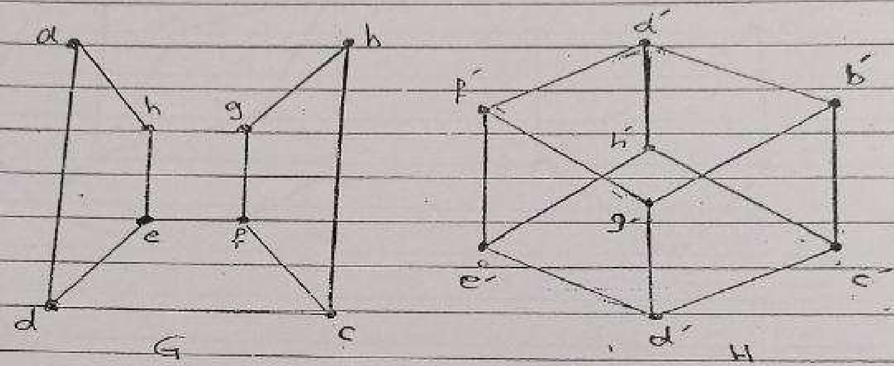
$f(y_1) = d$, $f(y_2) = e$, $f(y_3) = g$, $f(y_4) = h$

1. f is one to one
2. f is onto
3. if $\{x, y\} \in E(G)$, then $\{f(x), f(y)\} \in E(G)$

$\therefore$ f is an isomorphism.

Questions:

1. Determine whether the graphs shown in the figure isomorphic or not.



Solution:

Both the graphs have same number of edges and same number of vertices.

Define a map $f : G \to H$ by

$f(a) = a'$, $f(d) = d'$
$f(b) = b'$ $\quad f(e) = e'$
$f(c) = c'$ $\quad f(f) = f'$
$f(g) = g'$ $\quad f(h) = h'$

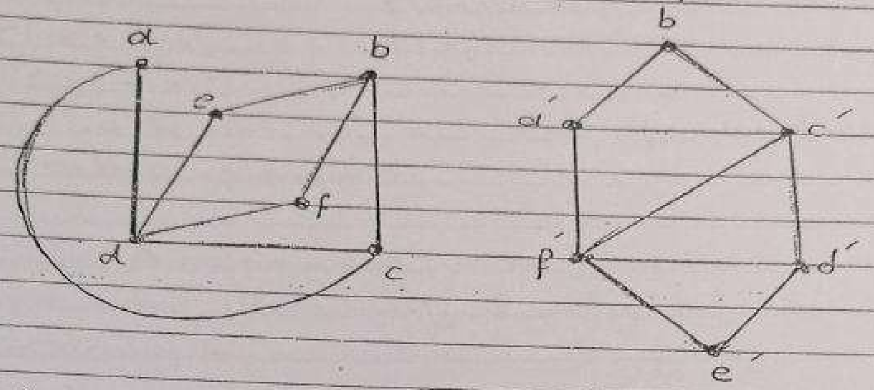
clearly f is one-to-one onto.

Also

if $\{x, y\} \in E(G) \Rightarrow \{f(x), f(y)\} \in E(H)$

Therefore the given two graphs are isomorphic

2. Determine whether the graphs shown in the figure are isomorphic or not.

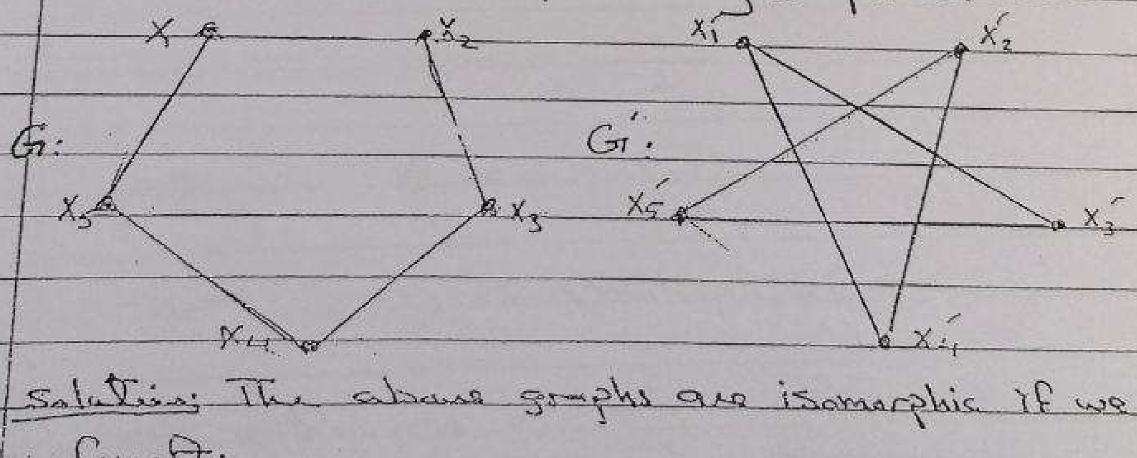


Solution:

Both the graphs have same number of vertices and edges.

Definition: A graph G is called self-complementary if it is isomorphic to its complement.

Example: A self-complementary graph is:



Solution: The above graphs are isomorphic if we define a function

$f : G \to G'$ by

$f(x_1) = x_1'$ $\quad$ $f(x_4) = x_4'$
$f(x_2) = x_2'$ $\quad$ $f(x_5) = x_5'$
$f(x_3) = x_3'$

f is one to one, onto and $\forall \{x_i, x_j\} \in E(G)$, $\{f(x_i), f(x_j)\} \in E(\bar{G})$

$\therefore$ Self complementary

Theorem: A simple graph G on n-vertices is self complementary if and only if either n or $n-1$ divisible by 4.

$\Rightarrow$)

Proof: Suppose that G is a graph on n-vertices. Then

the number of edges in $G = m$

Since G is self complementary, we have

$|E(G)| = |E(\bar{G})| = m$

Therefor $|E(G)| + |E(\bar{G})| = \dfrac{n(n-1)}{2}$

$\Rightarrow \quad m + m = \dfrac{n(n-1)}{2}$

$\Rightarrow \quad 2m = \dfrac{n(n-1)}{2}$

$\Rightarrow \quad 4m = n(n-1)$

$\Rightarrow$ $n(n-1)$ is a multiple of 4

$\Rightarrow$ either n or $n-1$ is divisible by 4

$\Leftarrow$) $4 \mid n(n-1)$

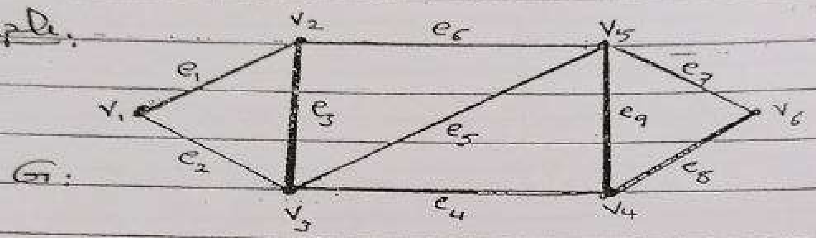
Walk:

Definition: In a graph G, a walk is a sequence of vertix and edges in a such away one vertex to next via an edge.

Walk is starting and ending with vertices, each edges in the sequence is incident between two vertices.

A walk is $V_1 e_1 V_2 e_2 V_3 e_3 \dots V_n$

Example:



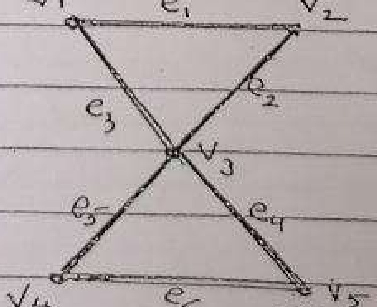
Walk sequence: $V_1 e_1 V_2 e_3 V_3 e_5 V_5 e_9 V_4 e_8 V_6$

In a walk sequence starting and ending vertex are called Terminal vertex.

and remaining are called intermediate vertex

Open and Closed walk:

If a walk begins and end with same vertex is called Closed walk.



$V_1 e_1 V_2 e_2 V_3 e_4 V_5 e_6 V_4 e_5 V_3 e_3 V_1$

other wise it is called open walk.

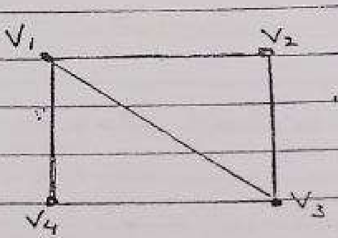
Note: In a walk vertex may be repeat

Cycle Path:

Definition: If there is a path containing one or more edges which start from a vertex 'v' and terminates into the same vertex, then a path is called cycle

Example:

G:



$v_1 v_2 v_3 v_4 v_1$

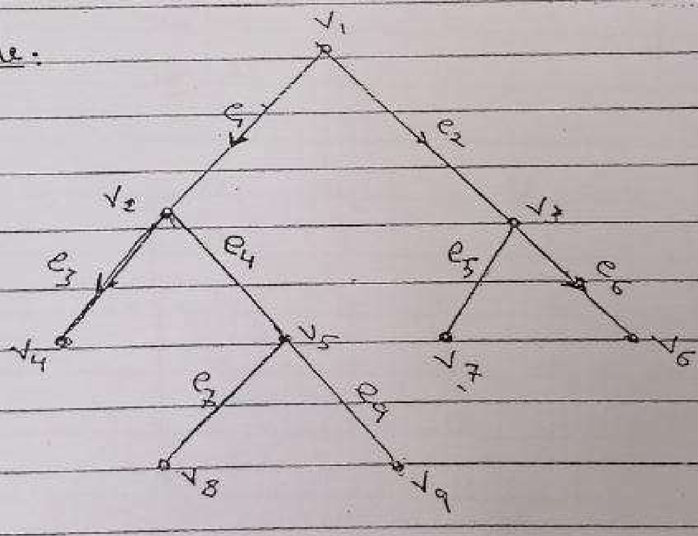
$v_1 v_3 v_2 v_1$

$v_1 v_3 v_4 v_1$

Definition: A graph which does not have any cycle is called an cycle graph

Example:

G:



G is an cycle graph

" Path Matrix "

Suppose that G is a simple graph with n - vertices, then the (n×n) matrix

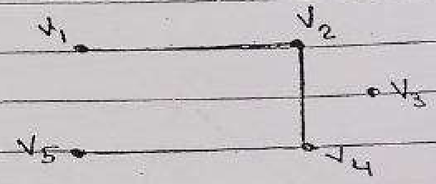
$$P = [P_{ij}]_{n\times n}$$

defined by:

$$P_{ij} = \begin{cases} 1 & \text{, if there is a path from } v_i \text{ to } v_j \\ 0 & \text{, otherwise.} \end{cases}$$

it is called a path matrix of the graph G.

<u>Example</u>: Let G:



$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$P = \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

"Incidence Matrix"

Suppose that G is a simple graph with m vertices and n edges, then the incidence matrix

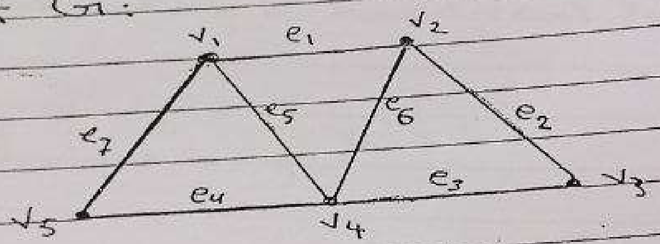
$$I = [a_{ij}]_{m \times n}$$

defined by:

$$a_{ij} = \begin{cases} 1, & \text{if the vertex } (i) \text{ belong to edge } j \\ 0, & \text{otherwise} \end{cases}$$

Example.

Let G:



$$I = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The Incidence matrix of a graph G.

Class

Product of graphs

Def:

Suppose that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs, then the product of graphs G_1 and G_2 is given by

$$G_1 \times G_2 = (V, E)$$

where $V = V_1 \times V_2$

and edges set E can be found out from the following relation:

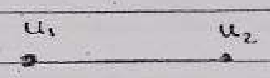
If (v_1, v_2) and (u_1, u_2) be two vertices of $G_1 \times G_2$, then there is an edge between (u_1, u_2) and (v_1, v_2) if:

متجاور

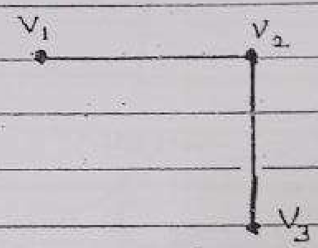
1. $u_1 = v_1$ and u_2 is adjecent to v_2 or
2. $u_2 = v_2$ and u_1 is adjecent to v_1

Example:

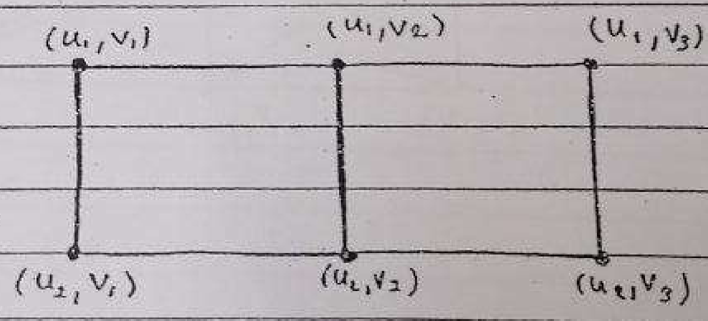
G_1:



G_2:



$G_1 \times G_2$:



Composition of two graphs

Def.

Suppose that $G_1=(V_1,E_1)$, $G_2=(V_2,E_2)$ be two graphs, then the composition of G_1 and G_2 is denoted by $G_1[G_2]$

and it's defined as $G_1[G_2]=(V,E)$

where $V=V_1 \times V_2$ and the edges set E can be found out from the following relation:

1. if (u_1,u_2) and $(v_1,v_2) \in V$, then there is an edge between them if
   u_1 is adjecent to v_1 or
   $u_1=v_1$ and u_2 is adjecent to v_1

Example: