قسم الرياضيات / مادة نظرية الفوضى / المرحلة الثالثة / الكورس الثاني

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- periodic point
- one-dimensional chaos
- Two dimensional chaos
- Baker's function
- Henon map
- Horse shop map
- Fractals
- Lyapunve dimension
- mapping by function on the Complex dimension
- Complex function "attracting-repelling
- Julia set and mandelbrot set

Iterates of functions

Def:- let $f$ be a function and let $x_0$ be in the domain of $f$, then

$f(x_0)$ is the first iterate of $x_0$ for $f$.

$f(f(x_0))$ is the second iterate of $x_0$ for $f$

$\vdots$

$f^n(x_0)$ is the $n^{th}$ iterate of $x_0$ for $f$

Def:- the orbit of $x$ under $f$ is the set of points $\{x_0, f(x_0), f^2(x_0), \ldots\}$

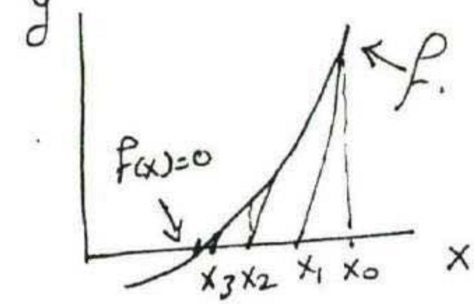
The starting point $x_0$ for the orbit is called the initial value of the orbit.

Example :- Consider the Newton Raphson method which is used for approximating a zero of a function. Iterates form the basis of this method.

Let $f$ be a function, assume that near a zero, say $z$ of $f$ the derivative of $f$ is nonzero (i.e. $f' \neq 0$).

Then for any positive integer $n$, we use $x_n$ to define $x_{n+1}$ as follows:-

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = g(x_n)$$




Clearly, the sequance $\{x_n\}_{n=0}^{\infty}$, which is form the orbit of $x_0$ under $g$, genera by newton Raphson method $\{x_0, x_1 = g(x_0), x_2 = g(x_1) = g^2(x_0), \cdots\}$ consists of the iterates of $x_0$ for $g$.

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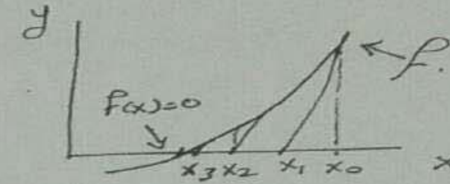



Clearly, the sequence $\{x_n\}_{n=0}^{\infty}$, which is form the orbit of $x_0$ under $g$, generated by newton Raphson method $\{x_0, x_1 = g(x_0), x_2 = g(x_1) = g^2(x_0), --- \}$ consists of the iterates of $x_0$ for $g$.

Graphical Analysis of iterates / Graphical representation of an orbit

Sketch the graph of the function $f$ together with diagonal line. Sketching the the orbit of a given initial condition $x_0$ is done as follows:

1. Locate $x_0$ on the x-axis,
2. Draw the vertical dotted line through $(x_0, 0)$ crosses the graph of $f$ at $(x_0, f(x_0))$.
3. Draw the horizontal dotted line through $(x_0, f(x_0))$ crosses the line $y = x$ at the point $(f(x_0), f(x_0)) = (x_1, x_1)$.
4. Applying the same process in 2 and 3 with $x_1$ replacing $x_0$ to obtain the point $(x_2, x_2)$.

Containing in the same manner, we determine the location of $(x_1, x_1), (x_2, x_2)$

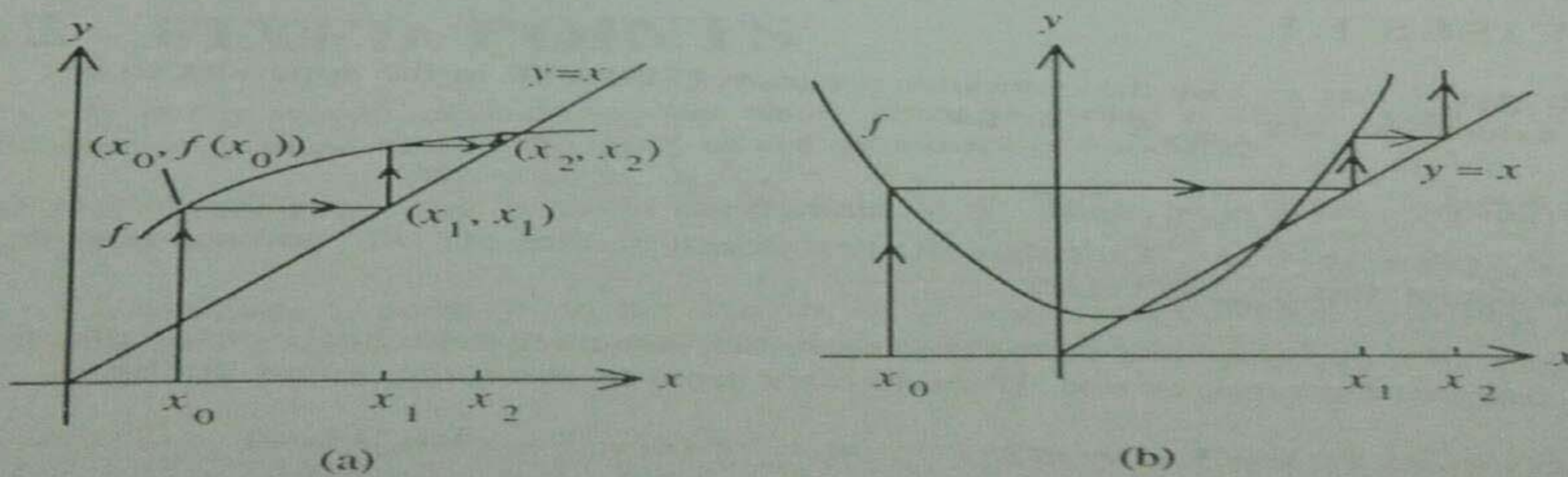



Figure 1.2

$(x_1, x_1)$. By applying the same process with $x_1$ replacing $x_0$, we obtain the point

Example: let $f(x) = 2x(1-x)$ and $x_0 = 0.1$ then by using the graphical technique we can show the following:

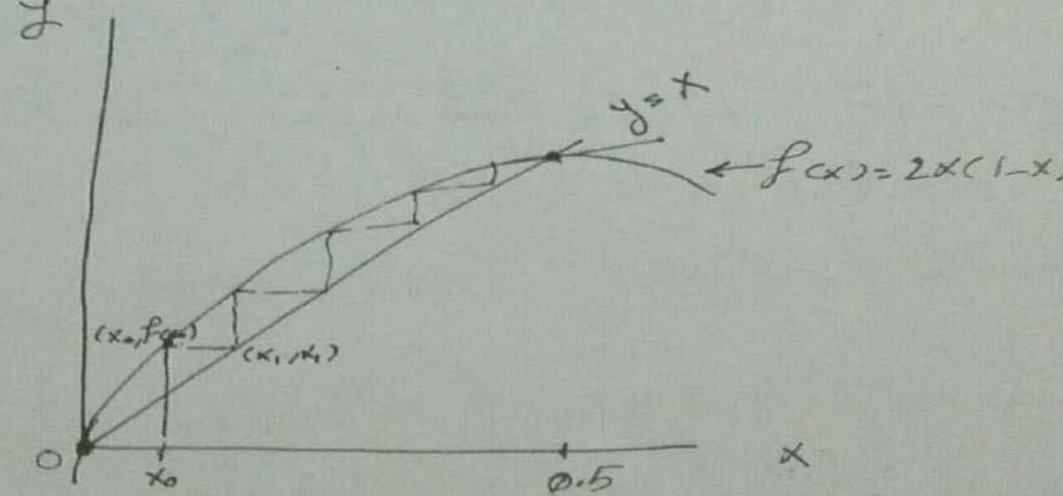
* we find the fixed points by solving the equation

$y = 2x(1-x)$

$$f(x) = x$$

$$2x(1-x) = x$$

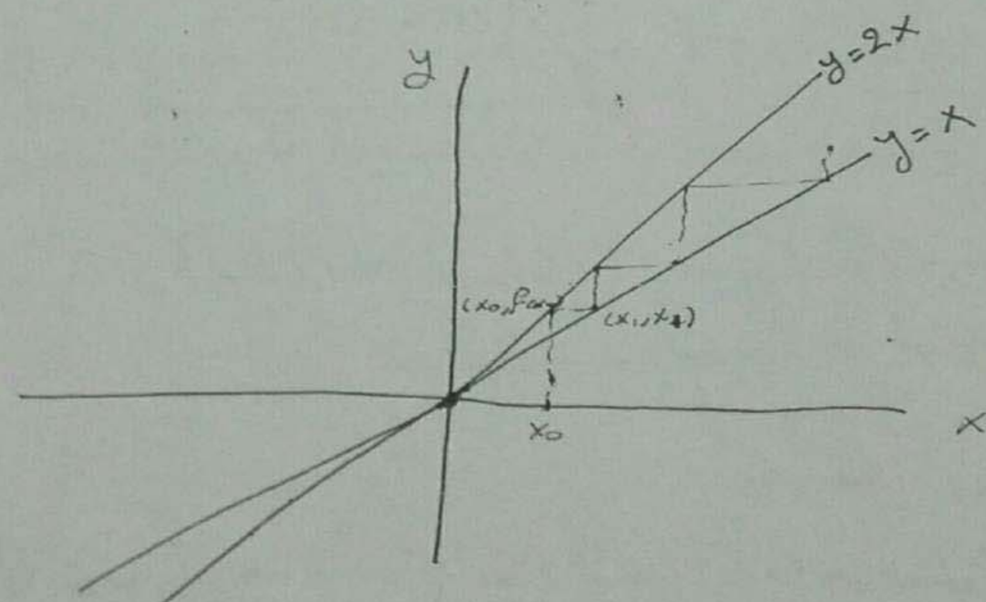
Thus there are two solutions $\underline{x = 0}$ and $\underline{x = 0.5}$ which are the two fixed points of $f$.




It is clear from the figure (Cobweb plot) that: The orbit starting at $x_0 = 0.1$ is convergent to fixed point $x = 0.5$.

Example: let $f(x) = 2x$ and $x_0 = 0.01$, then by using the graphical technique we can show that:-

* we find the fixed point $f(x) = x \Rightarrow 2x = x$ iff $x = 0$ is the only fixed point




orbit $\{x_0, f(x_0), \ldots\} = \{0.01, 0.02, 0.04, \ldots\}$

clearly the only fixed point of $f(x) = 2x$ is $x = 0$, The cobweb plot ~~illustrate~~ illustrate the orbit at $x_0 = 0.01$ under $f(x) = 2x$, go away from the fixed point of $f$.

## 1.2 FIXED POINTS

A point whose iterates are the same point is called a fixed point. Fixed points are very important in the study of the dynamics of functions.

**DEFINITION 1.2.** Let $p$ be in the domain of $f$. Then $p$ is a **fixed point** of $f$ if $f(p) = p$.

Graphically, a point $p$ in the domain of $f$ is a fixed point of $f$ if and only if the graph of $f$ touches (or crosses) the line $y = x$ at $(p, p)$ (Figure 1.4).

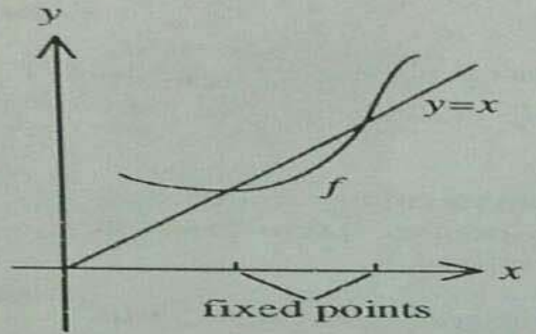


Figure 1.4

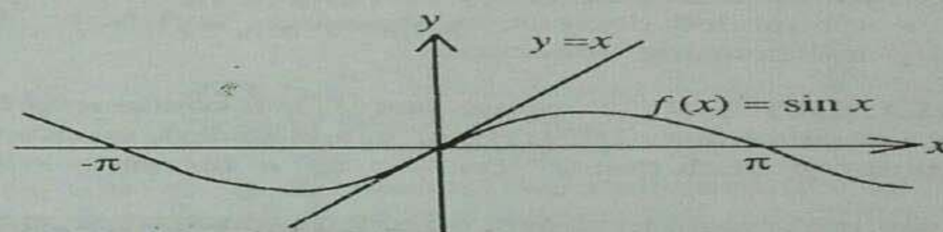


Figure 1.5

From Figure 1.5 we might conjecture that the origin is the only point at which the graph of $\sin x$ and the line $y = x$ touch each other. We will prove that this is true. In the solution we will use the Mean Value Theorem, which says that if $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there is a $c$ in $(a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \text{ or equivalently, } f(b) - f(a) = f'(c)(b - a)$$

EXAMPLE 1. Let $f(x) = \sin x$. Show that $0$ is the unique fixed point of $f$.

*Solution.* To begin, we observe that $f(x) \neq x$ if $|x| > 1$, since $|\sin x| \leq 1$ for all $x$. Next, if $0 < x \leq 1$, then the Mean Value Theorem implies the existence of $z$ between $0$ and $x$ such that

$$\sin x = \sin x - \sin 0 = f'(z)(x - 0) = x \cos z$$

Since $0 < \cos z < 1$ for such $z$, it follows that $0 < \sin x = x \cos z < x$. The fact that $f(-x) = -f(x)$ implies that $x \neq f(x)$ for all $x < 0$. Finally, $\sin 0 = 0$, so we conclude that $0$ is the unique fixed point of $f$. $\square$

The next theorem will be very important to us. For convenience we will write $f^{[n]}(x) \to p$ for "$f^{[n]}(x)$ approaches $p$" (as $n$ increases without bound).

**THEOREM 1.3.** Suppose that $f$ is continuous at $p$, and let $x$ be in the domain of $f$. If $f^{[n]}(x) \to p$ as $n$ increases without bound, then $p$ is a fixed point of $f$.

*Proof.* By hypothesis, $f^{[n]}(x) \to p$, so that $f^{[n+1]}(x) \to p$. Since $f^{[n]}(x) \to p$, the continuity of $f$ at $p$ yields $f(f^{[n]}(x)) \to f(p)$. However $f^{[n+1]}(x) = f(f^{[n]}(x))$, so that by substituting $f^{[n+1]}(x)$ for $f(f^{[n]}(x))$, we find that $f^{[n+1]}(x) \to f(p)$. The uniqueness of the limit of a given sequence implies that $f(p) = p$. Consequently $p$ is a fixed point of $f$. ∎

From calculus we know that a bounded sequence $\{x_n\}_{n=0}^{\infty}$ that is increasing converges to the least number $z$ such that $x_n \le z$ for all $n$. A similar statement holds for a bounded decreasing sequence, and hence for any monotone (that is, increasing or decreasing) sequence.

**COROLLARY 1.4.** Suppose that $f$ is a continuous function defined on a closed interval. Assume that $\{f^{[n]}(x)\}_{n=0}^{\infty}$ is a bounded, monotone sequence. Then there is a fixed point $p$ such that $f^{[n]}(x) \to p$ as $n$ increases without bound.

*Proof.* By the comment above, bounded monotone sequences always converge. Thus this result is an immediate consequence of Theorem 1.3. ∎

Now we will use Corollary 1.4 to prove that the iterates of any real $x$ for the sine function converge to $0$ — a result deduced by graphical analysis in Section 1.1.

EXAMPLE 2. Let $f(x) = \sin x$. Show that the iterates of any $x$ converge to $0$.

*Solution.* Let $x$ be an arbitrary number. To show that the sequence $\{f^{[n]}(x)\}_{n=0}^{\infty}$ is bounded, we observe that $-1 \le \sin x \le 1$ for each number $x$. Thus the sequence lies in $[-1, 1]$, and hence is bounded. Next we observe that

$$f(-x) = \sin(-x) = -\sin x = -f(x)$$

Therefore if we can show that the sequence converges to $0$ for each $x$ in $[0, 1]$, then the same happens for each $x$ in $[-1, 0]$, and hence for all $x$. Thus we only need to show that the sequence converges for each $x$ in $[0, 1]$.

Since $f(0) = 0$, let $0 < x \le 1$. We will show next that $\{f^{[n]}(x)\}_{n=0}^{\infty}$ is a decreasing sequence. As in the solution of Example 1, the Mean Value Theorem yields a $z$ between $0$ and $x$ such that

$$\sin x < x \cos z < x$$

Therefore $0 < \sin x < x < 1$ for $0 < x \leq 1$, which means that

$$f(x) < x \text{ for } 0 < x \leq 1$$

It follows that for any $n \geq 0$,

$$f^{[n+1]}(x) = f(f^{[n]}(x)) = \sin f^{[n]}(x) < f^{[n]}(x)$$

We conclude that $\{f^{[n]}(x)\}_{n=0}^{\infty}$ is a decreasing sequence when $0 < x \leq 1$. Since the sequence is also bounded, Corollary 1.4 implies that the sequence must converge to a fixed point, which by Example 1 is $0$. Consequently $\{f^{[n]}(x)\}_{n=0}^{\infty}$ converges to $0$ for all $x$. ❑

Theorem 1.3 provides information concerning the Newton-Raphson method described in Section 1.1. Recall that the method involves calculating a sequence $\{x_n\}_{n=0}^{\infty}$ created by letting $x_0$ be an initial value, and defining

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Here we assume that $f'(x_n) \neq 0$ for all $n$. In Section 1.1 we indicated that if $\{x_n\}_{n=0}^{\infty}$ converges, then its limit is a zero of $f$. Now we can support this assertion. Let

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Then $x_1 = g(x_0)$, $x_2 = g(x_1) = g^{[2]}(x_0)$, and in general, $x_n = g^{[n]}(x_0)$. Thus $\{x_n\}_{n=0}^{\infty}$ is the sequence of iterates of $x_0$ for $g$. Theorem 1.3 tells us that if the sequence converges, then it converges to a fixed point $z$ of $g$. In that case,

$$z = g(z) = z - \frac{f(z)}{f'(z)}$$

so that

$$\frac{f(z)}{f'(z)} = 0, \text{ or equivalently, } f(z) = 0$$

This means that $z$ is a zero of $f$, as we wished to prove.

## Attracting and Repelling Fixed Points

By applying graphical analysis we can see diverse behavior for the iterates of various points. Indeed, in Figure 1.6(a) the iterates of $x$ approach the fixed point

$p$, whereas in Figure 1.6(b) the iterates tend toward $\infty$. The iterates of $x$ in Figure 1.6(c) have each of these characteristics, depending on the $x$. We are led to the following definition.

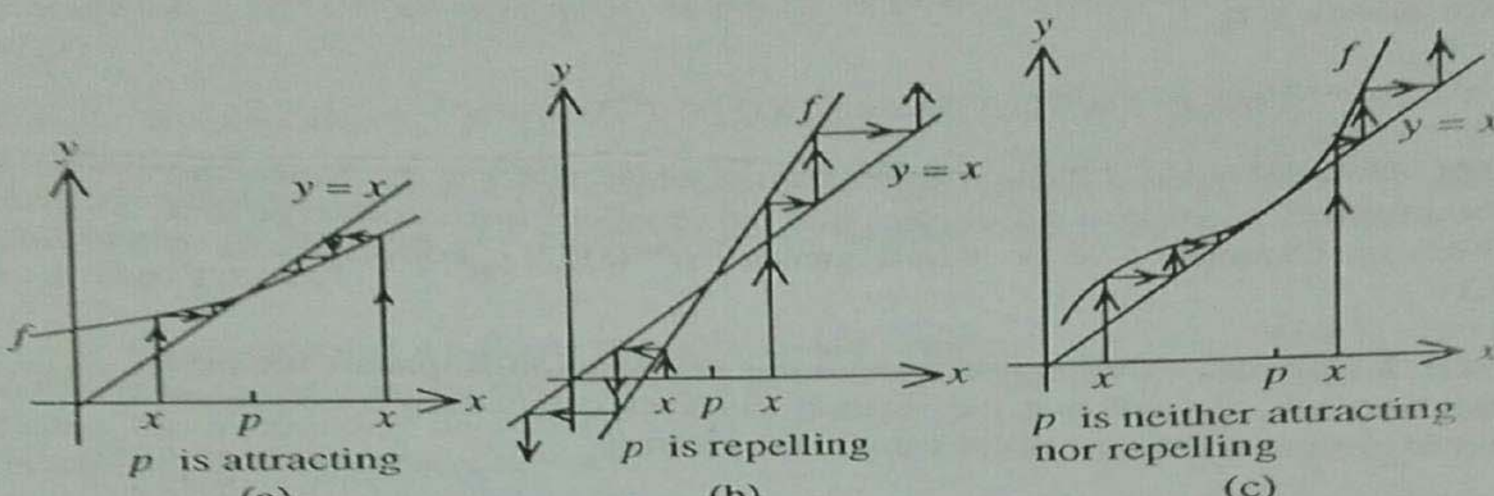



Figure 1.6

**DEFINITION 1.5.** Let $p$ be a fixed point of $f$.

a. The point $p$ is an **attracting fixed point** of $f$ provided that there is an interval $(p - \varepsilon, p + \varepsilon)$ containing $p$ such that if $x$ is in the domain of $f$ and in $(p - \varepsilon, p + \varepsilon)$, then $f^{[n]}(x) \to p$ as $n$ increases without bound. (Such a point is also called **asymptotically stable** in the literature.)

b. The point $p$ is a **repelling fixed point** of $f$ provided that there is an interval $(p - \varepsilon, p + \varepsilon)$ containing $p$ such that if $x$ is in the domain of $f$ and in $(p - \varepsilon, p + \varepsilon)$ but $x \neq p$, then $|f(x) - p| > |x - p|$.

It follows from the definitions above that the fixed point in Figure 1.6(a) is attracting, and that the one in Figure 1.6(b) is repelling. That not every fixed point is attracting or repelling is demonstrated in Figure 1.6(c), where points to the left of $p$ are attracted to $p$ and points to the right of $p$ are repelled from $p$. Other kinds of fixed points that are neither attracting nor repelling can occur.

**LAW OF INDUCTION:** Assume that for each integer greater than or equal to an initial integer $n_0$, a statement, formula or equation, $S(n)$, is given. Suppose that

i. $S(n_0)$ is true.

ii. For any integer $n \geq n_0$, if $S(n)$ is true, then $S(n+1)$ is true.

Then $S(n)$ is true for all integers $n \geq n_0$.

Step (ii) in the axiom is frequently called the **inductive step**. Now we are ready to state and prove Theorem 1.6.

**THEOREM 1.6.** Suppose that $f$ is differentiable at a fixed point $p$.

a. If $|f'(p)| < 1$, then $p$ is attracting.

b. If $|f'(p)| > 1$, then $p$ is repelling.

c. If $|f'(p)| = 1$, then $p$ can be attracting, repelling, or neither.

*Proof.* To begin our proof of (a), we notice that since $|f'(p)| < 1$, the definition of derivative implies that there is a positive constant $A < 1$ and an open interval $J = (p - \varepsilon, p + \varepsilon)$ such that if $x$ is in $J$ and $x \neq p$, then

$$\left| \frac{f(x) - f(p)}{x - p} \right| \leq A$$

Therefore $|f(x) - f(p)| \leq A\,|x - p|$, for all $x$ in $J$. For each such $x$, this means that

$$|f(x) - p| = |f(x) - f(p)| \leq A|x - p| \tag{1}$$

so that $f(x)$ is in $J$ because $0 < A < 1$. Thus $f(x)$ is at least as close to $p$ as $x$ is. Let $x$ be fixed in $J$. If $f^{[n]}(x) = p$ for some $n$, then $f^{[n]}(x) \to p$ as $n$ increases without bound, so we will assume henceforth that $f^{[n]}(x) \neq p$ for all $n$. Next we will use the Law of Induction to prove that

$$|f^{[n]}(x) - p| \leq A^n\,|x - p| \quad \text{for all } n \geq 1 \tag{2}$$

By (1), the inequality holds for $n = 1$. Next, we assume that (2) holds for a given $n > 1$. Then $f^{[n]}(x)$ is in $J$ since $0 < A^n < A < 1$. Therefore by (1) with $f^{[n]}(x)$ substituted for $x$, and then by (2), we find that

$$|f^{[n+1]}(x) - p| = |f(f^{[n]}(x)) - p| \leq A\,|f^{[n]}(x) - p| \leq A(A^n|x - p|)$$

so that $|f^{[n+1]}(x) - p| \leq A^{n+1}|x - p|$. By the Law of Induction we deduce that (2) holds for all integers $n \geq 1$. Since $A^n \to 0$ as $n$ increases without bound, it follows that $f^{[n]}(x) \to p$ for every $x$ in $J$. Thus (a) is proved. The proof of (b) is analogous. Part (c) is addressed in Exercise 10. ∎

EXAMPLE 3. Let $\mu > 0$ be a constant, and let

$$f(x) = \mu x(1 - x) = \mu x - \mu x^2, \text{ for } 0 \leq x \leq 1$$

a. Find the values of $\mu$ for which 0 is an attracting fixed point.
b. Find the values of $\mu$ for which there is a nonzero fixed point.
c. Find the values of $\mu$ for which the nonzero fixed point is attracting.

*Solution*. Notice that $x$ is a fixed point of $f$ if $x = \mu x - \mu x^2$. Thus either $x = 0$ or else $1 = \mu - \mu x$, which implies that $x = 1 - 1/\mu$. If $0 < \mu \leq 1$, then we have $1 - 1/\mu \leq 0$, so there is only one fixed point in the interval $[0, 1]$, namely 0. By contrast, when $\mu > 1$, there are two distinct fixed points in $[0, 1]$: 0 and $1 - 1/\mu$. Next we will determine which fixed points are attracting and which are repelling. Since $f'(x) = \mu - 2\mu x$, it follows that

$$f'(0) = \mu \quad \text{and} \quad f'(1 - 1/\mu) = \mu - 2\mu(1 - 1/\mu) = 2 - \mu$$

Theorem 1.6 tells us that 0 is attracting if $0 < \mu < 1$ and is repelling if $1 < \mu$. It also tells us that $1 - 1/\mu$ is attracting if $1 < \mu < 3$, and is repelling if $\mu > 3$. Finally, it is possible to show that 0 is attracting if $\mu = 1$ (Exercise 15), and that $1 - 1/\mu$ is attracting if $\mu = 3$ (Exercise 16). ❑

## Basins of Attraction

If a fixed point $p$ of $f$ is attracting, then all points near to $p$ are "attracted" toward $p$, in the sense that their iterates converge to $p$. The collection of *all* points whose iterates converge to $p$ is called the basin of attraction of $p$.

**DEFINITION 1.7.** Suppose that $p$ is a fixed point of $f$. Then the **basin of attraction** of $p$ consists of all $x$ such that $f^{[n]}(x) \to p$ as $n$ increases without bound, and is denoted by $B_p$.

EXAMPLE 4. Let $f(x) = x^2$. Find the basin of attraction $B_0$ of the fixed point 0.

*Solution*. If $|x| < 1$, then $f^{[n]}(x) = x^{(2^n)} \to 0$ as $n$ increases without bound, so that $x$ is in $B_0$. By contrast, if $|x| \geq 1$, then $|f^{[n]}(x)| \geq 1$, so that $x$ is not in $B_0$. Thus $B_0$ consists of all $x$ such that $|x| < 1$, that is, $B_0 = (-1, 1)$. (We could also draw the same conclusion by using graphical analysis.) ❑

## Eventually Fixed Points

Finally we introduce the notion of eventually fixed point, which will be of use in later examples.

**DEFINITION 1.8.** Let $x$ be in the domain of $f$. Then $x$ is an **eventually fixed point** of $f$ if there is a positive integer $n$ such that $f^{[n]}(x)$ is a fixed point of $f$.

A fixed point is trivially an eventually fixed point. However, if $f(x) = \sin x$, then $f(\pi) = 0$ and $f(0) = 0$, so that $\pi$ is an eventually fixed point that is not a fixed point. In order not to create confusion, when we refer to $x$ as an eventually fixed point, we will generally assume that $x$ is *not* a fixed point.

EXAMPLE 5. Let $T$ be defined by

$$T(x) = \begin{cases} 2x & \text{for } 0 \le x \le 1/2 \\ 2 - 2x & \text{for } 1/2 < x \le 1 \end{cases}$$

Show that $1/8$ is an eventually fixed point.

*Solution.* A routine check shows that

$$T(\frac{1}{8}) = \frac{1}{4}, \quad T(\frac{1}{4}) = \frac{1}{2}, \quad T(\frac{1}{2}) = 1, \quad T(1) = 0, \quad T(0) = 0$$

Therefore $1/8$ is an eventually fixed point. ❑

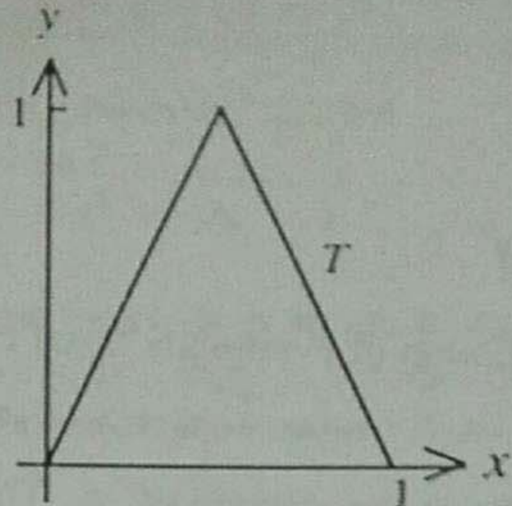



Figure 1.7

## EXERCISES 1.2

In Exercises 1–6, find the fixed points, and determine whether each is attracting or repelling.

1. $f(x) = 4x - x^2$
2. $f(x) = x^3 - x/3$
3. $f(x) = \sqrt{x}$
4. $f(x) = e^{x-1}$
5. $f(x) = \arcsin x$
6. $f(x) = 1/x$
7. Let $g(x) = x^2 + 1/4$. Show that if $|x| > 1/2$, then $|g^{[n]}(x)| \to \infty$ as $n$ increases without bound.

In Exercises 22–25, use either algebra or graphical analysis to find the largest open interval in the basin of attraction of the fixed point 0.

22. $f(x) = \sin^3 x$

23. $f(x) = x^4$

24. $f(x) = x^5 + x^3$

25. $f(x) = (.5)x(1 - x)$

## 1.3 PERIODIC POINTS

Periodicity is a notion common in everyday language. For example, Halley's comet has a period of approximately 76 years. Similarly, the longer a pendulum is, the longer its period is. The notion of periodicity is central to the study of dynamics.

**DEFINITION 1.9.** Let $x_0$ be in the domain of $f$. Then $x_0$ **has period** $n$ (or is a **period-$n$ point**) if $f^{[n]}(x_0) = x_0$, and if in addition, $x_0$, $f(x_0)$, $f^{[2]}(x_0), \ldots,$ $f^{[n-1]}(x_0)$ are distinct. If $x_0$ has period $n$, then the orbit of $x_0$, which is

$$\{x_0, f(x_0), f^{[2]}(x_0), \ldots, f^{[n-1]}(x_0)\}$$

is a **periodic orbit** and is called an **$n$-cycle**.

By Definition 1.9, fixed points are periodic points — with period 1. If a point has period 1, then we will refer to it as a fixed point (rather than a periodic point).

To illustrate a 2-cycle, let $h(x) = -x^3$. Then $\{-1, 1\}$ is a 2-cycle because $h(-1) = 1$ and $h(1) = -1$. Next, we will exhibit a 3-cycle for the tent function $T$.

EXAMPLE 1. The tent function $T$ is given by

$$T(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 1/2 \\ 2 - 2x & \text{for } 1/2 < x \leq 1 \end{cases}$$

Show that $\{2/7, 4/7, 6/7\}$ is a 3-cycle for $T$.

$$T\left(\frac{2}{7}\right) = \frac{4}{7}, \quad T\left(\frac{4}{7}\right) = 2 - 2\left(\frac{4}{7}\right) = \frac{6}{7}, \quad \text{and} \quad T\left(\frac{6}{7}\right) = 2 - 2\left(\frac{6}{7}\right) = \frac{2}{7}$$

confirming that $\{2/7, 4/7, 6/7\}$ is a 3-cycle for $T$. □

Not only does $T$ have a 3-cycle; it has $n$-cycles for every positive integer $n$. This is one of the reasons why the tent function is featured in the study of dynamics. We will take a closer look at the tent function in Section 1.4.

Graphically, an $n$-cycle of a function is represented by a closed loop. Figure 1.9(a) shows the 2-cycle $\{-1, 1\}$ for the function $-x^3$, and Figure 1.9(b) shows the 3-cycle for the tent function in Example 1 above.

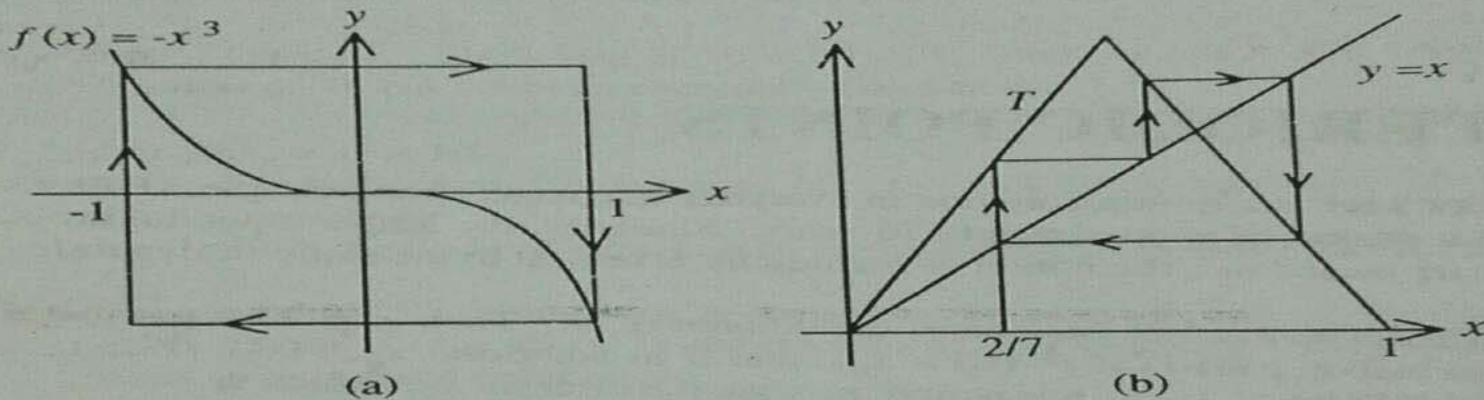



Figure 1.9

It is important to notice that if $f(x) = z$ and $f(z) = x$, then

$$f^{[2]}(x) = f(f(x)) = f(z) = x$$

so that $x$ is a fixed point of $f^{[2]}$. By the same token, $z$ is a fixed point of $f^{[2]}$. Thus, if $\{x, z\}$ is a 2-cycle for $f$, then $x$ and $z$ are both fixed points of $f^{[2]}$. Conversely, if $x$ is a fixed point of $f^{[2]}$ that is not a fixed point of $f$, then there is a point $z$ different from $x$ such that $\{x, z\}$ is a 2-cycle of $f$, so that $x$ is a period-2 point of $f$. Therefore

$\{x, z\}$ is a 2-cycle for $f$ if and only if $f(x) = z$,
where $x$ and $z$ are distinct fixed points of $f^{[2]}$ (1)

For example, assume that $f(x) = x^2 - 1$, so that $f^{[2]}(x) = (x^2 - 1)^2 - 1 = x^4 - 2x^2$.

Obviously $0$ is a fixed point of $f^{[2]}$ that is *not* a fixed point of $f$. Thus there must be a $z$ such that $\{0, z\}$ is a 2-cycle for $f$. Since $f(0) = -1$ we deduce that $\{0, -1\}$ is a 2-cycle for $f$. More generally, $\{x_0, x_1, x_2, \ldots, x_{n-1}\}$ is an $n$-cycle of $f$ if and only if $x_k$ is a fixed point of $f^{[n]}$, for $k = 0, 1, 2, \ldots, n-1$.

## Attracting Periodic Points

Suppose that $x$ is a period-$n$ point of $f$. Then $x$ is a fixed point for $f^{[n]}$. Therefore we have a natural way of defining attracting and repelling periodic points.

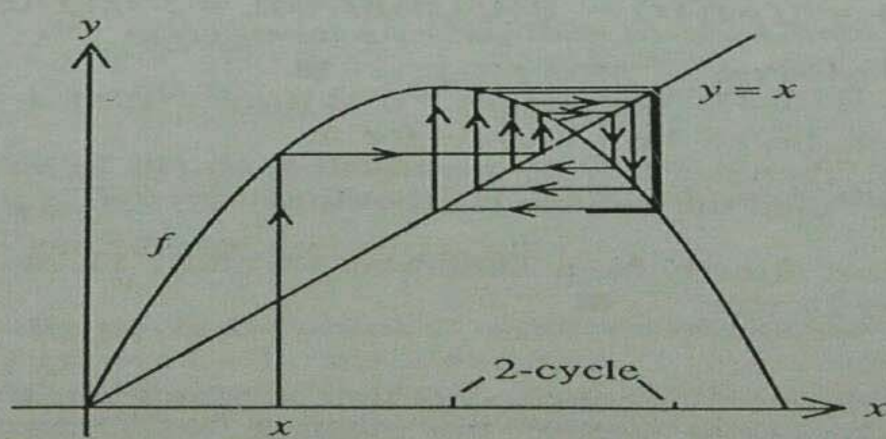
**DEFINITION 1.10.** Let $x$ be a period-$n$ point for a function $f$. Then $x$ is an **attracting** period-$n$ point if $x$ is an attracting fixed point of $f^{[n]}$; also $x$ is a **repelling** period-$n$ point if $x$ is a repelling fixed point of $f^{[n]}$.

Suppose that $f$ is continuous at a period-$n$ point $x$. If $x$ is attracting (repelling), then each point in $\{x, f(x), f^{[2]}(x), \ldots, f^{[n-1]}(x)\}$ is an attracting (repelling) period-$n$ point, so we say that the $n$-cycle $\{x, f(x), f^{[2]}(x), \ldots, f^{[n-1]}(x)\}$ is **attracting (repelling)**.

In particular, if $n = 2$ then the period-2 point $x$ is attracting if and only if there is an interval $(x - \varepsilon, x + \varepsilon)$, such that whenever $y$ is in $(x - \varepsilon, x + \varepsilon)$,

$$f^{[2n]}(y) \to x \quad \text{and} \quad f^{[2n+1]}(y) \to f(x)$$

as $n$ increases without bound. Figure 1.10 shows a number $x$ that is attracted to a 2-cycle of $f$.



EXAMPLE 2. Let $f(x) = -x^{1/3}$. Show that 1 is an attracting period-2 point of $f$.

*Solution.* First notice that $f(1) = -1$ and $f(-1) = 1$. Therefore the point 1 has period 2. Next, observe that

$$f^{[2]}(x) = f(f(x)) = -(-x^{1/3})^{1/3} = x^{1/9}, \text{ so that } (f^{[2]})'(1) = \frac{1}{9}$$

Theorem 1.6 then implies that 1 is an attracting fixed point of $f^{[2]}$, so that 1 is an attracting period-2 point by Definition 1.10. $\square$

One could also prove that 1 is an attracting point of period 2 by showing that there is an interval $J$ around 1 such that whenever $x$ is in $J$, $|f^{[2]}(x) - 1| < |x - 1|$.

In Section 1.2 we gavè a criterion for attracting and repelling fixed points that involves the derivative. Similarly, there is a criterion for attracting and repelling cycles that involves the derivative. Before we state it in Theorem 1.12, we have a preliminary result.

**THEOREM 1.11.** Let $\{x, z\}$ be a 2-cycle of $f$. If $f^{[2]}$ is differentiable at $x$ and at $z$, then

$$(f^{[2]})'(x) = f'(x)f'(z) = (f^{[2]})'(z) \tag{2}$$

*Proof.* Using the Chain Rule and the fact that $f(x) = z$, we find that

$$(f^{[2]})'(x) = (f \circ f)'(x) = [f'(f(x))][f'(x)] = f'(x)f'(z)$$

By symmetry we have $(f^{[2]})'(z) = f'(x)f'(z)$. $\blacksquare$

**THEOREM 1.12.** Let $\{x, z\}$ be a 2-cycle for $f$.

a. If $|f'(x)f'(z)| < 1$, then the 2-cycle is attracting.
b. If $|f'(x)f'(z)| > 1$, then the 2-cycle is repelling.

*Proof.* The result follows directly from Theorems 1.6 and 1.11, and the definition of an attracting (repelling) 2-cycle. $\blacksquare$

If $|f'(x)f'(z)| = 1$, then we cannot conclude anything about whether the cycle $\{x, z\}$ is attracting, repelling or neither. For example, let $f(x) = 1/x$. Then $f^{[2]}(x) = x$, so that $\{x, 1/x\}$ is a 2-cycle for each $x \neq 0$. Evidently the 2-cycle is neither attracting nor repelling, although $|f'(x)f'(z)| = |(f^{[2]})'(x)| = 1$ for all $x \neq 0$.

When $|f'(x)f'(z)| \neq 1$, the criterion can be effective in telling us if $\{x, z\}$ is attracting or repelling.

EXAMPLE 3. Let $f(x) = x^2 - 3x + 2$. Show that $\{0, 2\}$ is a repelling 2-cycle.

*Solution.* Since $f(0) = 2$ and $f(2) = 0$, it follows that $\{0, 2\}$ is a 2-cycle. The fact that $f'(x) = 2x - 3$ implies that $f'(0) = -3$ and $f'(2) = 1$, so that

$$f'(0)f'(2) = (-3)(1) = -3$$

Therefore Theorem 1.12 implies that $\{0, 2\}$ is a repelling 2-cycle of $f$. ❑

Figure 1.11 displays the graph of $f$, with 2-cycle $\{0, 2\}$. By analyzing the iterates of $x$, which is close to 0, we can see why the 2-cycle is repelling.

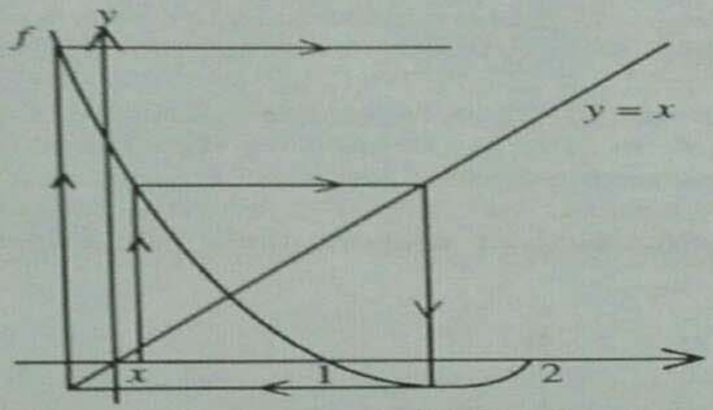



Figure 1.11

If $\{x, f(x), \ldots, f^{[n-1]}(x)\}$ is an $n$-cycle, then by the Chain Rule,

$$(f^{[n]})'(x) = [f'(f^{[n-1]}(x))]\,[f'(f^{[n-2]}(x))] \cdots [f'(f(x))]\,[f'(x)] \tag{3}$$

If the absolute value of the right-hand side of (3) is $< 1$ ($> 1$), then the $n$-cycle is attracting (repelling). We remark that if $x$ is a fixed point, then (3) becomes

$$(f^{[n]})'(x) = [f'(x)]^n \tag{4}$$

**An eventually periodic point** is a point some iterate of which is periodic. For example, let $f(x) = x^2 - 1$. It follows that 1 is an eventually periodic point, since $f(1) = 0$, $f(0) = -1$, and $f(-1) = 0$. Henceforth we will generally assume (as we do with eventually fixed points), that when we refer to a point as eventually periodic, the point is not periodic.

## EXERCISES 1.3

1. Let $f(x) = -\frac{1}{2}x^2 - x + \frac{1}{2}$. Show that 1 is an attracting period-2 point.

2. Show that $\{2/9, 4/9, 8/9\}$ is a repelling 3-cycle for the tent function $T$.

3. Let $f(x) = 1/x$. Show that if $x \neq -1$, 0, or 1, then $x$ is a period-2 point.

4. Let $f(x) = 1/(1-x)$. Show that if $x \neq 0$ or 1, then $x$ is a period-3 point.

5. Let $f(x) = 3.2x - 3.2x^2$. Use a calculator or the program ITERATE to find a 2-cycle, and show that it is attracting.

6. Let $f(x) = 3.84x - 3.84x^2$. Use a calculator or the program ITERATE to find a period-3 point, and determine whether it is attracting or not.

7. Let $T$ be the tent function. Find a point that is not periodic but is eventually periodic with period
   a. 3 $(\frac{1}{7})$ b. 4 $(\frac{1}{17})$ c. 5 $(\frac{7}{11})$

8. Let $f(x) = \cos x$. Determine whether $f$ has any period-$n$ points with $n > 1$.

9. Let $f(x) = x^2 - 1$. Show that the basin of attraction of the 2-cycle $\{-1, 0\}$ consists of all numbers in the interval $((1-\sqrt{5})/2, (1+\sqrt{5})/2)$ except for those whose iterates are eventually the fixed point $(1-\sqrt{5})/2$.

Exercises 10–19 involve the baker's function $B$.

10. Sketch the graphs of $B^{[2]}$ and $B^{[3]}$.

11. Find the fixed points and the period-2 points of $B$. ← fix points 0 & 1
Period 2-points $\frac{1}{3}$, $\frac{2}{3}$

12. Determine whether the following points are fixed, eventually fixed, periodic or eventually periodic, and indicate their periods if they are periodic.
    a. 3/7 b. 3/16 c. 1/10 d. 1/11

13. For each positive integer $n$, determine the number of fixed points of $B^{[n]}$.

14. Show that $x$ in $[0, 1]$ is an eventually fixed point of $B$ if and only if $x$ is a dyadic rational.

15. Show that $x$ in $[0, 1]$ is an eventually periodic point of $B$ if and only if $x$ is a rational.

16. Show that if $p$ is an odd positive integer, then $k/p$ is periodic, for $k = 1, 2, \ldots, p-1$. Is the converse true? Explain your answer.

17. Show that if $1/2 < x \leq 1$, then $B(x) \leq x^2$.

18. Let $m$ be an arbitrary positive integer, and assume that $x$ is not a dyadic rational. Show that there is an integer $n \geq m$ such that
a. $B^{[n]}(x) < 1/2$ b. $B^{[n]}(x) > 1/2$

19. Let $c$ be a constant, and let $f(x) = x^3 - 3x + c$. Determine the values of $c$ for which $\{0, c\}$ is a 2-cycle. Is such a 2-cycle attracting for any such value of $c$? Explain your answer.

20. Let $f(x) = ax^3 - bx + 1$, where $a$ and $b$ are constants. Determine the values of $a$ and $b$ for which $\{0, 1\}$ is an attracting 2-cycle.

21. Let $m$ be a positive integer. Prove that if $(f^{[m]})'(x) = 0$, then there is an iterate $x_k$ of $x$ such that $f'(x_k) = 0$.

22. Let $|f'(x)| < 1$ for all $x$. Show that $f$ cannot have any period-2 points.

23. a. Let $f$ be increasing. Show that there are no period-$n$ points for $n > 1$.
b. Let $f$ be decreasing. Show that there are no period-$n$ points for $n > 2$.
c. Find an example of a decreasing function $f$ that has a fixed point and a period-2 point. $f(x) = -x$

24. Let $f$ be a linear function. Show that there are no period-$n$ points for $n > 2$.

# 1.4 FAMILIES OF FUNCTIONS

In Example 3 of Section 1.2 we discussed the function $f$ defined by

$$f(x) = \mu x(1 - x), \text{ for } 0 \leq x \leq 1$$

where $\mu$ was a fixed positive constant. We found that $f$ has one or two fixed points, depending on whether $0 < \mu \leq 1$ or $\mu > 1$. Thus the number of fixed points depends on the value of $\mu$. In order to emphasize the fact that the function depends on $\mu$, we will henceforth designate $f$ by $Q_\mu$, so that

$$Q_\mu(x) = \mu x(1 - x), \text{ for } 0 \leq x \leq 1$$

The family $\{Q_\mu\}$ for $\mu > 0$ is the **quadratic family**, so named because each of the functions in the family is a quadratic function.

A collection of functions such as $\{Q_\mu\}$ is called a **parametrized family of functions** (or a **one-parameter family**), and $\mu$ is the **parameter** for the family. Other parametrized families that we will encounter are

$$g_\mu(x) = x^2 + \mu, \text{ for all } x$$

$$T_\mu(x) = \begin{cases} 2\mu x & \text{for } 0 \le x \le 1/2 \\ 2\mu(1-x) & \text{for } 1/2 < x \le 1 \end{cases} \quad \text{where } 0 < \mu \le 1$$

$$E_\mu(x) = \mu e^x, \text{ for all } x$$

$$S_\mu(x) = \mu \sin x, \text{ for } 0 \le x \le \pi$$

Notice that $\mu$ is constant and $x$ is the variable for each function in the parametrized families listed above. For the family $\{T_\mu\}$, $\mu$ is restricted to the interval $(0, 1]$ in order that the range of the functions $T_\mu$ will be contained in the domain $[0, 1]$.

There are names for the families listed above. The family $\{T_\mu\}$ is the **tent family** (because the functions $T_\mu$ are from the same mold as the tent function $T$), $\{E_\mu\}$ is the **exponential family**, and $\{S_\mu\}$ is the **sine family**. We give the family $\{g_\mu\}$ no special name; later we will show that the family $\{g_\mu\}$ is a close relative of the quadratic family $\{Q_\mu\}$. When we wish to refer to a general parametrized family rather than a specific one, we will denote it by $\{f_\mu\}$.

The way in which the orbits in a parametrized family change as the parameter varies is called the **dynamics of the family**. In the present section we will study the dynamics of $\{g_\mu\}$ and $\{T_\mu\}$; we will devote the entire Section 1.5 to the dynamics of $\{Q_\mu\}$.

## The Family $\{g_\mu\}$

The family $\{g_\mu\}$ consists of the functions defined by

$$g_\mu(x) = x^2 + \mu, \text{ for all } x$$

which are among the simplest nonlinear differentiable functions. The dynamics of the family $\{g_\mu\}$ vary according to the value of $\mu$. If $0 \le \mu$, we can describe in detail the orbit $\{g_\mu^{[n]}(x)\}_{n=0}^{\infty}$ for any real number $x$, whereas if $\mu < 0$, the orbits can be very complicated. As a result, in this section we will limit the discussion to the members of $\{g_\mu\}$ for which $0 \le \mu$.

Among nonnegative parameters for $\{g_\mu\}$, two values are especially noteworthy: $\mu = 0$ and $\mu = 1/4$. For $\mu = 0$ we have $g_0$, which is the simplest function in the family and is defined by $g_0(x) = x^2$ (Figure 1.15(a)). A moment's reflection reveals that the fixed points of $g_0$ are 0 and 1. Next, we notice that $|g_0(x)| = x^2 < |x|$ if $|x| < 1$, and $|g_0(x)| = x^2 > |x|$ if $|x| > 1$. It follows that 0 is an attracting fixed point whose basin of attraction is $(-1, 1)$, and that 1 is a repelling fixed point. Since all iterates of $x$ approach 0 if $|x| < 1$ and are unbounded if $|x| > 1$, there can be no periodic points besides 0 and 1.

Next we turn to $g_{1/4}$ (Figure 1.15(b)). Notice that

$$g_{1/4}(x) - x = x^2 + \frac{1}{4} - x = \left(x - \frac{1}{2}\right)^2 \begin{cases} = 0 & \text{if } x = 1/2 \\ > 0 & \text{if } x \neq 1/2 \end{cases}$$

Therefore $g_{1/4}$ has one and only one fixed point: 1/2. Moreover, the graph of $g_{1/4}$ lies above the line $y = x$ except at $x = 1/2$, and is tangent at the point $(1/2, 1/2)$. Using graphical analysis (or the Mean Value Theorem), one can show that the basin of attraction of the fixed point 1/2 is $[-1/2, 1/2]$, and that 1/2 repels points to the right. We call a fixed point that attracts on one side and repels on the other an **attracting-repelling fixed point**.

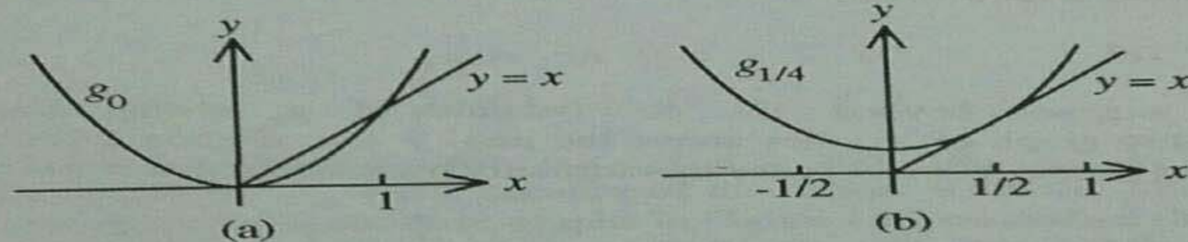



Figure 1.15

Because the graph of $g_\mu$ shifts upward as $\mu$ increases, and because the graph of $g_{1/4}$ is tangent to the graph of $y = x$, a glance at Figure 1.15(b) suggests that the graph of $g_\mu$ intersects the graph of $y = x$ if $0 < \mu \le 1/4$ and does not intersect it if $\mu > 1/4$. Therefore we will divide the analysis of $\{g_\mu\}$ for the remaining positive values of $\mu$ into two groups: $0 < \mu < 1/4$ and $\mu > 1/4$.

Case 1. $0 < \mu < 1/4$

The number $x$ is a fixed point of $g_\mu$ if and only if $x = g_\mu(x) = x^2 + \mu$, which is equivalent to $x^2 - x + \mu = 0$. Solving for $x$, we obtain

$$x = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\mu} \quad \text{or} \quad x = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\mu}$$

as the fixed points of $g_\mu$. Next we will determine which (if any) of these points is attracting. Since $g_\mu'(x) = 2x$, it follows that

$$\left| g_\mu'\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4\mu}\right)\right| = \left| 1 \pm \sqrt{1-4\mu} \right|$$

If we let

$$p_\mu = \frac{1}{2} - \frac{1}{2}\sqrt{1-4\mu} \quad \text{and} \quad q_\mu = \frac{1}{2} + \frac{1}{2}\sqrt{1-4\mu}$$

then $|g_\mu'(p_\mu)| = \left|1 - \sqrt{1-4\mu}\right| < 1$ for $0 < \mu < 1/4$. By Theorem 1.6, $p_\mu$ is an attracting fixed point. By contrast, $|g_\mu'(q_\mu)| = \left|1 + \sqrt{1-4\mu}\right| > 1$ for $0 < \mu < 1/4$, so that again by Theorem 1.6, $q_\mu$ is a repelling fixed point. It turns out that if $0 < \mu < 1/4$, then the basin of attraction of $p_\mu$ is the open interval $(-q_\mu, q_\mu)$, and the iterates of each number $x$ such that $|x| > q_\mu$ march off toward $\infty$. Thus every number other than $\pm q_\mu$ has the property that its iterates approach $p_\mu$ or are unbounded. We mention that as the parameter $\mu$ approaches $1/4$, the fixed points $p_\mu$ and $q_\mu$ are drawn toward each other, and actually coalesce when $\mu = 1/4$ (compare Figure 1.16(a) with 1.16(b)).

Case 2. $\mu > 1/4$

As $\mu$ increases beyond $1/4$, the dynamics of $g_\mu$ change dramatically, because the entire graph of $g_\mu$ lies above the line $y = x$ (Figure 1.16(c)). Thus there is no fixed point. We can prove this formally by noticing that if $\mu > 1/4$, then

$$g_\mu(x) - x = x^2 - x + \mu > x^2 - x + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2 \geq 0$$

so that $g_\mu(x) > x$ for all $x$. Moreover, since the iterates of each number $x$ form an

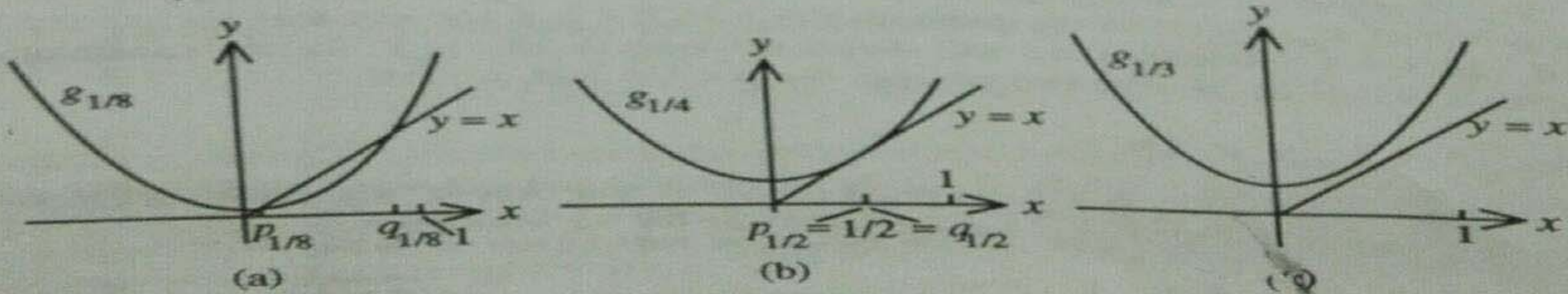



Figure 1.16

increasing sequence that diverges to $\infty$, it follows that $g_\mu$ has no periodic points.

Something very special has happened for $\mu = 1/4$, because as $\mu$ increases and passes through $1/4$, $g_\mu$ first has two fixed points and then none. We say that the family has a bifurcation at 1/4. More generally we have the following definition.

**DEFINITION 1.13.** A parametrized family $\{f_\mu\}$ has a **bifurcation at** $\mu_0$, or **bifurcates at** $\mu_0$, if the number or nature (attracting vs. repelling) of periodic points of $f_\mu$ changes as $\mu$ passes through $\mu_0$. In this case $\mu_0$ is said to be a **bifurcation point** for the family.

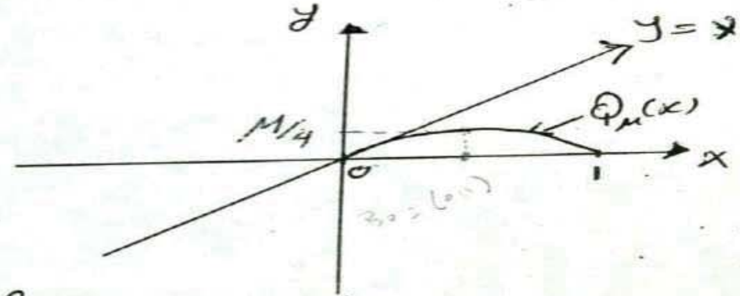
The term "bifurcate" comes from Latin words meaning "two branches." From Definition 1.13 we infer that $\{g_\mu\}$ bifurcates at the number 1/4. Bifurcation points signal changes in dynamics of a parametrized family. We will discuss bifurcation points for each of the parametrized families that we encounter, and will devote Section 1.6 to bifurcations.

Consider the family $\{Q_\mu\}$ which consists the functions defined by

$$Q_\mu(x) = \mu x(1-x) = \mu x - \mu x^2 \qquad 0 \le x \le 1,\quad 0 < \mu \le 4$$

$Q_\mu(x)$ has exactly two fixed points which are $x=0$ and $x = 1 - \frac{1}{\mu} = P_\mu$

$\boxed{1}$ If $0 \le \mu \le 1$, $0$ is the only fixed point of $Q_\mu$. So, we will find the basin of attraction of $0$.




$$Q_\mu'(x) = \mu - 2\mu x = 0$$

if $x = \frac{1}{2}$ c.p.

$Q_\mu(\frac{1}{2}) = \frac{\mu}{4}$ is an exterme value of $Q_\mu$

Now, we will use the following lemma (*) :-

Suppose that $f$ is continuous defined on the closed interval and $\{f^n(x)\}$ is bounded monotonic, then there exists a fixed point $p$ such that $f^n(x) \longrightarrow p$ as $n$ increases without bound.

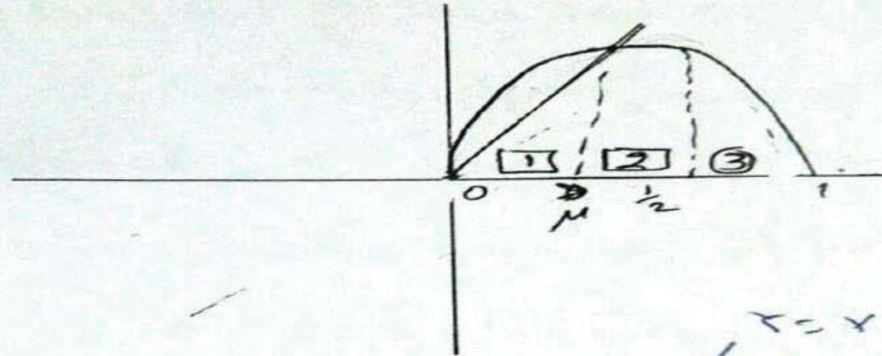
$0 < Q_\mu(x) = \mu x(1-x) < \mu x < x$

$\therefore \{Q_\mu^n(x)\}$ is decreasing sequence and $Q_\mu(x) < 1$

$\therefore \{Q_\mu(x)\}$ is bounded

[2] For $1 < \mu \le 2$, there are two fixed points $0$ and $1 - \frac{1}{\mu} = P_\mu$

in this case $P_\mu = 1 - \frac{1}{\mu} < \frac{1}{2}$




a. For $0 < x < P_\mu$

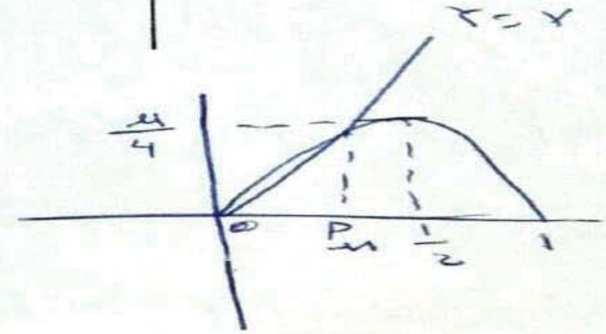
$x < P_\mu \Rightarrow x < 1 - \frac{1}{\mu}$

$\Rightarrow \frac{1}{\mu} < 1 - x$

$\Rightarrow 1 < \mu(1-x)$

$\Rightarrow x < \mu x(1-x) = Q_\mu(x)$

$\Rightarrow Q_\mu(x) < Q_\mu^2(x)$




$\therefore \{Q_\mu^n(x)\}$ is an increasing sequance

$\because x < P_\mu$ and $Q_\mu(x)$ increasing, therefore

$Q_\mu(x) < Q_\mu(P_\mu) = P_\mu$

$\therefore \{Q_\mu^n(x)\}$ is bounded monotonic so, by lemma (*)

$Q_\mu^n(x) \longrightarrow P_\mu$ as $n$ increases without bound.

b. $P_\mu < x < \frac{1}{2}$

clear that $\{Q_\mu^n(x)\}$ is decreasing seq.

$x > 1 - \frac{1}{\mu} \Rightarrow x > Q_\mu(x) \Rightarrow Q_\mu(x) > Q_\mu^2(x)$.

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∵ $Q_\mu(x)$ decreasing and $x > p_\mu$

⇒ $Q_\mu(x) < Q_\mu(p_\mu) = p_\mu$

⇒ $\{Q_\mu^n(x)\}$ is bounded.

∴ by lemma (*), $\{Q_\mu^n(x)\} \longrightarrow p_\mu$ as $n$ increases without bound.

c. If $\frac{1}{2} < x < 1$

From the figure

$Q_\mu(x) < x$

∴ $\{Q_\mu^n(x)\}$ is decreasing sequance

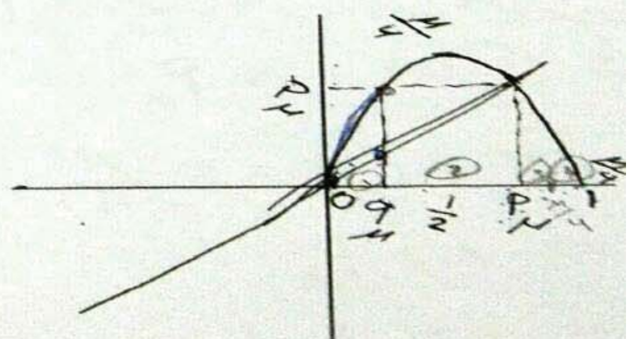
∵ $x > \frac{1}{2} > p_\mu$

⇒ $Q_\mu(x) < Q_\mu(p_\mu) = p_\mu$ (Since $Q_\mu(x)$ is decreasing sequance)

∴ $\{Q_\mu^n(x)\} \longrightarrow p_\mu$ (by lemma (*))

$\boxed{3}$ $2 < \mu \leq 3$

$\Rightarrow p_\mu = 1 - \frac{1}{\mu} > \frac{1}{2}$



Assume that $q_\mu$ is the unique number in $(0, \frac{1}{2})$ such that $Q_\mu(q_\mu) = Q_\mu(p_\mu) = p_\mu$, $q_\mu$ and $p_\mu$ are symmetric with respect to $x = \frac{1}{2}$.

a. For $0 < x < q_\mu$

In this case we have the graph of $q_\mu$ lies above the line $y = x$, that

$Q_\mu(x) > x$

$\therefore \{Q_\mu^n(x)\}$ increasing seq.

So, $Q_\mu(x) < Q_\mu(q_\mu) = P_\mu$

$\therefore \{Q_\mu^n(x)\}$ is bounded

$\therefore Q_\mu^n(x) \longrightarrow P_\mu$ as $n$ increases without bound

b. $q_\mu < x \leq P_\mu$

From the figure

$P_\mu \leq Q_\mu(x) \leq \frac{\mu}{4}$

$\therefore x < Q_\mu(x) \quad \forall x \in (q_\mu, P_\mu]$

$\therefore \{Q_\mu^n(x)\}$ increasing seq. and $x < P_\mu$

$\therefore Q_\mu(x) < Q_\mu(P_\mu) = P_\mu$

$\therefore Q_\mu^n(x) \longrightarrow P_\mu$ as $n$ increases without bound

c. For $P_\mu < x \leq \frac{\mu}{4}$

$\Rightarrow x > P_\mu$

$\Rightarrow x > 1 - \frac{1}{\mu}$

$x > Q_\mu(x)$

$\therefore \{Q_\mu^n(x)\}$ decreasing seq.

$\therefore x > P_\mu$

$\therefore Q_\mu(x) < Q_\mu(P_\mu) = P_\mu$

$\therefore \{Q_\mu^n(x)\}$ is bounded

So, $Q_\mu^n(x) \longrightarrow P_\mu$ as $n$ increases without bound.

d. $\frac{\mu}{4} < x \leq 1$

$x > \frac{\mu}{4} > P_\mu$

$\therefore \{Q_\mu^n(x)\}$ is decreasing seq.

So, $Q_\mu(x) < Q_\mu(P_\mu) = P_\mu$

$\therefore \{Q_\mu^n(x)\}$ is bounded and $Q_\mu^n(x) \longrightarrow P_\mu$ as $n$ increases without bound.

$\therefore$ For each $x \in (0,1)$, $Q_\mu^n(x) \longrightarrow P_\mu$ as $n$ increases without bound, that is $B_{P_\mu} = (0,1)$

$\boxed{H}$ For $3 < \mu \leq 4$

$|Q_\mu'(P_\mu)| = |\mu - 2\mu(1-\frac{1}{\mu})| = |2-\mu|$

when $\mu$ increases to 3. $|Q_\mu'(P_\mu)| < 1$

when $\mu$ increases, Further $|Q_\mu'(p_\mu)| > 1$

So, $\mu_0 = 3$ is a bifurcation point for the quadratic family $\{Q_\mu^n(x)\}_{n=0}^{\infty}$

Bifurcation :

In the study of the family of quadratic function, the values of $\mu$ at which the family bifurcates (that mean, where periodic points arise or disappear, as well as where periodic points become or cease to be attracting) play a prominent role.

One method of displaying the points at which a parameterized family of functions $\{f_\mu\}$ bifurcates is called a bifurcation diagram.


It is designed to give information about the behavior of higher iterates of arbitrary members of the domain of $f_\mu$ for all values of $\mu$.

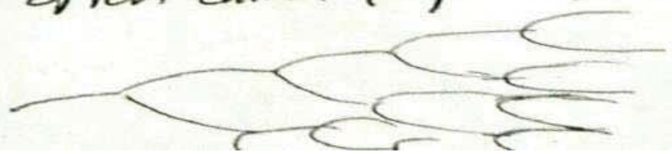
Bifurcation Diagram of $\{f_\mu\}$

Is a graph for which the horizontal axis represents values of $\mu$ and the vertical axis represents higher iterates of the variable $x$.

For each value of $\mu$, the diagram includes all points of the from $(\mu, f_\mu^n(x))$ for value of $u$ Larger that say 50 or 100.

Periodic - doubling Bifurcation :

At this type of bifurcation an attracting period-$n$ cycle becomes repelling and gives ~~bi~~ birth to an attracting $2n$-cycle. The graph near the bifurcation point in this type is a resemble a pitchfork, therefore this kind of bifurcation is often called (a pitchfork bifurcation).



The general properties of families that accompany period-doubling bifurcation:

let $\{f_\mu\}$ be a parameterized family, and assume that:-

1. $f_{\mu^*}$ has a fixed point $p$, (i.e) $f_{\mu^*}(p)=p$

2. $f'_{\mu^*}(p) = -1$.

3. The graph of $f_\mu^2$ crosses the line $y=x$ when $\mu<\mu^*$, is tangent to the line $y=x$ when $\mu=\mu^*$

and, snakes the line $y=x$ when $\mu>\mu^*$


$\mu<\mu^*$ $\mu=\mu^*$ $\mu>\mu^*$

or $\dfrac{\partial^2 f(\mu^*, p)}{\partial x \partial \mu} \neq 0$

Example:
Show that the Quadratic family $\{Q_\mu\}$ has the following properties relat to the bifurcation point $\mu=3$:

~~$Q_3(2/3) = 2/3$ so that~~

First, we have $Q_\mu(x) = \mu x(1-x)$

the fixed points are $x=0$, $x = 1-\frac{1}{\mu}$

$x = 1 - \frac{1}{3}$

$x = \frac{2}{3}$

1- $Q_3\left(\frac{2}{3}\right) = 3\cdot\frac{2}{3}\left(1-\frac{2}{3}\right)$

$= \frac{2}{3}$

so that, $2/3$ is a fixed point

2- Since $Q_\mu'(x) = \mu - 2\mu x$, it follows that

$Q_3'\left(\frac{2}{3}\right) = 3 - 6 * \frac{2}{3}$

$= -1$

$Q_\mu^2(x) = Q(\mu x - \mu x^2)$

$\mu^2x^2 - 2\mu^2x^3 + \mu^2x^4$

$= \mu(\mu x - \mu x^2) - \mu(\mu x - \mu x^2)^2$

$= \mu^2 x - \mu^2 x^2 - \mu^3 x^2 + 2\mu^3 x^3 - \mu^3 x^4$

$(Q_\mu^2)'(x) = \mu^2 - 2\mu^2 x - 2\mu^3 x + 6\mu^3 x^2 - 4\mu^3 x^3$

$\cancel{(Q_3^2)'\left(\frac{2}{3}\right) = 9 - 18\cdot\frac{2}{3} - 2 * 27 * \frac{2}{3} + 6 * 27 \cdot \frac{4}{9} - 4 * 27 * \frac{8}{27}}$

$(Q_\mu^2)''(x) = -2\mu^2 - 2\mu^3 + 12\mu^3 x - 12\mu^3 x^2$

$= -2 * 9 - 2 * 27 + 12 * 27 \cdot \frac{2}{3} - 12 * 27 * \frac{4}{9}$

$= -18 - 54 + 216 -$

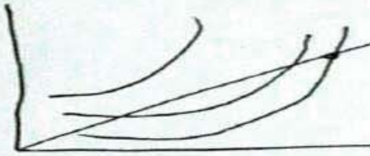
$=$

As a result, $(Q_3^2)''$ changes sign at $2/3$ so that $\left(\frac{2}{3}, \frac{2}{3}\right)$ is an inflection point.

Clearly, the period-doubling bifurcation of $Q_\mu$ occures at $\mu^* = 3$ and $p = \frac{2}{3}$.

2- Tangent Bifurcation

A family $\{f_\mu\}$ has a tangent bifurcation (or saddle or fold bifurcation) at $\mu^*$. if a pair of fixed points are born as a curve in the graph of $f_\mu$ becomes tangent to and then crosses the line $y=x$ when $\mu$ passes through $\mu^*$.




A parameterized family $\{f_\mu\}$ has a tangent bifurcation at $\mu^*$ if:-

1- $f_{\mu^*}(p)=p$, so $p$ is a fixed point of $f_\mu^*$

2- $f'_{\mu^*}(p)=1$, so the graph of $f_{\mu^*}$ is tangent to the line $y=x$ at $(p,p)$

3- $f_\mu(p)$ is monotone function of $\mu$ near $\mu^*$ or $A$- $\frac{\partial f}{\partial \mu}(\mu^*,p)\neq 0$

4- The graph of $f_\mu$ is concave upward (or downward) at $(p,p)$ or $\frac{\partial^2 f}{\partial x^2}(\mu^*,p)\neq 0$

Example

show that $\{E_\mu\}$ where $E_\mu(x)=\mu e^x$, $\mu>0$, satisfies (1-4) relative to the bifurcation point $\mu=\frac{1}{e}$.

Show that $\{E_\mu\}$ where $E_\mu(x) = \mu e^x$, $\mu > 0$, Satisfies (1)-(4) relative to the bifurcation point $\mu = \frac{1}{e}$.

Solution:-

To find the F.p.

$$\frac{1}{e} e^x = x$$

$$e^{-1} e^x = x$$

$$e^{x-1} = x$$

$x - 1 = \ln x$ iff $x = 1$

Since $(E_{1/e})(1) = E_{1/e}(1) = \frac{1}{e} e = 1$

$\therefore f_{\mu^*}(p) = p$

2- $f'(x) = E_\mu'(x) = \mu e^x$

$$E_{1/e}(1) = \frac{1}{e} e^1 = 1$$

so (1) and (2) are satisfied with $p = 1$

3- $A = \frac{\partial E}{\partial \mu}(\mu^*, p) = e^x = e^1 = 2.7 \neq 0$

(i.e) $E_\mu$ is monotone for the graph of $E_\mu$ is ~~concave~~

4- $\frac{\partial^2 E}{\partial x^2}(\mu^*, p) = \mu e^x \Big|_{(\frac{1}{e}, 1)} = e^0 = \cancel{2.7}\ 1 \neq 0$ , (i.e)

the graph of $E_\mu$ is concave upward because

$E_\mu''(x) = E_\mu(x) = \mu e^x > 0 \quad \forall x$ and $\forall \mu$

$\therefore$ (4) is true as well

the $\{E_\mu\}$ has a tangent bifurcation of $\mu^* = \frac{1}{e}$

① if $AB < 0$, the two fix. pt apear if $\mu < \mu^*$

② if $AB > 0$, the two fixed pt apear if $\mu > \mu^*$

## Period-3 points

what the presence of a period-3 point, or any period-$n$ point implies about the existence of other periodic points?

The answer will derive from two famous theorms those of Li-York 1975 and Sharkovsky 1964.

**Theorem** (Maximum-Minimum)

suppose that $f$ is continuous on the interval $[a,b]$. Then $f$ has a maximum value and minimum value

**Theorem** (Intermediate Value theorem)

Suppose that $f$ is continuous on the interval $[a,b]$ and let $p$ be any number between $f(a)$ and $f(b)$. Then there is a number $c$ in $[a,b]$ such that $f(c)=p$.

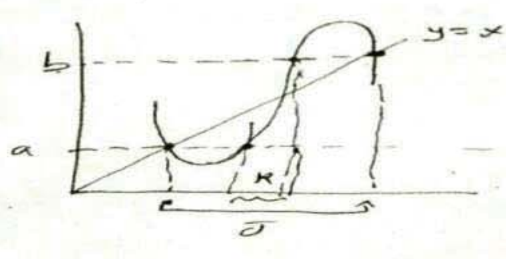
Now, in order to prove the Li-YorK theorem we need the following lemmas:

**lemma 1)**

let $f$ be continuous on an interval $J$. Let $f(J)$ denote the collection of all values $f(x)$ for $x \in J$. Then $f(J)$ is also an interval.

**lemma 2)**

let $f$ be continuous on a closed interval $J$ and assume that $f(J) \supseteq [a,b]$. Then there is a closed interval $K$ such that $J \supseteq K$ and $f(K)=[a,b]$.



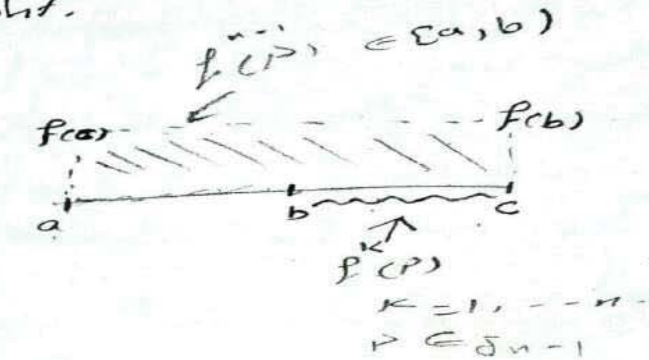
lemma 3)

Suppose that $J$ is a closed interval, and assume that $f$ is contirious on $J$ and $f(J) \supseteq J$. Then $f$ has a fixed point in $J$.

lemma 4)

let $f$ be continuous and suppose that $f(a) = b$, $f(b) = c$ and $f(c) = a$.
Then $f$ has a fixed point and a period 2-point.

Proof:




~~[illegible]~~

assume that $a < b < c$

$\because f(b) = c$ and $f(c) = a$

$\therefore f[b,c] \supseteq [a,c]$ — lemma 2

$\therefore f[b,c] \supseteq [a,c] \supseteq [b,c]$

So, by lemma (3), $f$ has a fixed point in $[b,c]$.

To show that $f$ has a period-2 point:

let $a^*$ is the largest number such that $a \le a^* < b$ and $f(a^*) = b$
since $f$ is continuous on $[a,b]$ and $f(b) = c > b$, such that an $a^*$ exists

then $f[a^*,b] \supseteq [b,c]$ ← $f$ cont.

$\Rightarrow f^2[a^*,b] \supseteq f[b,c] \supseteq [a,c] \supseteq [a^*,b]$

$\therefore$ by lemma (3) with $J = [a^*,b]$, there is a fixed point $p$ of $f^2$ in $[a^*,b]$

$\because a^* < b < c$, $f(a^*) = b$ and $f(b) = c$

$\therefore p \neq a^*$ and $p \neq b$ — if $p = a^* \Rightarrow f(p) = f(a^*) = b$, $p = f^2(p) = f(b) = c$

Also; if $a^* < x < b$ then $f(x) > b$ (from definition of $a^*$)

so, $f(p) > b > p$

$\therefore$ $p$ is not a fixed point of $f$

so $p$ is a period 2-point of $f$.

Theorem (Li-Yorke)

suppose that $f$ is continuous on the closed interval $J$ with $J \supseteq f(J)$. If $f$ has a period-3 point then $f$ has points of all other periods.

proof:- Assume that $a < b < c$ and $f(a) = b$, $f(b) = c$ and $f(c) = a$

According to lemma (4) $f$ has points of period 1 and 2.

by assumption $f$ has points of period-3.

Now, Let $n > 3$, we will show that $f$ has points of period-$n$.

let $J_0 = [b, c]$

$\because$ $f(J_0) = f[b, c] \supseteq [a, c] \supseteq [b, c] = J_0$ *

also, due to lemma (2) there exists a closed interval $J_1$, such that

$J_0 \supseteq J_1$ and $f(J_1) = J_0 = [b, c]$ *

$\Rightarrow f^2(J_1) = f(J_0) \supseteq J_0$

so, by lemma (2) there is a closed interval $J_2$ such that

$J_1 \supseteq J_2$ and $f^2(J_2) = J_0$

$\Rightarrow f^3(J_2) = f(f^2(J_2)) = f(J_0) \supseteq J_0$

Again by lemma (2), there is a closed interval $J_3$, such that

$J_2 \supseteq J_3$ and $f^3(J_3) = J_0$

Inductively, we obtain a nested sequence of closed intervals $J_0, J_1, \ldots, J_{n-2}$ with

$$\left.\begin{array}{l} [b,c] = J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots \supseteq J_{n-2} \text{ and} \\ f^k(J_k) = J_0 = [b,c] \quad , \text{for } k = 1, 2, \ldots, n-2 \end{array}\right\} \cdots (1)$$

Thus, in particular

$$f^{n-2}(J_{n-2}) = [b,c]$$

$$\Rightarrow f^{n-1}(J_{n-2}) = f(f^{n-2}(J_{n-2})) = f[b,c] \supseteq [a,c] \supseteq [a,b]$$

~~$f^{n-1}(J_{n-2})$~~

So that by lemma (2), there is a closed interval $J_{n-1}$ such that

$$J_{n-2} \supseteq J_{n-1} \text{ and } f^{n-1}(J_{n-1}) = [a,b] \quad \text{--- (2)}$$

Consequently by (1) and (2), we have

$$f^n(J_{n-1}) = f(f^{n-1}(J_{n-1})) = f[a,b] \supseteq [b,c] \supseteq J_{n-2} \supseteq J_{n-1}$$

Therefore, by lemma (3), there is a point $p \in J_{n-1} \subseteq [a,b]$ there is a fixed point of $f^n$.

we will show now, that $p$ has period $n$ as follows :-

First:-) $f^k(p) \in [b,c]$ for $k = 1, 2, \dots, n-2$ $\qquad f^n(p) = p$

because for each $k$, $p \in J_k$ and $[b,c] = f^k(J_k)$ by (1) $\qquad f^{n-1}(p) \neq p$

Second:-) from (2), we have $f^{n-1}(J_{n-1}) = [a,b]$ $\qquad p \in J_{n-1} \subseteq J_{n-2} \subseteq \dots \subseteq J_0$

$\because p \in J_{n-1} \Rightarrow f^{n-1}(p) \in [a,b]$

Now, if $f^{n-1}(p) = b \Rightarrow p = f^n(p) = f(f^{n-1}(p)) = f(b) = c$

$\therefore f(p) = f(c) = \underline{a}$

However, Since $J_1 \supseteq J_2 \supseteq \dots \supseteq J_{n-2} \supseteq J_{n-1}$ and $p \in J_{n-1}$

$\Rightarrow f(p) \in [b,c] = f^1(J_1)$, so that $f(p) \neq a$

this Contradiction implies that $\qquad \underline{f(p)} \in f(J_1) = [b,c]$

$f^{n-1}(p) \neq b \Rightarrow f^{n-1}(p) \in [a,b)$

$\therefore$ all the first $(n-2)$ intervals of $p$ lie in $[b,c]$, the $(n-1)$ iterate lies in $[a,b)$ and the $n^{th}$ iterate lies again in $[b,c]$.

Consequantly $p$ really does have period $-n$.

~~Remarks~~:

Definition :- The sharovsky ordering of the positive integers is define by

$3 \to 5 \to 7 \to \cdots\ 2\cdot 3 \to 2\cdot 5 \to 2\cdot 7 \to \cdots\ 2^2\cdot 3 \to 2^2\cdot 5 \to 2^2\cdot 7 \cdots\ 2^3 \to 2^2 \to 2 \to 1$

there $m \to n$ signifies that $m$ appears before $n$ in the sharkovsky ordering.

Example :- $17 \to 14$ due to $14 = 2 * 7$

$40 \to 64$ due to $40 = 2^3 \times 5$ and $64 = 2^6$

Remark :- since every ~~position~~ positive integer can be written as $2^k \times$ (odd integers)

for a suitable nonnegative integer $k$ and a suitable odd integer, the sharkovsky ordering is an ordering of the collection of all positive integers.

Skarkovsky Theorem

let $f$ be a continous function defined on the interval $J$, and suppose that $J \supseteq f(J)$.

If $f$ has a point with period $m$ then $f$ has a point with period $n$ for all $n$ such that $m \to n$.

~~Theorem : (Sharkovsky~~

~~Proof~~

# The Schwarzian Derivative

Recall than if $f$ is differentiable function defined on an interval $J$. then $x$ is a critical point in the interior of $J$ if $f'(x)=0$

Def:-

let $f$ be defined on the interval $J$, and assume that the third derivative $f'''$ is continuous on $J$. Define $Sf$ by

$$(Sf)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

Then, $(Sf)(x)$ is the schwarzian derivative of $f$ at $x$ whenever it exists as a number or as $-\infty$ or $\infty$.

Example:- let $Q_\mu(x) = \mu x(1-x)$. Show that $(SQ_\mu)(x) < 0$ for $x \in [0,1]$

Solution:-

we have $Q_\mu'(x) = \mu(1-2x)$ , $x=\frac{1}{2}$

$Q_\mu''(x) = -2\mu$ and $Q_\mu'''(x) = 0$

~~Theorem~~ therefore , if $x \neq \frac{1}{2}$ , then

$$(SQ_\mu)(x) = -\frac{3}{2}\left(\frac{-2\mu}{\mu(1-2x)}\right)^2$$

$$= \frac{-6}{(1-2x)^2} < 0$$

* لأنه إذا عوضنا $x=\frac{1}{2}$ ~~سوف~~ راح تطلع $\frac{6}{0}$ = رح تصير قيمة غير معرفة.

also

$$\lim_{x \to \frac{1}{2}} (SQ_\mu)(x) = -\infty$$

$$(SQ_\mu)(\tfrac{1}{2}) = -\infty$$

Lemma :-

Suppose that $Sf < 0$ and $Sg < 0$ then $S(f \circ g) < 0$.

proof :- By using the chain Rule, we get

$$(f\circ g)'(x) = [f'(g(x))][g'(x)]$$

$$(f\circ g)''(x) = [f''(g(x))][g'(x)]^2 + [f'(g(x))][g''(x)]$$

$$(f\circ g)'''(x) = [f'''(g(x))][g'(x)]^3 + 2[f''(g(x))][g'(x)][g''(x)]$$
$$+ [f''(g(x))][g'(x)][g''(x)] + [f'(g(x))][g'''(x)]$$
$$= [f'''(g(x))][g'(x)]^3 + 3[f''(g(x))][g'(x)][g''(x)] + [f'(g(x))]$$
$$[g'''(x)]$$

Then

$$S(f\circ g)(x) = \frac{(f\circ g)'''(x)}{(f\circ g)'(x)} - \frac{3}{2}\left(\frac{(f\circ g)''(x)}{(f\circ g)'(x)}\right)^2$$

$$= \frac{[f'''(g(x))][g'(x)]^3 + 3[f''(g(x))][g'(x)][g''(x)] + [f'(g(x))][g'''(x)]}{[f'(g(x))][g'(x)]}$$

$$- \frac{3}{2}\left(\frac{[f''(g(x))][g'(x)]^2 + [f'(g(x))][g''(x)]}{[f'(g(x))][g'(x)]}\right)^2$$

$$= \underbrace{[(Sf)(g(x))]}_{<0}\underbrace{[g'(x)]^2}_{\geq 0} + \underbrace{(Sg)(x)}_{<0} < 0.$$

problem (H.w) : Show that If $Sf < 0$, then $Sf^n < 0$ for any positive integer n

Singer's Theorem

Let $f$ be defined on a closed interval $J$; and suppose that $J \supseteq f(J)$. Assume that $Sf < 0$ and that $f$ has $n$ critical points. Then $f$ has at most $n+2$ attracting cycles.

Corollary :- let $0 < \mu < 4$, each function in the quadratic family $\{Q_\mu\}$ has at most one attracting cycle.

proof :-

if $0 < \mu \le 1$, then the basin of attraction of $0$ is $[0,1]$

So, $0$ is the only attracting periodic point.

Assume that $1 < \mu < 4$

$\because$ $Q_\mu$ has the unique critical point $(\frac{1}{2})$, Singer's theorem implies that there can be at most 3 attracting cycles, one each associated with intervals of the form $[0,L)$, $(L,R)$ and $(R,1]$ where $0 < L < R \le 1$

$\because$ $0$ is a repelling fixed point and $Q_\mu(1) = 0$

$\therefore$ neither $[0,L)$ nor $(R,1]$ appears as a basin of attraction for cycles of $Q_\mu$.

$\therefore$ $Q_\mu$ has at most one attracting cycle.

# One dimensional Chaos

Till now we were mainly focus on periodic point and attracting periodic points, which are indicate a regularity, predictability and stability in the dynamics of a function of a parameterized family of function.

Now, onward we will focus on a contrasting dynamical action points whose iterates separate from one another. This kind of behavior is symptomatic of what we call chaotic dynamic or just chaos.

**Notation :-**

We will write $f: A \longrightarrow B$ to indicate that the domain of $f$ is $A$ and the range of $f$ is contained in $B$.

**Def :-**

let $J$ be an interval, and suppose that $f: J \longrightarrow J$. Then $f$ has (Sensitive dependence on initial conditions at $x$) or just (Sensitive dependence at $x$), if there is an $\epsilon > 0$ such that for each $\delta > 0$, there is a $y$ in $J$ and a positive integer $n$ s.t.

$$|x-y| < \delta \text{ and } |f^n(x) - f^n(y)| > \epsilon$$

**Example**

Consider the following function (Barker's function)

$$B(x) = \begin{cases} 2x & \text{for} \quad 0 \le x \le \frac{1}{2} \\ 2x-1 & \text{for} \quad \frac{1}{2} < x \le 1 \end{cases}$$

Show that after 10 iterates, the iterates of $\frac{1}{3}$ and 0.333 are farther than 1/2 apart ?

1st Solution)

| iterates (n) | $B^n(\frac{1}{3})$ | $B^n(0.333)$ |
|---|---|---|
| | 1/3 | 0.333 |
| 1 | 2/3 | 0.666 |
| 2 | 1/3 | 0.332 |
| 3 | 2/3 | 0.664 |
| 4 | 1/3 | 0.328 |
| 5 | 2/3 | 0.656 |
| 6 | 1/3 | 0.312 |
| 7 | 2/3 | 0.624 |
| 8 | 1/3 | 0.248 |
| 9 | 2/3 | 0.496 |
| 10 | 1/3 | 0.992 |

Clearly, the $10^{th}$ iterates of 1/3 and 0.333 are farther apart than distance 1/2.

(i.e)

$$|B^{10}(\tfrac{1}{3}) - B^{10}(0.333)| > \tfrac{1}{2}$$

$$|\tfrac{1}{3} - 0.992| = |-0.658|$$

$$= 0.658 > \tfrac{1}{2}$$

## Lyapunov Exponents

let $J$ be a bounded interval, and consider a function $f: J \to J$ having a continuous derivatives, we assume that for each $x$ in the interior of $J$ ($x \in int(J)$) and each small enough $\epsilon > 0$, there is a number $\lambda(x)$ s.t. for each positive integer $n$

$$|f^n(x+\epsilon) - f^n(x)| \simeq [e^{\lambda(x)}]^n \epsilon$$

$$\Rightarrow e^{n\lambda(x)} \simeq \left| \frac{f^n(x+\epsilon) - f^n(x)}{\epsilon} \right|$$

$$\Rightarrow e^{n\lambda(x)} = \lim_{\epsilon \to 0} \left| \frac{f^n(x+\epsilon) - f^n(x)}{\epsilon} \right| \Rightarrow e^{n\lambda(x)} = |(f^n)'(x)|$$

If $(f^n)'(x) \neq 0$, then by taking logarithms and dividing by $n$, we obtain

$$\lambda(x) = \frac{1}{n} \ln |(f^n)'(x)| \qquad \text{---} \; (*)$$

**Def.)**

let $J$ be a bounded interval and $f: J \to J$ continuously differentiable on $J$. Fix $x$ in $J$, and let $\lambda(x)$ be defined by

$$\lambda(x) = \lim_{n \to \infty} \frac{1}{n} \ln |(f^n)'(x)|$$

$$|(T^n)'(x)| = \lim_{\epsilon \to 0} \left| \frac{T^n(x+\epsilon) - T^n(x)}{\epsilon} \right| = 2^n$$

Thus the lyapunov exponent of $T$ is

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \ln |(T^n)'(x)|$$

$$= \lim_{n\to\infty} \frac{1}{n} \ln 2^n \Rightarrow \lim_{n\to\infty} \frac{n}{n} \ln 2$$

$$= \ln 2$$

Remarks)

1- lyapunov exponent $\lambda(x)$ can be considered to measure " the average loss of information " of successive iterates of points near $x$.

2. If $y$ is near $x$, and if the iterates of $x$ and $y$ remain close together, then $\lambda(x)$ is negative. (not sensitive)

3. If the iterates of $x$ and $y$ separate from one another then $\lambda(x)$ is positive. (sensitive)

- To make the general calculation of $\lambda(x)$ simple let $x_0 = x$ and $x_k = f^k(x)$ for $k = 1, 2, --$.

$$(f^n)'(x) = [f'(f^{n-1}(x))][f'(f^{n-2}(x))] --- [f'(f(x))][f'(x)]$$

$$= [f'(x_{n-1})][f'(x_{n-2})] --- [f'(x_1)][f'(x_0)]$$

Thus

$$\ln |(f^n)'(x)| = \ln | f'(x_{n-1}) f'(x_{n-2}) --- f'(x_1) f'(x_0)|$$

$$= \ln [ |f'(x_{n-1})| |f'(x_{n-2})| --- |f'(x_1)| |f'(x_0)| ]$$

$$= \sum_{k=0}^{n-1} \ln | f'(x_k)|$$

Therefore, Eq. (*) can be rewritten as

$$\lambda(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln | f'(x_k) |$$

Example)

Let $Q_\mu(x) = \mu x(1-x)$ for $0 \le x \le 1$, where $1 < \mu < 3$ and $\mu \ne 2$, show that $\lambda = \ln|2-\mu|$ ?

Solution) ~~becau~~ let $x \in (0,1)$ arbitrary point

we have $Q_\mu$ has one fixed point $P_\mu = 1 - \frac{1}{\mu}$ which attracts all points in $(0,1)$.

Therefore

$x_k = Q_\mu^k(x) \xrightarrow[k\to\infty]{} P_\mu$     iteration ⟶ النقطة Fixed point

$\because Q_\mu'(x) = \mu - 2\mu x$

$Q_\mu'(x_k) \longrightarrow Q_\mu'(P_\mu) = \mu - 2\mu(1-\frac{1}{\mu})$     تأخذ المشتقة لأنها تقترب الى fixed point.

$= 2-\mu$

by hypothesis $\mu \ne 2$, so that

$\ln|Q_\mu'(x_k)| \longrightarrow \ln|2-\mu|$ as $k$ increasing $k \longrightarrow \infty$

let $\epsilon > 0$, because the natural logarithum is continuous and $x_k$ approchs $P_\mu$ as $k$ increases, there is apositive integer $N$ such that     cont. الاستمرارية بالتعريف

if $k > N$ then     $x_n \to x_0$, $\forall \epsilon > 0, \exists K$ s.t $\forall n > K$ $|x_n - x_0| < \epsilon$ $\Rightarrow -\epsilon < x_n - x_0 < \epsilon \Rightarrow x_0 - \epsilon < x_n < x_0 + \epsilon$

$\ln|2-\mu| - \epsilon < \ln|Q_\mu'(x_k)| < \ln|2-\mu| + \epsilon$

Consequantly for $n > N$     induction L.H.S

$$\frac{n-N}{n}(\ln|2-\mu| - \epsilon) = \frac{1}{n}\sum_{k=N+1}^{n}(\ln|2-\mu| - \epsilon)$$

$$< \frac{1}{n}\sum_{k=N+1}^{n}\ln|Q_\mu'(x_k)|$$

$$< \frac{1}{n}\sum_{k=N+1}^{n}(\ln|2-\mu| + \epsilon)$$

$$= \frac{n-N}{n}\ln|2-\mu| + \epsilon \qquad \text{--- (1)}$$

If $n$ is sufficiently large and much larger than $N$ then $\frac{n-N}{n} \simeq 1$
So, inequality (1) becomes

$$\ln|2-\mu| - \epsilon < \frac{1}{n}\sum_{k=N+1}^{n} \ln|Q'_\mu(x_k)| < \ln|2-\mu| + \epsilon$$

Moreover for large enough $n$, we have s.t $\frac{1}{n} \longrightarrow 0$

$$\left|\frac{1}{n}\sum_{k=0}^{n} \ln|Q'_\mu(x_k)|\right| < \epsilon \quad (**)$$

Therefore, by using eq $(**)$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} \ln|Q'_\mu(x_k)| = \lim_{n\to\infty}\Big(\underbrace{\frac{1}{n}\sum_{k=0}^{N} \ln|Q'_\mu(x_k)|}_{=0} + \frac{1}{n}\sum_{k=N+1}^{n} \ln|Q'_\mu(x_k)|\Big)$$

$$\simeq \lim_{n\to\infty}(0 + \ln|2-\mu|)$$

$$= \ln|2-\mu|$$

$\therefore \lambda = \ln|2-\mu|$
is lyapunov exponent of $Q_\mu(x)$.

Remark:-

$\because \ln|2-\mu| < 0$ for $1<\mu<2$ and $2<\mu<3$
and since
$\ln|2-\mu| \longrightarrow -\infty$ as $\mu \longrightarrow 2$, thus $\lambda<0 \quad \forall\, \mu\in(1,3)$

Definition) A function $f$ is chaotic if it satisfies at least one of the following conditions:-

1- $f$ has a positive lyapunov exponent at each point in its domain that is not eventually periodic