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التحليل الرياضي   Real Analysis

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Real Analysis

Third Class

Chapter (1)

Real and rational numbers

The axiom of real numbers -:

Let $(F, +,\cdot)$ be a triple consist of a non-empty set with the operation of addition and multiplication. We say the triple $(F, +,\cdot)$ is a field if it satisfies the following properties:-

1) $a + b = b + a$ and $a \cdot b = b \cdot a$ (Commutative la)
2) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (Associative law)
3) $a \cdot (b + c) = a \cdot b + a \cdot c$ (Distributive law)
4) There is distinct real number 0 and 1 s.t $a + 0 = a$ and $a \cdot 1 = a \quad \forall a$
5) For each a there's a real number $-a$ such that $a + (-a) = 0$ and if $a \neq 0$ there is a real number $\frac{1}{a}$ such that $a \cdot \frac{1}{a} = 1$

Example-:

The real numbers from a field and the rational numbers (which are the real number that can be written as $= \frac{a}{b}$, where a and b integers and $b \neq 0$)

The order relation-:

The real numbers ordered by the relation $<$, which has the following properties:-

6) For each pair of real numbers a and b exactly one of the following is true $a = \mathrm{b}$, $a < b$, $a > b$
7) If, $a < b$ and $b < c$, then) $a < c$ (transitive)
8) If $a < b$, then $a + c < b + c$ for any c and if $c > 0$, then $a \cdot c < b \cdot c$.

A field with an order relation satisfying (8) ,(7) ,(6) is an order field. Thus the real numbers form ordered field. The rational numbers also forms an ordered field .

Supremum of a set :-

A set S of real numbers is bounded above if there is a real number b such that $x \leq b$ for each $x \in S$. In this case, b is an upper bound of S. If b is an upper bound of S, then so is any larger number, because of property (7)

If b' is an upper bound of S, but no number less than b', then b' is a supremum of S , and we write $b' = \sup(S)$.

Example:-

If S is the set of negative numbers, then any non-negative number is an upper bound of S, and $\sup(S) = 0$.

If S_1 is the set of negative integers, then any number a such that $a \geq -1$ is an upper bound of S, and $\sup(S_1) = -1$

The example shows that a supremum of a set may or may not be in the set since S_1 contains it's supremum but S dose not

Infimum of a set :-

A set S of real numbers is bounded below if there is a real number a such that, $x \geq a$ for each $x \in S$. In this case a is a lower bound of S so is any smaller number because of property (7). If a' is a lower bound of S but no number greater than a', then a' is an infimum of S, and we write $a' = \inf(S)$.

Remark :-

If S is a non-empty set of real numbers, we write $\sup(S) = \infty$ to indicate that S is unbounded above and $\inf(S) = -\infty$ to indicate that S is unbounded below.

Example:-

Let, $S = \{x: x < 2\}$, then $\sup(S) = 2$ and . $\inf(S) = -\infty$

Example:-

Let, $S = \{x: x \geq 2\}$, then $\sup(S) = \infty$ and $\inf(S) = -2$.

If S is the set of all integers, then $\sup(S) = \infty$ and $\inf(S) = -\infty$

H.W): Find $\sup(S)$ and $\inf(S)$, state whether they are in S.

1- $S = \{x: x^2 \leq 5\}$

2- $S = \{x: x^2 > 9\}$

3- $S = \{x: |2x + 1| < 7\}$

The relation between the field of rational of numbers and real number:

Proposition (1-1):-

Every orderd field contains a subfield similar to field of rational numbers.

Proof:- Let $(F, +, \cdot)$ be an orderd field $1 \in F$ (1 is the identity element with respect to $(\cdot)$ operation) ($0 \in F$, is the identity of $+$) $1 + 1 + 1 + \cdots + 1 = n \cdot 1 = n \in F, \quad n \in Z^+$

Claim (1) $n \cdot 1 = 0 \quad iff \quad n = 0$

Proof (1)⇒) Suppose the result is not true i.e there exists a positive integer $k \geq 1$.and $k \cdot 1 = 0$ It's clear that $k > 1 \Rightarrow k - 1 > 0$ and $(k-1) \cdot 1 > 0$
$0 < (k-1) \cdot 1 < k \cdot 1 = 0$ C! (since $0 < 1$)

∴ The result is not true.

⇐)Trivial.

Claim (2) $n \cdot 1 = m \cdot 1 \quad iff \quad n = m$

Proof :⇐) if $n = m$ clearly $n \cdot 1 = m \cdot 1$.

⇒) If $n \cdot 1 = m \cdot 1 \quad \Rightarrow \quad n \cdot 1 + (-m \cdot 1) = 0 \quad \Rightarrow \quad (n + (-m) \cdot 1) = 0$.
Then by (1) $n - m = 0 \quad \Rightarrow \quad n = m$. Thus $\mathrm{N} \subset \mathrm{F}$ (F Contains a copy of Z).

$\forall\, \mathrm{n} \in \mathrm{F}$ (∵ F.is a group), $\exists -n \in F$ such that $n + (-n) = 0$, hence $\mathrm{Z} \subset \mathrm{F}$ (F Contains a copy of Z)

$\forall\, \mathrm{n} \neq 0, \mathrm{n} \in \mathrm{F}$ (∵ F is a field), ∴ $\exists\, \frac{1}{n} \in F$ such that $\left(\frac{1}{n}\right) \cdot n = 1$.

$\forall\, \mathrm{m} \in \mathrm{F}$, $\left(\frac{1}{n}\right) \cdot m = \frac{m}{n} \in \mathrm{F}$ (binary operation).

$\mathrm{Q} \subset \mathrm{F}$ (F Contains a copy of Q).

Corollary (1-2):-

$\mathrm{Q} \subseteq \mathrm{R}$

$(R, +, \cdot, \leq)$ orderd field, $1 + 1 + 1 + \cdots + 1 = n \cdot 1 = n \in R$,

.

Q/ Is $Q = R$.

To answer this question, we beginning by this proposition:

Proposition (1-3):-

The equation $x^2 = 2$ has no solution in Q.

Proof: Suppose the result is not true i.e the equation $x^2 = 2$ has a root in Q say $\frac{a}{b}$, $b \neq 0$, $a, b \in \mathrm{Z}$ and the greatest common divisor $(a, b) = 1$, $\frac{a^2}{b^2} = 2 \Rightarrow a^2 = 2b^2$.

- If a and b are odd, then $a^2(odd) = 2b^2(even)$ C! .
- If a is odd and b is even, $(i.e\ b = 2m,\ m \in Z\ )$, then $a^2(odd) = 2(2m)^2 = 8m^2 = 2(4m^2)(even)$ C! .
- If a is even and b is odd, $(i.e\ a = 2n,\ n \in Z\ )$ (, then $(2n)^2 = 2b^2 \Rightarrow 4n^2 = 2b^2 \Rightarrow 2n^2(even) = b^2(odd)$ C!.
- If a and b are even, $(i.e\ a = 2n,\ n \in Z\ ,\ b = 2m,\ m \in Z\ )$ (,then $4n^2 = 8m^2 \Rightarrow n^2(even)\ or\ (odd) = 2m^2(even)$ C!..

So that there is no rational number satisfy this equation.

H.W:-

The equation $x^2 = 3$ has no solution in Q.

Proposition (1-4):-

The equation $x^2 = 2$ has only one real positive root.

Proof: Let $S = \{x \in Q : x > 0\ ,\ x^2 < 2\} \neq \emptyset$. S is bounded above (2,3, upper bound of S). $S \neq \emptyset$, since $1 \in S$, ($1 > 0$ and $1^2 < 2$,

Since R is complete orderd field, then by (completeness property: Every non empty subset of R has an upper bound, then it has $l.u.b = sup$), then S has a least upper bound say y. $y = l.u.b(S) = \sup(S)$

Claim:- $y^2 = 2$, (i.e the least upper bound of S is a root of equation $x^2 = 2$).

If not, then either $y^2 > 2$ or $y^2 < 2$.

1. If $y^2 < 2$, take $0 < h < 1$,

$(y + h)^2 = y^2 + 2yh + h^2 < y^2 + 2yh + h$

$(y + h)^2 < y^2 + h(2y + 1)$

Choose h satisfies: $0 < h < \frac{2-y^2>0}{2y+1>0} < 1$

$\Rightarrow\ h(2y + 1) < 2 - y^2 \ \Rightarrow\ y^2 + h(2y + 1) < 2$

Hence $(y + h)^2 < y^2 + h(2y + 1) < 2$. Thus $(y + h)^2 < 2$

2. If $y^2 > 2$, take $0 < k < 1$

$(y - k)^2 = y^2 - 2yk + k^2 > y^2 - 2yk + k$

$(y - k)^2 > y^2 - k(2y + 1)$

Choose k satisfies: $0 < k < \frac{y^2-2}{2y+1} < 1$

$\Rightarrow\ k(2y + 1) < y^2 - 2 \ \Rightarrow\ 2 < y^2 - k(2y + 1) < (y - k)^2$

Hence $2 < (y - k)^2$, since $y > y - k$

Uniqness:

Let $\exists\, z \in R$ such that $z^2 = 2$ and $z \neq y$. Then either $z < y$ or $z > y$
$(2 < 2)$ $C!$

Thus $z = y$.

Corollary (1-5):-

$Q \subsetneq R$. (The field of rational numbers Q is proper subfield of the field of real numbers R).

Proof: $\sqrt{2} \in R$, from (1.4).

$\sqrt{2} \notin Q$, from (1.3).

Corollary (1-6):-

Q is not complete orderd field.

Proof: Let $S = \{x \in Q : x > 0\ ,\ x^2 < 2\} \subset Q$.

$Sup(S) = l.u.b(S) = \sqrt{2} \notin Q \quad C!$.

Thus Q is not complete orderd field.

Remark (1-7):-

Let $Q' = R - Q$ denote the set of irrational numbers, $R = Q \cup Q'$. Q' is complete orderd field $\left(\sqrt{2} \in Q'\right) \Rightarrow \ (Q \neq Q')$.

Now, we study the set Q' and how we distribute the elements of Q and the element of Q' in R. We start by the following theorem:

Theorem (1-8) : (Archimedean property):-

For each real numbers a and b, $a > 0$ there exists a positive integer n such that $na > b$

Proof: Suppose the result is not true $\forall n \in Z^+ \ na \leq b$

Consider the set $S = \{ka : k \in Z^+\} \neq \emptyset$, $(1.a \in S)$, then S is bounded above by b. Since $S \subseteq R$, then by the completeness of real numbers S has a least upper bound in R say $y = l.u.b.(S) = Sup(S)$

Since $a > 0$, then $y - a < y$, hence $y - a$ is not upper bound of S, then $\exists\, m \in Z^+$ such that $m.a \geq y - a$, $(m.a \in S)$, $a(m+1) \geq y$ $C!$,
$m + 1 \in Z^+$, $(m+1).a \in S$, $y = Sup(S)$.

Thus the result is true.

Corollary (1.9):- $\forall \epsilon > 0$, there exists a positive integer n such that $\frac{1}{n} < \epsilon$.

Proof: Take $b = 1$, $a = \epsilon$. By (1.8) $\exists\, n \in Z^+$ such that $(n\epsilon > 1) \div n$,
hence $\frac{1}{n} < \epsilon$.

Theorem (1.10)-: (The density of rational numbers)

For each real numbers a and b with $a < b$, there exists a rational number r between a and b $(a < r < b)$

Proof: $0 < a < b$ → $0 < b - a < 1$

→ $b - a \geq 1$ $\cdots (1)$

(1) If $b - a \geq 1$

Define $S = \{n \in N : n \cdot 1 > a\} \neq \emptyset, (1, a \in R\ , \exists n \in Z^+\ s.t\ n \cdot 1 > a\ Arch.)$.

Choose k be the smallest positive integer satisfies $k \cdot 1 > a$ $i.e$ $k - 1 < k$

$\Rightarrow$ $k - 1 \leq a$ $\cdots (2)$

From (1) and (2) $b - a \geq 1$ $\Rightarrow$ $b \geq a + 1$

$k-1 \leq a \;\Rightarrow\; k \leq a+1 \;\Rightarrow a < k \leq a+1 < b \quad \therefore k$ is the rational number between a and b.

If $b-a<1$, then $\exists\, n \in Z^{+}$ such that $n(b-a)>1 \;\Rightarrow\; nb-na>1$, hence by the previous result, $\exists\, n \in Z^{+}$ such that $na < k < nb\,] \div n$

$\Rightarrow a < \frac{k}{n} < b$, $\therefore \frac{k}{n}$ is the rational number between a and b.

(2) $a < 0 < b$

$\therefore$ 0 is the rational number between a and b.

(3) $a < b < 0 \;\;] \times (-1) \;\Rightarrow\; 0 < -b < -a \;\Rightarrow\; -a > -b > 0$

And by (1) there exists a rational number $-b < r < -a \quad \Rightarrow$

$a < -r < b$

$\therefore r$ is the rational number

Corollary (1-11)-:

For each real numbers a and b there exists an infinite countable set of rational numbers between a and b

Proof: $a < b$ by (1.10) $\exists\, r_1 \in Q \quad s.t \quad a < r_1 < b$.

$a < r_1$ by (1.10) $\exists\, r_2 \in Q \quad s.t \quad a < r_2 < b$

And $\exists\, r_2' \in Q \quad s.t \quad r_1 < r_2' < b$

Generally $\exists\, r_n \in Q$ between a and r_{n-1} and r_n' between r_{n-1} and b.

Thus we have infinite countable set between a and b

Theorem(1.12):- The density of irrational number

For each real numbers a and b with $a < b$, there exists an irrational number s between a and b.

Proof: Suppose the result is not true i.e between a and b there is only rational number by (1.10) $(a < r < b)$

$\sqrt{2} \notin Q$, $\sqrt{2} \in Q'$ $\Rightarrow$ $a + \sqrt{2} < b + \sqrt{2}$ $\Rightarrow$

$a + \sqrt{2} < r + \sqrt{2} < b + \sqrt{2}$

$r + \sqrt{2} \in Q'$, If $(r \in Q\ ,\ s \in Q', then\ r + s \in Q')$, hence a contradiction

Corollary (1.13)-:

For any real numbers a and b there exists an infinite countable set of irrational numbers between a and b.

Proof :$a < b$ by (1.12) $\exists\, s_1 \in Q'$ $s.t$ $a < s_1 < b$.

$a < s_1$ by (1.12) $\exists\, s_2 \in Q'$ $s.t$ $a < s_2 < b$

And $\exists\, s_2' \in Q'$ $s.t$ $s_1 < s_2' < b$

Generally $\exists\, s_n \in Q'$ between a and s_{n-1} and s_n' between s_{n-1} and b.

we have infinite countable set $\{s_1 , s_2, s_2', \cdots\}$ between a and b

Example:- $.1.25 < 1.50$

$1.50 - 1.25 = 0.25$, by Arch., then $\exists\, n \in Z^+$ $s.t$ $n(0.25) > 1$

$10(1.25) < k < 10(1.50)$ $(choose\ n = 10) \Rightarrow$ $12.5 < k < 15$ $\Rightarrow$
$k = 13$. The number is $\frac{13}{10}$

Chapter(2)

The sequences of real numbers

Definition(2.1) :-

Let $f: N \to R$ be a function, then $f(n) = a_n \quad \forall\, n \in Z$, is called a sequence of real numbers which will be denoted by $\langle a_n \rangle$ or $\{a_n\}$.

$$\langle a_n \rangle = a_1 , a_2 , a_3 , \dots , a_n , \dots$$

Examples:-

1. $\langle \frac{1}{n} \rangle = 1 , \frac{1}{2} , \frac{1}{3} , \frac{1}{4} , \dots , \frac{1}{n} , \dots$
2. $\langle \frac{1}{2^n} \rangle = \frac{1}{2} , \frac{1}{4} , \frac{1}{8} , \dots , \frac{1}{2^n} , \dots$
3. $\langle (-1)^n \rangle = -1 , 1 , -1 , \dots , (-1)^n , \dots$
4. $\langle 3^n \rangle = 3 , 9 , 81 , \dots , 3^n , \dots$
5. $\langle \frac{1}{2} \rangle = \frac{1}{2} , \frac{1}{2} , \frac{1}{2} , \dots , \frac{1}{2} , \dots$
6. $\langle \frac{n}{n+1} \rangle = \frac{1}{2} , \frac{2}{3} , \frac{3}{4} , \dots , \frac{n}{n+1} , \dots$

Converging sequences:

Definition(2.2) :-

Let $\langle a_n \rangle$ be a sequence of real numbers ,we say that $\langle a_n \rangle$ is converging sequence if there exists a real number a_0 satisfies for all $\in > 0$ $(0 < \in <)$ there exist a positive integer $k = k(\in)$ (depend on $\in$) such that $|a_n - a_0|$ i.e if $a_n \to a_0$, then $\lim_{n\to\infty} a_n = a_0$.

Otherwise the sequence is divergence.

Proposition (2-3):-

If the sequence $\langle a_n \rangle$ is convergence sequence, then the limit point is unique.

Proof: Suppose that $a_n \to a_0$ and $a_n \to b_0$ such that $a_0 \neq b_0$, then $0 < d = |a_n - a_0|$. Since $a_n \to a_0$

$\forall \in > 0$, in particular take $\in = \frac{d}{2}$, $\exists\, k_1\left(\frac{d}{2}\right)$ such that $|a_n - a_0| < \frac{d}{2}$ $\forall\, n > k_1$.

Since $a_n \to b_0$

$\forall\ \frac{d}{2} > 0$, $\exists\, k_2\left(\frac{d}{2}\right)$ such that $|a_n - b_0| < \frac{d}{2}$ $\forall\, n > k_2$.

$0 < d = |a_0 - b_0| = |a_0 - a_n + a_n - b_0|$.

$. \leq d = |a_n - n_0| + |a_n - b_0|$.

$< \frac{d}{2} + \frac{d}{2} = d\ \ C!\ (d < d)$, $\forall\, n > k = max\{k_1 , k_2\}$.

Examples:-

1) Is $\langle \frac{1}{n} \rangle$ converge to 0

$\langle \frac{1}{n} \rangle = 1 , \frac{1}{2} , \frac{1}{3} , \frac{1}{4} , \dots , \frac{1}{n} , \dots$

Let $\in > 0$, to find $k(\in)$ such that: $\left|\frac{1}{n} - 0\right| < \in$ $\forall\, n > k$

Proof: $\left|\frac{1}{n}\right| = \frac{1}{n}$, since $n \in Z^+$.

By Archimedean $\forall \in > 0$, $\exists\, k \in Z^+$ such that $\frac{1}{k} < \in$, $\frac{1}{n} < \frac{1}{k} < \in \quad \forall\, n > k_2$

Thus $\left|\frac{1}{n} - 0\right| < \in \quad \forall\, n > k$

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2) Is $\langle a_n \rangle = \langle 3 \rangle$ converge to 3,

$f : N \to R$, $f(n) = a_n = 3$, $\langle 3 \rangle = 3{,}3{,}3, \ldots$

$\forall \in > 0$, $\exists\, k = 0$, $|3 - 3| = 0 \quad \forall\, n > 0$.

3) Let $\langle a_n \rangle$ be define by:

$$a_n = \{\begin{matrix} -2 & n > 10^7 \\ n & n \le 10^7 \end{matrix}$$

$\langle a_n \rangle = 1{,}2{,}3{,}4{,}5, \ldots , 10^7 , -2 , -2 , \ldots$

This sequence convergence to (-2).

$\forall \in > 0$, $\exists\, k = 10^7$, $|\, a_n - (-2)| < \in \quad \forall\, n > 10^7$.

4) Let $\langle a_n \rangle = \langle (-1)^n \rangle$ be a divergence sequence.

$\langle (-1)^n \rangle = -1 , 1 , -1 , 1 , \ldots$

If $a_0 = -1$, then for all $\in > 0$, $(-1 - \in , -1 + \in)$ contain all odd terms but doesn't contain any even term and since the even terms are infinite, then $a_n \nrightarrow -1$.

If $a_0 = 1$, then for all $\in > 0$ $(1-\in, 1+\in)$ contain all even terms but doesn't contain any odd term and since the odd terms are infinite, then $a_n \nrightarrow 1$.

If $a_0 \neq 1$ or $a_0 \neq -1$

$0 < d_1 = |a_0 - 1|$, $0 < d_2 = |a_0 - (-1)|$.

If we choose $\in \leq min\{d_1, d_2\}$, then any open interval $(a_0-\in, a_0+\in)$ doesn't contain any term of the sequence and hence $a_n \nrightarrow a_0$.

Thus $\langle(-1)^n\rangle$ is a divergence sequence.

H.W: Which of the following sequence convergence or divergence.

1. $\langle\frac{n}{n+1}\rangle$.
2. $\langle\frac{1}{2^n}\rangle$.
3. $\langle 3^n\rangle$.

Bounded sequences:

Definition(2.4) :-

A sequence $\langle a_n\rangle$ of real numbers is said to be a bounded sequence ,if there exists a real number M such that $|a_n| \leq M \quad \forall n$, $M \leq a_n \leq M$.

. Examples-:

1. $\langle a_n\rangle = \langle\frac{1}{n}\rangle$ is bounded sequence since $-1 \leq 0 \leq \frac{1}{n} \leq 1$.
2. $\langle a_n\rangle = \langle 3\rangle$is bounded sequence since $-3 \leq 3 \leq 3$.
3. $\langle a_n\rangle = \{\begin{matrix} -2 & n > 10^7 \\ n & n \leq 10^7 \end{matrix}$

$\langle a_n\rangle = 1,2,3,4,5, \dots, 10^7, -2, -2, \dots$

This sequence is bounded since $-10^7 \leq a_n \leq 10^7$.

4. $\langle a_n \rangle = \langle (-1)^n \rangle = -1, 1, -1, 1, \dots$ is bounded sequence since $-1 \leq a_n \leq 1$.

4. $\langle a_n \rangle = \langle 2^n \rangle = 2, 4, 8, 16, \dots, 2^n, \dots$ is not bounded sequence since $0 \leq 2^n \leq ?$. (bounded below but not bounded above).

Proposition (2-5):-

Every convergence sequence is a bounded sequence.

Proof: Let $\langle a_n \rangle$ be a convergence sequence, that convergence to a_0

i.e $a_n \to a_0$

$\forall \in > 0$, $\exists\, k = k(\in)$, such that , $|a_n - a_0| < \in < 1 \quad \forall n > k$.

$|a_n| - |a_0| \leq |a_n - a_0| < 1 \quad \forall n > k.$

Then $|a_n| - |a_0| \leq 1 \quad \forall n > k$

Hence $|a_n| \leq |a_0| + 1 \quad \forall n > k.$

$|a_1|, |a_2|, \dots, |a_k|, |a_{k+1}|, |a_{k+2}|, \dots \leq |a_0| + 1 \quad \forall n > k$

Take $M = \{|a_1|, |a_2|, \dots, |a_k|, |a_{k+1}|, |a_{k+2}|, \dots, |a_0| + 1 \}.$

$|a_n| \leq M \quad \forall n.$

Example:-

$\langle 2^n \rangle = 2, 4, 8, 16, \dots, 2^n, \dots$ is not bounded sequence and by this theorem is divergence.

Remark(2.6):-

The converse of proposition (2.5) is not true in general, as the following example shows.

Example:-

$\langle(-1)^n\rangle$ is bounded sequence which is a divergence sequence.

Monotonic sequences:

Definition(2.7) :-

Le $\langle a_n\rangle$ be a sequence, we say that $\langle a_n\rangle$ is a non- decreasing sequence ,if $a_n \leq a_{n+1} \quad \forall\, n$.

$\langle a_n\rangle$ is an increasing sequence, if $a_n < a_{n+1} \quad \forall\, n.$

$\langle a_n\rangle$ is a non- increasing sequence, if $a_n \geq a_{n+1} \quad \forall\, n.$

And $\langle a_n\rangle$ is a decreasing sequence ,if $a_n > a_{n+1} \quad \forall\, n.$

And we say that $\langle a_n\rangle$ is a monotonic sequence ,if $\langle a_n\rangle$ satisfies one of the above conditions .

Examples-:

1) $\langle\frac{1}{n}\rangle = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots$ is decreasing sequence.
2) $\langle\frac{n}{n+1}\rangle = \frac{1}{2}, \frac{2}{3}, \frac{31}{4}, \dots, \frac{n}{n+1}, \dots$ is an increasing sequence.
3) $\langle 3\rangle = 3, 3, 3, \dots, 3, \dots$ is a non- increasing sequence and a non- decreasing sequence.
4) $\langle(-1)^n\rangle = -1, 1, -1, 1, \dots$ is not monotonic sequence.

Proposition (2-8):-

Every bounded monotonic sequence is convergence sequence.

Proof: Let $\langle a_n \rangle$ be a sequence in , since $\langle a_n \rangle$ is bounded sequence. $\exists\, M$, such that $|a_n| \leq M \quad \forall n$.
$S = \{a_n : n \in N\}$ bounded (above and below).

1) Suppose $\langle a_n \rangle$ is a non- decreasing sequence,

Since S is bounded above, then by completeness of real number S has a least upper bound say y.

$y = \sup(S) = l.u.b(S) \quad a_n \leq y \quad \forall\, n \in N.$

Claim: $a_n \to y$

$y - \frac{\epsilon}{2} \leq y$ *then* $y - \frac{\epsilon}{2}$ is not an upper bound.

$\exists\, k \in Z^+$ such that $a_k > y - \frac{\epsilon}{2}$

$y - \frac{\epsilon}{2} < a_k \leq a_n < y + \frac{\epsilon}{2}$

$y - \frac{\epsilon}{2} < a_n < y + \frac{\epsilon}{2} \quad \forall\, n > k$

$|a_n - y| < \frac{\epsilon}{2} \qquad \forall\, n > k.$

(2) Suppose $\langle a_n \rangle$ is a non- increasing sequence,

i.e $\exists\, M$, such that $|a_n| \leq M \quad \forall n$.
Since S is bounded below, where $S = \{a_n : n \in N\}$, then by completeness of real number S has greatest lower bound, say a_0.

Claim: $a_n \to a_0$ ($\forall \epsilon > 0$, $\exists\, k \in Z^+$ such that $|a_n - a_0| < \epsilon \;\; \forall\, n > k$).

$a_0 = \inf(S) = g.l.b(S) \quad a_n \leq a_0 \quad \forall\, n \in N$ ... (1).

$a_0 + \in$ is not a lower bound (since $a_0 < a_0 + \in$).

$\exists\, k \in Z^+$ such that $a_k < a_0 + \in$ **...(2)**

Since $\langle a_n \rangle$ is a non- increasing sequence, then $a_n < a_k$ ...(3).

From (1), (2), (3) $a_0 - \in \leq a_n < a_k < a_0 + \in \quad \forall\, n > k$.

$a_0 - \in \leq a_n \leq a_0 + \in \quad \forall\, n > k.$

Then $|a_n - a_0| < \frac{\in}{2} \quad \forall\, n > k.$

Thus $\langle a_n \rangle$ is converges.

Examples:-

1. $\langle a_n \rangle = \langle \frac{1}{n} \rangle = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots$ ..

$S = \{a_n : n \in N\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\right\}.$

This sequence is decreasing and bounded (below, above).

$a_n \to g.l.b(S) = \{0\}.$

2. Converges $\nRightarrow$ monotonic.

5. Let $\langle a_n \rangle = \begin{Bmatrix} \langle a_n \rangle & n > 10^2 \\ -1 & n \leq 10^2 \end{Bmatrix}$

$\langle a_n \rangle = 1,2,3,4,5, \dots, 10^2, -1, -1, -1, \dots$

It is converges but not monotonic sequence.

Cauchy sequences:

Definition(2-10) :-

A sequence $\langle a_n \rangle$ is called a Cauchy sequence if $\forall \in > 0$ there exist a positive integer $k = k(\in)$ such that $|a_n - a_0| < \in \quad \forall\, n, m > k$.

Proposition (2-11)

Every convergence sequence in R or Q is a Cauchy sequence.

Proof: Let $\langle a_n \rangle$ be a convergence sequence, that convergence to a_0

i.e $a_n \to a_0$

$\forall \in > 0$, $\exists\, k = k(\in)$ such that $|a_n - a_0| < \frac{\in}{2} \quad \forall\, n > k$.

$|a_n - a_m| = |a_n - a_0 + a_0 - a_m|$.

$\leq |a_n - a_0| + |a_m - a_0|$.

$< \frac{\in}{2} + \frac{\in}{2} = \in \qquad \forall\, n > k\ ,\ \forall\, m > k$.

Thus $|a_n - a_m| < \in \qquad \forall\, n, m > k$

Remark(2-12) :

The converse of Proposition (2-11) is not true in general in the field of rational number.

We need the following lemma:

Lemma (2-13):-

For any real number r, there exists a sequence of rational number converge to r.

Proof: Let $r \in R$ $\qquad\qquad r - 1 < r + 1$

By the density of rational numbers $\exists\, r_1 \in Q$ such that

$r - 1 < r_1 < r + 1$, then $r - \frac{1}{2} < r + \frac{1}{2}$

And by the density of rational numbers $\exists\, r_2 \in Q$ such that

$r - \frac{1}{2} < r_2 < r + \frac{1}{2}$.

Continue in this way we get a sequence of rational numbers $\langle r_n \rangle$

$r - \frac{1}{n} < r_n < r + \frac{1}{n} \qquad \forall\, n \in N$ ... (*)

<u>Claim :</u> $r_n \to r$ from (*) $|r_n - r| < \frac{1}{n}$.

(Arch.) $\forall \in > 0$, $\exists\, k = k(\in)$ such that $\frac{1}{k} < \in$

$|r_n - r| < \frac{1}{n} < \frac{1}{k} < \in \quad \forall\, n > k \quad (\forall\, n > k \Rightarrow \frac{1}{n} < \frac{1}{k}$

Thus $|r_n - r| < \in$

i.e $r_n \to r$

Remark (2.12) :-

The converge of proposition (2.11) in general is not true in Q.

Proof: Let $r = \sqrt{2} \notin Q$

then by lemma (2.13) $\exists$ a sequence of rational numbers $\langle r_n \rangle$ such that $r_n \to \sqrt{2}$, since $r_n \to \sqrt{2}$, then by proposition (2.11) $\langle r_n \rangle$ is a Cauchy sequence, but $\langle r_n \rangle$ is not converges in Q

H.W:

(1) $\langle \frac{1}{n} \rangle \quad R - \{0\}$

(2) For any real number there exists a sequence of irrational numbers converge to r.

Theorem (2.14):- (The nested intervals theorem)

Let $\langle I_n \rangle$ be a sequence of closed intervals such that $I_{n+1} \subseteq I_n \ \forall\, n$. Then $\cap_n I_n \neq \emptyset$.

Moreover if $\langle |I_n| \rangle$ converges to zero, then $\cap_n I_n$ consists of only one point.

Proof: Let $\langle I_n \rangle = [a_1, b_1], [a_2, b_2], \dots, [a_n, b_n], \dots$

Let $S_1 = \{a_1, a_2, \dots, a_n, \dots\}$, $S_2 = \{b_1, b_2, \dots, b_n, \dots\}$

$\forall\, n \ \ I_{n+1} \subseteq I_n \Rightarrow if\ n \leq m \Rightarrow a_n \leq a_m \ ,\ b_n \leq b_m$.

$if\ n > m \Rightarrow a_n < a_m < b_n < b_m$. Then $a_n < b_n$.

So that each element in S_2 is an upper bound of S_1. Thus S_1 is bounded above.

By completeness of real numbers S_1 has a least upper bound say y.
$y = \sup(S_1)$

$a_n \leq y \quad \forall\, n \in N$ and $y < b_n \ \ \forall\, n \in N \ \ (y = \mathrm{l.u.b}(S_1))$

$a_n < y < b_n \quad \forall\, n$. Hence $y \in \cap_n I_n$. Thus $\cap_n I_n \neq \emptyset$.

-If $\langle |I_n| \rangle \to 0$

Suppose, there exists another point z, such that $z \in \cap_n I_n$ and $y \neq z$

$0 < d = |y - z|$

Since $\langle |I_n| \rangle \to 0$ then $\exists\, k \in Z^+$ such that $|I_k| < d$

$0 < d = |y - z| \leq |I_k| < d \quad C!$

Thus $y = z$

Remark (2.15):

In general theorem (2.14) is not true if the interval is not closed. As the following example show:

Example: $I_n = \left(0\,,\, \frac{1}{n}\right) \ \forall\, n$

$\cap_n I_n = \emptyset$?

If $\cap_n I_n = \{y\}$

$\forall\, y > 0\,,\ \exists\, k \in Z^+$ such that $0 < \frac{1}{k} < y \quad C!$?

i.e $y \notin I_k = \left(0\,,\, \frac{1}{k}\right)$ thus $\cap_n I_n = \emptyset$

.

Completeness of real numbers

Every Cauchy sequence in R is converging in R.

Proposition (2.17):

Every Cauchy sequence is a bounded sequence.

Proof: let $\langle a_n \rangle$ be a Cauchy sequence, **i.e** $\forall \in > 0\,,\ \exists\, k = k(\in)$ such that $|a_{n-}a_m| <\in \quad \forall\, n\,, m > k$

In particular take $m = k + 1$

$|a_n| - |a_{k+1}| \leq |a_{n-}a_{k+1}| < \in < 1 \quad \forall\, n > k$

$|a_n| < |a_{k+1}| + 1 \quad \forall n > k$

Take $M = \max\{ |a_{k+1}| + 1 , |a_1| , |a_2| , \dots , |a_k| \}$

Thus $|a_n| \leq M \quad \forall n$.

Proposition (2.18):

Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be two convergence sequences such that $a_n \to a_0$ and $b_n \to b_0$, then:

1. $a_n \mp b_n \to a_0 \mp b_0$.
2. $a_n \cdot b_n \to a_0 \cdot b_0$.
3. $c \cdot a_n \to c \cdot a_0 \quad \forall c \in R$
4. $\frac{a_n}{b_n} \to \frac{a_0}{b_0} \qquad b_n \neq 0 \quad \forall n \ , \ b_0 \neq 0.$

Proof: (4)

Since $a_n \to a_0$ then $\forall \in > 0 , \exists\, k_1 = k_1(\frac{\in}{2})$ such that $|a_{n-}a_0| < \frac{\in |b_0|}{2} \quad \forall n > k_1$

Since $b_n \to b_0$ then $\forall \in > 0 , \exists\, k_2 = k_2(\frac{\in}{2})$ such that $|a_{n-}a_0| < \frac{\in M_2 |b_0|}{M_1} \quad \forall n > k_2$

Since $\langle a_n \rangle$ is converge, then $\exists\, M_1$ s.t $|a_n| \leq M_1 \quad \forall n$.

Since $\langle b_n \rangle$ is converge, then $\exists\, M_2$ s.t $|b_n| \leq M_2 \quad \forall n$.

$$\left| \frac{a_n}{b_n} - \frac{a_0}{b_0} \right| = \left| \frac{b_0\, a_n - a_n b_n + a_n b_n + a_0 b_n}{b_n b_0} \right|$$

$$\leq \frac{|a_n|\,|b_n - b_0|}{|b_n||b_0|} + \frac{|b_n||a_n - a_0|}{|b_n||b_0|} .$$

$< \frac{M_1}{M_2} \cdot \frac{\in M_2|b_0|}{M_1|b_0|} + \frac{\in|b_0|}{2|b_0|} \quad \forall\, n \;>\; k_1$

$= \;\frac{\in}{2} + \frac{\in}{2}$

$= \in \quad \forall\, n \;> k = max\,\{k_1\,,\; k_2\}.$

Countable sets

Q is countable set.

Proposition (2.19):

R is not countable set.

Proof: Let $S = \{a_1\,, a_2\,, \dots, a_n\,, \dots\} \subseteq R$ be a countable set $(S \neq R\,)$

Let I_1 be a closed interval in R such that $|I_1| < 1$ and $a_1 \notin I_1$.

Let I_2 be a closed interval in R such that $|I_2| < \frac{1}{2}$ and $a_2 \notin I_2$ and $I_1 \supseteq I_2$.

Let I_n be a closed interval in R such that $|I_2| < \frac{1}{n}$ and $a_n \notin I_n$ and

$I_{n-1} \supseteq I_n\; I_1\,, I_2\,, I_3\,, \dots, I_n\,, \dots,\quad |I_n| \;\to 0$ and $\left|\frac{1}{n}\;\right| \to 0$, by nested theorem $\cap_n I_n \;= \{y\}\quad,\; y \in R$

$y \in I_n \quad \forall\, n$ and $y \neq a_n \quad \forall n$. Then $y \notin S$. Thus $S \neq R$

Corollary (2.20):

The set of irrational number is uncountable set. (The union of two countable set is countable)

Proof: If not, then $R = Q \cup Q'$, then countable C!

Thus Q' is not countable.

Chapter (4)

The metric spaces:

Definition(4.1):

An order pair (X, d) is called a metric space if X is a non-empty set and d is a function

$$d: X \times X \rightarrow R$$

Satisfies:

1) $d(x, y) \geq 0 \quad \forall\, x, y \in X$

2) $d(x, y) = 0 \quad iff \quad x = y \quad \forall x, y \in X$

3) $d(x, y) = d(y, x) \quad \forall x, y \in X$

4) $d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X$

d is called the distance function, and the elements of X are called the element of the space.

Examples(4.2):

1) (R, d); R the set of real numbers and $d: R \times R \to R$

is defined by $d(x, y) = |x - y| \quad \forall x, y \in R$

1. $d(x, y) = |x - y| \geq 0 \quad \forall x, y \in R$

2. $d(x, y) = |x - y| = 0 \ iff \ x - y = 0 \ iff \ x = y \quad \forall x, y \in R$

3. $d(x, y) = |x - y| = |-(y - x)| = |(-1)(y - x)| = |y - x| = d(y, x) \quad \forall x, y \in R$

4. $d(x, y) = |x - y| = |x - z + z - y| \leq |x - z| + |z - y|$

$\leq d(x, z) + d(z, y) \quad \forall x, y, z \in R$

$\therefore (R, d)$ is a metric space.

2) If $X = R^n$ such that

$R^n = \{x = (x_1, x_2, \cdots, x_n): x_i \in R\}$.

If $x = (x_1, x_2, \cdots, x_n) \in R^n$, $y = (y_1, y_2, \cdots, y_n) \in R^n$.

Defined: $d: R^n \times R^n \to R$ by:

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = \|x - y\| \quad \forall x, y \in R^n$$

1. $\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} \geq 0$

2. $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = 0 \quad iff \quad \sum_{i=1}^{n}(x_i - y_i)^2 = 0$

$iff \quad (x_i - y_i)^2 = 0 \quad iff \quad x_i = y_i \quad \forall\, i = 1, 2, \cdots, n \quad iff \quad x = y$

3. $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2} = d(y, x)$

To prove (4) we need the following:

Lemma (4.3): The Cauchy - Schwarz inequality

For each real numbers $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ we have:

$$|a_1 b_1 + a_2 b_2 + \cdots + a_n b_n| \leq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \cdot \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$$

Lemma (4.4):

For each real numbers $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ we have:

$$|(a_1 + b_1)^2 + (a_2 + b_2)^2 + \cdots + (a_n + b_n)^2| \leq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} + \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$$

Proof:

$$(a_1 + b_1)^2 + (a_2 + b_2)^2 + \cdots + (a_n + b_n)^2 = (a_1^2 + a_2^2 + \cdots + a_n^2) + 2(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n) + (b_1^2 + b_2^2 + \cdots + b_n^2)$$

By lemma (4.3)

$$\leq (a_1^2 + a_2^2 + \cdots + a_n^2) + 2\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \cdot \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$$

$$+(b_1^2 + b_2^2 + \cdots + b_n^2)$$

$$\therefore \sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2 + \cdots + (a_n + b_n)^2}$$

$$\leq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} + \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$$

4. $d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$

Let $z = (z_1, z_2, \cdots, z_n)$

$= \sqrt{\sum_{i=1}^{n}(x_i - z_i + z_i - y_i)^2} \leq \sqrt{\sum_{i=1}^{n}(x_i - z_i)^2} + \sqrt{\sum_{i=1}^{n}(z_i - y_i)^2}$

[By lemma (4.4)]

$\therefore d(x,y) \leq d(x,z) + d(z,y)$.

3) Let X is a non-empty set define:

$$d: X \times X \to R$$

By:

$$d(x,y) = \begin{cases} 0 & if \quad x = y \\ \dfrac{1}{3} & if \quad x \neq y \end{cases}$$

1) $d(x,y) \geq 0 \quad \forall x, y \in X$

2) $d(x,y) = 0 \quad iff \quad x = y \quad \forall x, y \in X$

3) $d(x,y) = d(y,x) \qquad \forall \ x, y \in X$

$$\frac{1}{3} = \frac{1}{3} \quad or \quad 0 = 0$$

4) $d(x,y) \leq d(x,z) + d(z,y) \qquad \forall x, y, z \in X$

4) If $X = R^2$ such that

$R^2 = \{x = (x_1, x_2) : x_1, x_2 \in R\}$

Defined $d: R^2 \times R^2 \to R$ by:

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|$$

$$x = (x_1, x_2), \qquad y = (y_1, y_2)$$

1) $d(x,y) = |x_1 - y_1| + |x_2 - y_2| \geq 0$
2) $d(x,y) = |x_1 - y_1| + |x_2 - y_2| = 0 \quad iff$
$|x_1 - y_1| = 0 \quad and \quad |x_2 - y_2| = 0 \quad iff$
$x_1 = y_1 \quad and \quad x_2 = y_2$
3) $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$
$= |y_1 - x_1| + |y_2 - x_2| = d(y,x)$
4) $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$

Let $z = (z_1, z_2)$

$= |x_1 - z_1 + z_1 - y_1| + |x_2 - z_2 + z_2 - y_2|$

$\leq |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2|$

$\leq d(x,z) + d(z,y)$

H.W: If $X = R^2$. Defined $d: R^2 \times R^2 \to R$ by:

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|$$

$$x = (x_1, x_2), \qquad y = (y_1, y_2)$$

Is (X,d) a metric space?

Remarks(4.5):

Let (X,d) be a metric space, then

1) For any $x, y, z \in X$, we have .

$$|d(x,z) + d(z,y)| \leq d(x,y)$$

2) For any $x_1, x_2, \cdots, x_n \in X$, we have

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n)$$

Proof(1):

$d(x,z) \leq d(x,y) + d(y,z) \qquad \cdots (1)$

$d(z,y) \leq d(z,x) + d(x,y) \qquad \cdots (2)$

From (1) we get: $d(x,z) - d(y,z) \leq d(x,y)$

From (2) we get: $-d(x,y) \leq d(z,x) - d(z,y)$

$\therefore \ |d(x,z) - d(z,y)| \leq d(x,y)$

Proof(2):

By induction on the element of X.

$n = 3$

$x_1, x_2, x_3 \in X$, then

$d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3) \qquad \cdots (3)$

Suppose the result is true for any $k = n - 1 < n$

i.e

$d(x_1, x_{n-1}) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-2}, x_{n-1})$

To prove is true for any n

$d(x_1, x_n) \leq d(x_1, x_{n-1}) + d(x_{n-1}, x_n)$ by (3)
$\qquad \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n)$

Basic principles of topology:

Definition(4.6):

Let (X, d) be a metric space, and $x_0 \in X$, $r \in R$, $r > 0$, then:

$$B_r(x_0) = \{ x \in X : d(x, x_0) < r\}$$

Is called a ball of radius r and center x_0.

$$D_r(x_0) = \{ x \in X : d(x, x_0) \leq r\}$$

Is called a disk of radius r and center x_0.

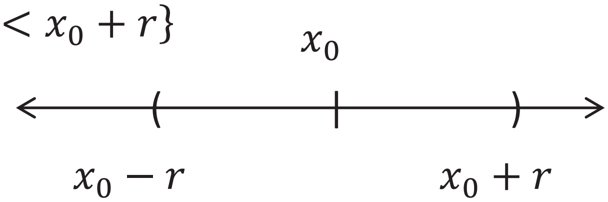
Examples:

1) (R, d) is a metric space.

$B_r(x_0) = \{ x \in R : |x - x_0| < r\}$

$= \{ x \in R : x_0 - r < x < x_0 + r\}$

$= (x_0 - r , x_0 + r)$

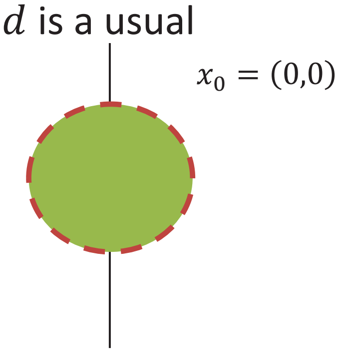


$D_r(x_0) = \{ x \in R : |x - x_0| \leq r\}$

$= [x_0 - r , x_0 + r]$

2) (R^2, d) is a metric space

$B_r(x_0) = \left\{ x \in R^2 : \sqrt{(x - x_0)^2 + (y - y_0)^2} < r \right\}$; d is a usual distance



$$= \{ x \in R^2 : (x - x_0)^2 + (y - y_0)^2 < r^2 \}$$

$r = 1$

(R^n, d) is a metric space; d is a usual distance

$$B_r(x_0) = \left\{ \mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_n) \in \mathrm{R}^2 \right.$$
$$\left. : \sqrt{(\mathrm{x}_1 - \mathrm{x}_0)^2 + (\mathrm{x}_2 - \mathrm{x}_0)^2 + \cdots + (\mathrm{x}_n - \mathrm{x}_0)^2} < \mathrm{r} \right\}$$

3) (R^2, d) is a metric space

Where $d: R^2 \times R^2 \to R$ defined by $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$

$x = (x_1, x_2), \qquad y = (y_1, y_2)$

$B_1(0) = \{ (x, y) \in R^2 : |x - 0| + |y - 0| < 1\}$

$B_1(0,0) = \{ (x, y) \in R^2 : |x| + |y| < 1\}$

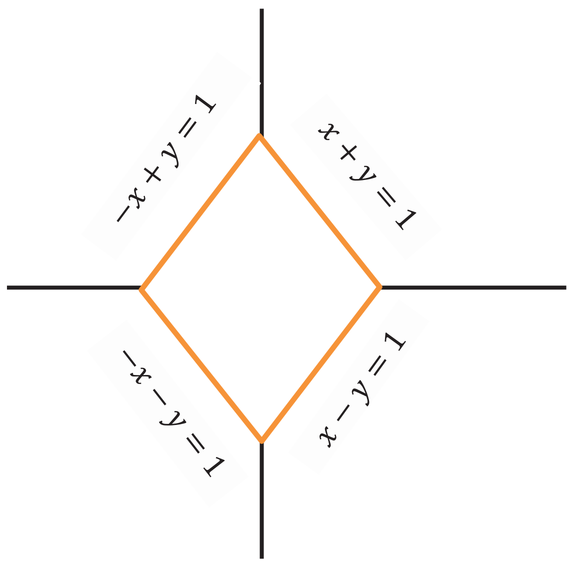
We have the following cases:

$x, y > 0 \qquad x + y = 1$

$x < 0 , y > 0 \qquad -x + y = 1$

$x > 0 , y < 0 \qquad x - y = 1$

$x < 0 , y < 0 \qquad -x - y = 1$



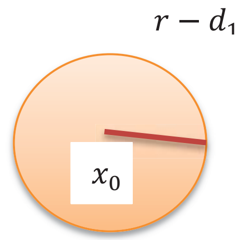
Definition(4.7):

Let (X, d) be a metric space and, $S \subseteq X$, S is called an open set if for each $x_0 \in S$ there exists $r > 0$, $(r \in R)$, , such that:

$$B_r(x_0) \subseteq S$$

Examples:

1) Every ball in any metric space is an open set.



$B_r(x_0) = \{ x \in X : d(x, x_0) < r\}$

Proof: let $y \in B_r(x_0)$

$0 < d(y, x_0) = d_1 < r$

Take $\epsilon = r - d_1 > 0$, to proof $B_\epsilon(y) \subseteq B_r(x_0)$

Let $z \in B_\epsilon(y) \overset{?}{\Rightarrow} z \in B_r(x_0)$

$d(z, y) < \epsilon$ $given$, to prove $d(z, x_0) < r$?

$d(z, x_0) \le d(z, y) + d(y, x_0)$

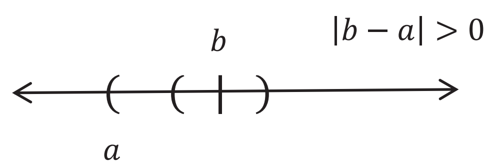
$< \epsilon + d_1$

$= r - d_1 + d_1$

$= r$

In particular every open interval in R is an open set, $(a, \infty), (-\infty, a)$ are open sets.

$\forall\, b \neq a$, $\exists\, d = |b - a|$



$(b - \epsilon\, ,\ b + \epsilon) \subseteq (a, \infty)$

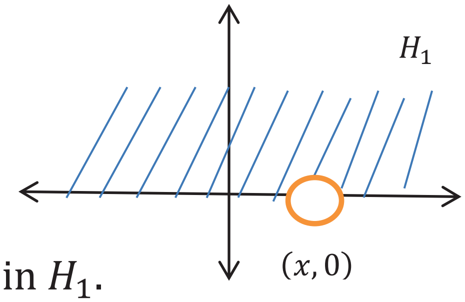
$[a, b)$ is not an open set.

$\exists\ (a-\epsilon, a+\epsilon) \not\subset [a, b)$

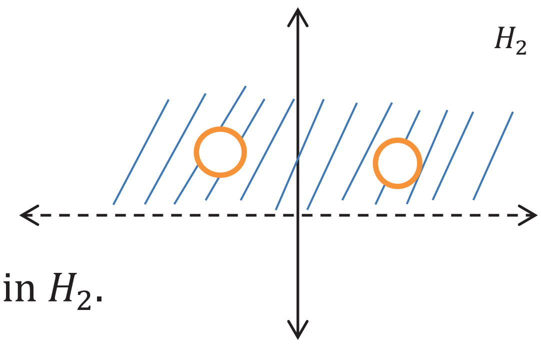
2) $H_1 = \{ (x, y) \in R^2 : x \in R\ ,\ y \geq 0\}$

is not open subset in R^2.

Since the ball with center $(x, 0)$ is not contain in H_1.



$H_2 = \{ (x, y) \in R^2 : x \in R\ ,\ y > 0\}$
is open subset in R^2



Since the ball with center (x, y) is contain in H_2.

3) The set of rational (irrational) number is not open set.

Since any interval in Q with center $\frac{a}{b} \in Q$, doesn't contain rational only (by density of irrational).

Also any interval in Q', doesn't contain irrational only because of the density of rational number) not open.

Proposition(4.8):

Let (X, d) be a metric space, and T be a collection of all open subset of X, then T satisfies the following:

1) $X, \emptyset \in T$.

2) The union of any number of open sets is open. (i.e The union of any element of T is again in T.

3) The intersection of a finite number of element of T is again in T.

Proof: 2) Let $\{T_n\}$ be any number of open sets in T.

To prove $\cup_n T_n \in T$ (i.e is open in T).

Let $x \in \cup_n T_n$, $\therefore \exists\, k \in N \quad s.t \quad x \in T_k$.

$\because\ T_k$ *is open* , $\therefore\ \exists\, r > 0$, $s.t\ \ B_r(x) \subseteq T_k$

$\therefore\ B_r(x) \subseteq \cup_n T_n$

$\therefore \cup_n T_n$ is open

3) Let $T_1, T_2, \cdots, T_n$ be a finite number of open sets in .

To prove $\cap_{i=1}^{n} T_i$ is open in T.

Let $x \in \cap_{i=1}^{n} T_i$, $\therefore\ x \in T_i \quad \forall\ i = 1,2,\cdots,n$.

$\because\ T_i$ *is open* , $\forall\ i = 1,2,\cdots,n$

$\therefore\ \exists\, r_1 \in R$, $s.t\ \ B_1(x) \subseteq T_1$, $\exists\, r_2 \in R$, $s.t\ \ B_2(x) \subseteq T_2$, $\cdots$

Take $r = \{r_1, r_2, \cdots, r_n\}$

$\therefore\ B_r(x) \subseteq\ \cap_{i=1}^{n} T_i$

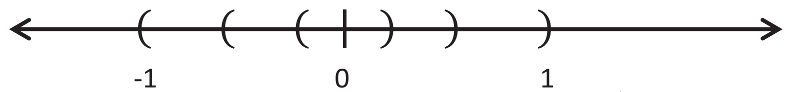
$\therefore\ \cap_{i=1}^{n} T_i$ is open

Remark(4.9):

The intersection of infinite number of open sets needn't be open. As the following example shows:

Example:

$\forall\, n \in N$, let $A_n = \left(\frac{-1}{n} , \frac{1}{n}\right) \subseteq R$, $\cap_n A_n = \{0\}$



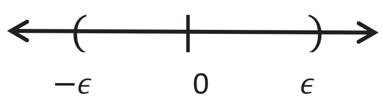
If $\exists\, x \neq 0$, $x > 0 \Rightarrow \exists\, k \in N$ $s.t$ $\frac{1}{k} < x$, $\therefore\ x \notin \left(\frac{-1}{k} , \frac{1}{k}\right)$.

If $\exists\, x \neq 0$, $x < 0$, $0 < -x \Rightarrow \exists\, t \in N$ $s.t$ $\frac{1}{t} < -x \Rightarrow$ $\frac{-1}{t} > x$, $\therefore\ x \notin \left(\frac{-1}{t} , \frac{1}{t}\right) \Rightarrow x \notin \cap_n A_n$

$\therefore\ \cap_n A_n$ is only zero.

Note:

$\{0\}$ is not open, since. $\forall\, \epsilon > 0$, $B_\epsilon(0) = (-\epsilon, \epsilon) \nsubseteq \{0\}$



Remark:

If (X, d) is a metric space, then we can define a topological space from this metric space by taking $T =$ the set of all open subsets of X and by proposition (4.8) we easily seen that (X, T) is a topological space

But if (X, T) is a topological space, then in general we couldn't get a metric space from this topological space as the following example shows:-

Example:

Let $X = \{a, b, c, d, e, f, \dots, z\}$ and $T = \{X, \emptyset\}$.

(X, T) is a topological space

But we cannot define a distance between the elements of X.

Proposition(4,12):-

Let (X, d) be a metric space and $S \subseteq X$, then S is open iff S is a union of balls

Proof:- $\Rightarrow$)let S be an open set

Then $\forall x \in S$, $\exists\, r_x > 0$ such that $B_{r_x}(x) \subseteq S$

$\therefore \cup_{x \in S} B_{r_x}(x) = S$.

$\Leftarrow$) $S = \cup_{i \in w} B_i$ are balls

$\because$ every ball is an open set $\Rightarrow S = \cup_{i \in w} B_i$ is open (by proposition (4.8)).

Definition (4.13):-

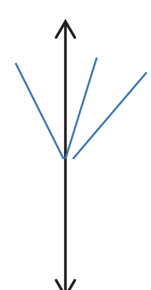
Let (X, d) be a metric space (topological space) and $E \subseteq X$, then E is closed in X if $X - E$ is open in X.

Examples:-

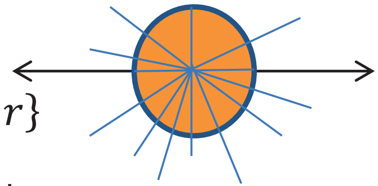
1- $[a, b] \subset R$, $[a, b]$ is closed

Since $R - [a, b] = (-\infty, a) \cup (b, \infty)$ is open.

The union of open set in a metric space is open.



In general any disk is a closed set.



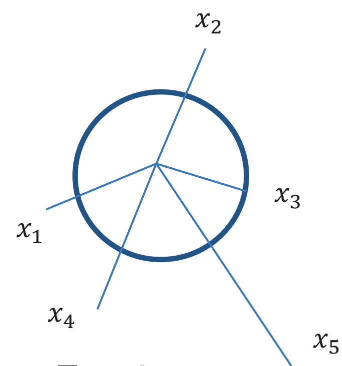
$$D_r(x_0) = \{x \in X: d(x, x_0) \leq r\}$$

$X - D_r(x_0) = \{x \in X: d(x, x_0) > r\}$ is an open set

2- Every finite subset E of a metric space (X, d) is a closed set.

Proof:- let $E = \{x_1, x_2, \cdots, x_n\} \subseteq X$

T.P $X - E$ is open.



Let $a \in X - E$, $\therefore a \neq x_i$, $\forall i = 1,2, \cdots, n$

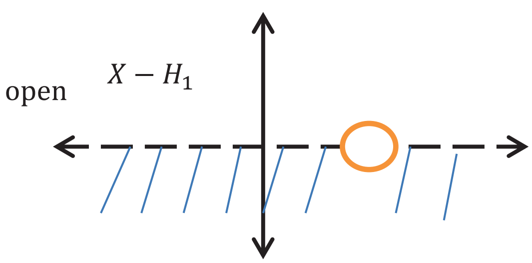
$\therefore \exists 0 < d_i$, $\forall i = 1,2, \cdots, n$

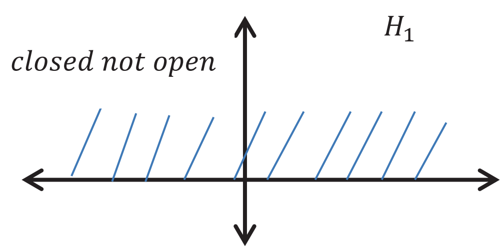
Take $r = \min\{d_1, d_2, \cdots, d_n\} \Rightarrow B_r(a) \nsubseteq E$

$\Rightarrow B_r(a) \cap E = \emptyset \Rightarrow B_r(a) \subseteq X - E \Rightarrow X - E$ is open
$\Rightarrow E$ is closed.

3- $H_1 = \{ (x, y) \in R^2 : x \in R , y \geq 0\}$

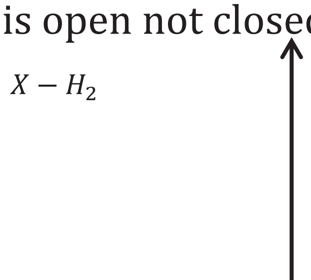
is closed not open subset in R^2.

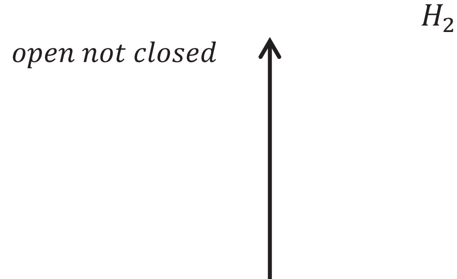




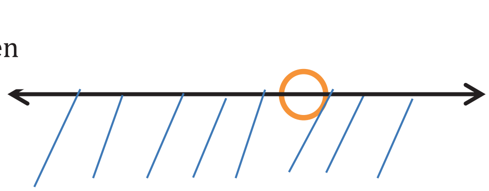
Since the ball with center $(x, 0)$ is not contain in H_1.

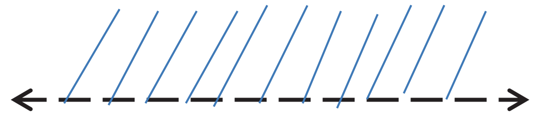
$H_2 = \{ (x, y) \in R^2 : x \in R , y > 0\}$
is open not closed subset in R^2





not open





Since the ball with center (x, y) is contain in H_2.

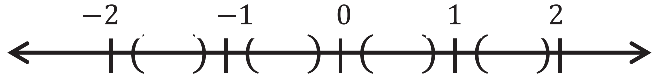
4- $Q \subset R$ is not closed $R - Q = Q'$ is not open

$\therefore$ Q is not closed

5- Z (Integers number) is closed

$R - Z = \cdots \cup (-1,0) \cup (0,1) \cup (1,2) \cup \cdots$

Balls $\Rightarrow$ open



$\therefore Z$ is closed

6- X , $\emptyset$ are closed sets

$X - X = \emptyset$.is open $\therefore X$ is closed, $X - \emptyset = X$ is open, $\therefore \emptyset$ is closed.

Proposition (4.14):-

Let (X, d) be a metric space (topological space) and let T be the collection of all closed subsets of X. Then T satisfies the followings:

1) X , $\emptyset$ $\in T$ (**i.e** , X and $\emptyset$ are closed)

2) The union of finite numbers of elements in T is an element is T (**i.e**, the union of finite numbers of closed set is again a closed set)

3) The intersection of finite or infinite numbers of elements of T is an element in T

(**i.e**, the intersection of finite or infinite numbers of closed set is closed)

Proof: (H.w)

Remark:

Let $X \neq \emptyset$ and $y_\alpha \subseteq X$ $\forall \alpha \in \wedge$ then

$X - \cup_{\alpha \in \wedge} y_\alpha = \cap_\alpha (X - y_\alpha)$

$X - \cap_{\alpha \in \wedge} y_\alpha = \cup_\alpha (X - y_\alpha)$.

Definition (4.15):

Let (X, d) be a metric space and $\emptyset \neq S \subseteq X$ and $p \in X$, we say.

that p is a cluster point for S, if every open set contain p contains another element q in S and $p \neq q$

i.e for any open set U ; $p \in U$ $(U - \{p\}) \cap S \neq \emptyset$

Note:-

We will denote the set of all cluster points of S by $l(S)$.

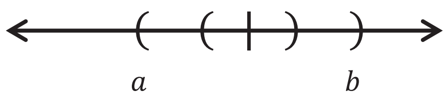
($\bar{S} = S \cup l(S)$ is called the closer of S)

$l(S) = \{ p : p \; is \; a$ cluster points of $S\}$

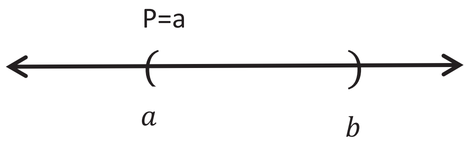
Example:-

1) $S = (a, b)$, $X = R$, find $l(S)$

p



(1)



(2)

(3)

$\therefore\ l(S) = [a,b] \quad \Rightarrow \qquad \bar{S} = [a,b]$

a) If $p \in S$, then any open interval $U,\ \exists\, p \in U$ we have:-$U - \{p\} \cap S \neq \emptyset$.

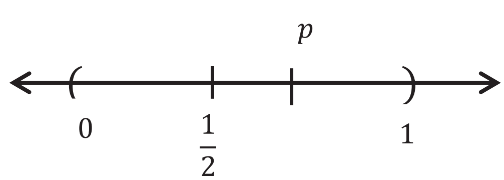
b) If $p = a$, then any open interval U contain $a = p$ satisfies :- $U \cap S \neq \emptyset$

c) For any $p \in R - [a,b]$, $p \neq a$, then $\exists\, U = (p - 4d, p + d)$, $U \cap S = \emptyset$ and $d = |a - p|$

2) Let $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots\right\} \subseteq R$

a) If $p = \frac{1}{n}$, $n \in N$, take $U = \left(\frac{1}{n+1}, \frac{1}{n-1}\right)$, $\frac{1}{n} \in U$, $\left(U - \left\{\frac{1}{n}\right\}\right) \cap S = \emptyset$

b) If $p \neq \frac{1}{n}$, $p \neq \{0\}$, if $p > 0$, $\exists\, U = \left(\frac{1}{n+1}, \frac{1}{n}\right)$, $\frac{1}{n} \in U$,



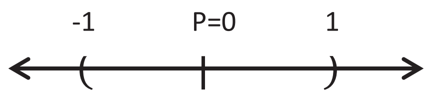
if $p > \frac{1}{n}$, $\exists\, U = \left(\frac{1}{n}, \frac{1}{n-1}\right)$ and $U \cap S = \emptyset$

c) If $p < 0$, $\exists\ U = (-\infty, 0)$, $p \in (-\infty, 0)$ and $U \cap S = \emptyset$

d) If $p = \{0\}$, then any open set (interval) contain 0, $0 \in \left(-\epsilon , \frac{\epsilon}{2}\right)$ and $U \cap S \neq \emptyset$.

Since $\forall\, \epsilon_2 > 0$, $\exists\, k \in Z^+$ $s.t$ $0 < \frac{1}{k} < \epsilon_2$, $\frac{1}{k} \in \left(-\epsilon , \frac{\epsilon}{2}\right)$

$\therefore\ l(S) = \{0\}$ only zero



$\bar{S} = S \cup l(S) = S \cup \{0\} = \left\{0,1 , \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots , \frac{1}{n}, \cdots\right\}$

$l(S) = \{0\} \nsubseteq S$

$\therefore\ S$ not closed.

3) Let (X, d) be a metric space and S be any finite subset of X, then $l(S) = \emptyset$.

<u>**Sol:**</u> let $S = \{x_1 , x_2, \cdots , x_n\} \subseteq X$, let $p \in X$, if $p \in S$, then $\exists\, t \in N$, $1 \leq t \leq n$ $s.t$ $p = x_t$.

Then $d(x_i, x_t) = d_i$ $\forall\, i = 1,2, \cdots , n$, $i \neq t$

$\epsilon = \min\{d_i : i = 1,2, \cdots , n$, $i \neq t\}$

$B_\epsilon(x_t) - \{x_t\} \cap S = \emptyset$

Now, $p \notin S$, $p \in X - S$

$p \neq x_i$ $\forall\, i = 1,2, \cdots , n$

$\therefore d(p, x_i) = r_i$ $\forall\, i = 1,2, \cdots , n$

Let $\epsilon < \min\{r_1, r_2, \cdots , r_n\}$ $\therefore\ B_\epsilon(p) \cap S = \emptyset$,

$\therefore p$ is not a cluster point

$\therefore\ l(S) = \emptyset \quad and \quad \bar{S} = S \cup \emptyset = S$.

4) Let Q be the set of rational numbers in R with the usual distance.

a) If $p \in Q$, then any open set (open interval) $U,\ s.t\ \ p \in U$ we have:- $(U - \{p\}) \cap Q \neq \emptyset$. (By the density rational number)

b) If $p \notin Q \quad \rightarrow p \in Q'$, then any open set (open interval) U such that $p \in U$, we have $U \cap Q \neq \emptyset$. (By the density irrational number)

$\therefore\ l(Q) = R \quad and \quad \bar{Q} = Q \cup\ l(Q)$

$= Q \cup R$.

(H.W)

Find $l(Q')$, $l(Z)$; $Z \subseteq R$.

Proposition (4.16):

Let (X, d) be a metric space and $\emptyset \neq S \subseteq X$, then S is closed iff S contains all its cluster points (i.e $\bar{S} = S$)

Proof: $\Rightarrow$)suppose the result is not true i.e $\exists$ a cluster point p for S such that $p \notin S$, $(p \in X - S)$.

$\because S$ is closed, then $X - S$ is open, hence $(X - S) \cap S = \emptyset \quad C!$ (p is a cluster point for S).

$\Leftarrow$) let $l(S) \subseteq S$,T.P S is closed i.e $X - S$ is open.

Let $x \in X - S$, $x \notin S$ $i.e$ $x \notin l(S)$, x is not a cluster point.

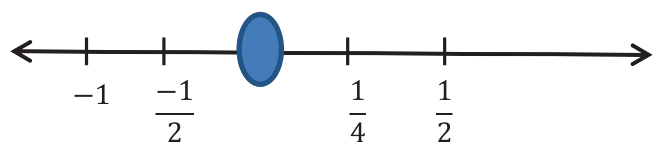
$\exists$ open set U_x ; $x \in U_x$ and $U_x \cap S = \emptyset$, $\therefore$ $U_x \subseteq X - S$.

In particular $\exists$ a ball $B(x) \subseteq X - S$ $\rightarrow$ $X - S$ is open

$\therefore$ S is closed.

Example:

$S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots\right\}$ is not closed.



$X - S$ is not open, $\exists$ $0 \neq x \in X - S$, $\exists$ any ball $B(x) \nsubseteq X - S$

$\therefore$ $0 \notin X - S$

Definition (4.17):

Let (X, d) be a metric space and $\emptyset \neq S \subseteq X$ and $p \in X$, define.

$$d(p, S) = \inf\{d(p, s): s \in S\}$$

is called the distant between the point p and the set S

Remark (4.18):

If $S \subseteq X$, (X, d) be a metric space and $p \in S$, then $d(p, S) = 0$

$$d(p, S) = \inf\,\{d(p, s): s \in S\}$$

If $p \in S$, then inf $\{d(p,p): \} = \inf\{0: positive\ number\} = o$.

The converse of remark (4.18) is not true in general as the following example show:

Example:

Let $S = (a, b)$, $X = R$.

$d(a, S) = \inf\{d(a, s): a < s < b\}$

$= \inf\{p - p + \epsilon, p - p + 2\epsilon, \cdots\}$

$= \inf\{\epsilon, \cdots\} = 0$

Proposition (4.19):

Let (X, d) be a metric space and $\emptyset \neq S \subseteq X, p \in X$, then $d(S, p) = 0$ iff $p \in S$ or p is a cluster point of S.

Proof: $\Rightarrow$) $d(S, p) = 0$ suppose that $p \notin S$ T.P p is a cluster point for S.

If p is not a cluster point for S.

$\exists$ a ball $B_r(p)$ such that $B_r(p) \cap S = \emptyset$

$\therefore d(s, p) > r$, $s \in S$ $C!$ (since $d(S, p) = 0$)

$\therefore p$ is a cluster point for S.

$\Leftarrow$) If $p \in S$ by remark (4.18) $d(S, p) = 0$.

If p is a cluster point for S, then for any open set U, $p \in U$

$(U-\{p\}\cap)S \neq \emptyset$

In particular $\exists$ a ball $B_\epsilon(p)$; $B_\epsilon(p) \cap S \neq \emptyset$

$\exists\, s \neq p \in S$, $s \in B_\epsilon(p)$, $d(s,p) < \epsilon$

$d(S,p) = \inf\{d(s,p) < \epsilon, +, +, \cdots\}$

$= 0$

Corollary (4.20):

Let (X,d) be a metric space and $\emptyset \neq S \subseteq X$, then

$\bar{S} = \{x \in X \colon d(S,x) = 0\}\, d(S,p) = 0.$

Proof: $\bar{S} = S \cup l(S)$ by proposition (4.19) ($d(S,x) = 0$ iff $x \in X$ or x is a cluster point for S).

Definition (4.21):

Let (X,d) be a metric space and $\langle x_n \rangle$ be a sequence in X , we say that $\langle x_n \rangle$ is a convergence sequence if there exists $x_0 \in X$ such that $\forall \epsilon > 0$, $\exists\, k = k(\epsilon)$ satisfise:

$$d(x_n, x_0) < \epsilon \qquad \forall\; n > k$$

i.e any ball with center x_0 and radius ϵ contain most of the terms of the sequence.

Proposition (4.22):

If $\langle x_n \rangle$ is a convergence sequence in X that converges to x_0, then x_0 is unique.

Proof: Suppose there exists another limit point y_0 for $\langle x_n \rangle$

i.e $x_n \to y_0$ and $x_0 \neq y_0$.

$0 < d = d(x_0, y_0)$ take $\epsilon = \frac{1}{2}d$

$\therefore\ \exists\ B_{\frac{1}{2}d}(x_0)$ and $B_{\frac{1}{2}d}(y_0)$ such that $B_{\frac{1}{2}d}(x_0) \cap B_{\frac{1}{2}d}(y_0) = \emptyset$

$\because\ x_n \to x_0$ and $x_n \to y_0$, then each of balls $B_{\frac{1}{2}d}(x_0)$ and $B_{\frac{1}{2}d}(y_0)$ contain most of the term of the sequence but $B_{\frac{1}{2}d}(x_0) \cap B_{\frac{1}{2}d}(y_0) = \emptyset$ a contradiction.

$\therefore x_n \to y_0$

Definition (4.23):

Let (X, d) be a metric space and $\langle x_n \rangle$ be a sequence in X, we say that $\langle x_n \rangle$ is a Cauchy sequence if $\forall \epsilon > 0$, $\exists\ k = k(\epsilon)$ such that:

$$d(x_n, x_m) < \epsilon \qquad \forall\ n, m > k$$

Proposition (4.24):

Every convergence sequence in a metric space X is a Cauchy sequence.

Proof: Let $\langle x_n \rangle$ be a convergence sequence that converge to x_0 i.e $x_n \to x_0$.

Let $\epsilon > 0$, $\because$ $x_n \to x_0$, then $\exists\ k = k(\frac{\epsilon}{2})$ such that $d(x_n, x_0)$

$$d(x_n, x_m) \leq d(x_n, x_0) + d(x_m, x_0)$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \qquad \forall\ n > k , \forall\ m > k$$

$$< \epsilon \qquad \forall\ n, m > k$$

Remark (4.25):

The converse of proposition (4.24) in general is not true.

Proof: Let $X = R - \{0\}$, $d(x, y) = |x - y| \quad \forall\ x, y \in R - \{0\}$

$\exists\ \langle \frac{1}{n} \rangle$ in $R - \{0\}$

$\frac{1}{n} \to 0 \notin R - \{0\}$

$\therefore\ \langle \frac{1}{n} \rangle$ is not a convergence sequence

By proposition (4.24) is a Cauchy sequence but not converges in $R - \{0\}$.

Definition (4.26):

A metric space (X, d) is called a complete metric space if every Cauchy sequence in X is a convergence sequence in X.

Theorem (4.27):

R^k is called a complete metric space $\forall\, k \geq 1$.

Proof: $k = 2$ let $\langle(x_n, y_n)\rangle$ be a Cauchy sequence in R^2.

$\forall\, \epsilon > 0$, $\exists\; k_1 = k_1(\frac{\epsilon}{2})$ such that

$$d\big((x_n, y_n), (x_m, y_m)\big) = \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2} < \frac{\epsilon}{2} \quad \forall n, m > k_1$$

$$= (x_n - x_m)^2 + (y_n - y_m)^2 < \frac{\epsilon^2}{4} \quad \forall\; n, m > k_1$$

$$\therefore\; (x_n - x_m)^2 < \frac{\epsilon^2}{4} \quad \forall\; n, m > k_1 \quad \cdots\; (1)$$

And $(y_n - y_m)^2 < \frac{\epsilon^2}{4} \quad \forall\; n, m > k_1 \quad \cdots\; (2)$

$$|x_n - x_m| < \frac{\epsilon}{2} \quad \forall\; n, m > k_1 \quad \cdots\; (3)$$

And $|y_n - y_m| < \frac{\epsilon}{2} \quad \forall\; n, m > k_1 \quad \cdots\; (4)$

$\therefore$ $\langle x_n \rangle$ is a Cauchy sequence in R and $\langle y_n \rangle$ is a Cauchy sequence in R.

$\because$ R is complete

$\therefore$ $x_n \to x_0 \in R$ and $y_n \to y_0 \in R$

$\exists\; k_2 = k_2(\frac{\epsilon}{2})$ such that $|x_n - x_m| < \frac{\epsilon}{2} \quad \forall\; n, m > k_2$

$\exists\; k_3 = k_3(\frac{\epsilon}{2})$ such that $|y_n - y_m| < \frac{\epsilon}{2} \quad \forall\; n, m > k_3$.

Claim: $(x_n, y_n) \to (x_0, y_0) \in R^2$.

$$\Big(d\big((x_n, y_n), (x_0, y_0)\big)\Big)^2 = (x_n - x_0)^2 + (y_n - y_0)^2$$

$$< \frac{\epsilon^2}{4} + \frac{\epsilon^2}{4} = \frac{\epsilon^2}{2} \qquad \forall n > k = \max\{k_1, k_2\}$$

H.W: In R^3

Definition (4.28):

Let (X, d) be a metric space and $\emptyset \neq S \subseteq X$, then (S, d_S).is a subspace of a metric space X, where $d_S = d|_S$

$$d: X \times X \rightarrow R$$

$$d_S: S \times S \rightarrow R$$

Proposition (4.29):

Let (X, d) be a metric space and $\emptyset \neq S \subseteq X$ if $\langle x_n \rangle$ is a sequence in S such that $\langle x_n \rangle$ converge to x_0, iff either $x_0 \in S$ or x_0 is a cluster point for S.

Proof: $\Rightarrow$) if that $x_0 \notin S$ T.P x_0 is a cluster point for S.

$\because$ $x_n \rightarrow x_0$, then any ball $B(x_0)$ contain most of the terms of the sequence, hence $B(x_0) \cap S \neq \emptyset$

$\therefore$ x_0 is a cluster point for S.

$\Leftarrow$) If $x_0 \in S$, then $\langle x_0 \rangle = x_0, x_0, x_0, \cdots \rightarrow x_0$.

If x_0 is a cluster point for S, then for any ball $B_{\frac{1}{n}}(x_0)$, $n \in N$, we have:

$$\left(B_{\frac{1}{n}}(x_0) - \{x_0\}\right) \cap S \neq \emptyset$$

Then $\forall\, n \in N$, $x_n \in \left(B_{\frac{1}{n}}(x_0) - \{x_0\}\right) \cap S$,

$\therefore \langle\, x_n \rangle$ is a sequence in S.

Claim: $\langle\, x_n \rangle$ converge to x_0.

$\forall\, n \in N \quad d(\, x_n,\, x_0) < \frac{1}{n}$.

$\forall\, \epsilon > 0$, $\exists\, k = k(\epsilon) \quad s.t \quad \frac{1}{k} < \epsilon$

$d(\, x_n,\, x_0) < \frac{1}{n} < \frac{1}{k} < \epsilon \qquad \forall\, n > k$

Proposition (4.30):

Let (X, d) be a complete metric space and $\emptyset \neq S \subseteq X$ if S is a closed set, then (S, d_S) is a complete metric space

Proof: let $\langle x_n \rangle$ be a Cauchy sequence in S. T.P $\langle x_n \rangle$ converge to $x_0 \in S$.

$\langle x_n \rangle$ is a Cauchy sequence in X.

$\because\ X$ is complete

$\therefore\ x_n \to x_0 \in X$, $\langle x_n \rangle \in S$.

By proposition (4.29) either $x_0 \in S$ or x_0 is a cluster point for S.

If $x_0 \in S$, then we are done.

If x_0 is a cluster point for S

Since S is closed, then by proposition (4.16) $x_0 \in S$.

Definition (4.31):

Let (X, d) be a metric space and $\emptyset \neq S \subseteq X$ and let

$$\hat{S} = \{d(x, y) : \ x, y \in S\}$$

$\hat{S}$ is bounded below since $d(x, y) \geq 0$

If $\hat{S}$ is bounded above, then we say that $\hat{S}$ is a bounded set and in this case (R is complete) we write

$$\mathrm{Sup}(\hat{S}) = \mathrm{Diam(S)} = \mathrm{D(S)}$$

Examples:

1. $S = (a, b) \subseteq R.$

   $\hat{S} = \{d(x, y): a < x < b \quad , \quad a < y < b\}$

   $\mathrm{Sup}(\hat{S}) = b - a = \mathrm{D(S)}$

   $\therefore$ S is a bounded set.

2. $S = [a, b] \subseteq R.$

   $\hat{S} = \{d(x, y): a \leq x \leq b \quad , \quad a \leq y \leq b\}$

   $\mathrm{Sup}(\hat{S}) = b - a = \mathrm{D(S)}$

   $\therefore$ S is a bounded set.

3. $Q \subset R.$

   $\hat{Q} = \{d(x, y): \quad x , y \in Q\}$

   Is not bounded above, hence Q is not bounded set.

Proposition (4.32):

Let (X,d) be a metric space and $\emptyset \neq S \subseteq X$, S is bounded if and only if $\forall\, x_0 \in S$, there exists $n \in N$ such that

$$d(x,x_0) < n \quad \forall\, x \in S$$

Proof: $\Rightarrow$) let S be a bounded.

Then $\hat{S} = \{d(x,y) : x,y \in S\}$

$\hat{S}$ is bounded set (above) $\exists\, n \in N$ such that $d(x,y) < n \quad \forall x,y \in S$.

In particular $x_0 \in S$, $d(x,x_0) < n \quad \forall x \in S$

$\Leftarrow$) let $x_0 \in S$, then $\exists\, n \in N$ such that $d(x,x_0) < n \quad \forall x \in S$.

$d(x,y) \leq d(x,x_0) + d(x_0,y) < n + n = 2n = M \quad \forall x,y \in S$

$\therefore\ d(x,y) < M$ (upper bound).

Cantor nested sets theorem(4.33):

Let (X,d) be a metric space and $\langle E_n \rangle$ be a sequence of bounded sets such that:

1) $E_1 \supseteq E_2 \supseteq \cdots \supseteq E_n \supseteq \cdots \quad \forall\, n.$
2) $\forall\, n \in N$, E_n is a non-empty closed sets.
3) The sequence $\langle \mathrm{diam}(E_n) \rangle$ converges to zero.

If (X,d) is a complete metric space, then $\cap_n E_n$ consist of only one point.

Proof: $\forall\, n \in N$, let $x_n \in E_n$ since $E_n \neq \emptyset \quad \forall\, n$.

Since $\mathrm{diam}(E_n) \to 0$, then $\forall\, \epsilon > 0$, $\exists\, k \in N$ such that $\mathrm{diam}(E_k) < \epsilon$.

$\forall\, n, m > k \quad , \quad x_n, x_m \in E_k$.

$x_n \in E_n \subseteq E_k$ *and* $x_m \in E_m \subseteq E_k$ from(1)

$\therefore d(x_n, x_m) < \text{diam}(E_k) < \epsilon \quad \forall\, n, m > k$, then $\langle x_n \rangle$ is a Cauchy sequence.

Since (X, d) is a complete metric space, hence $\langle x_n \rangle$ is converge to $x_0 \in X$.

Claim: $\cap_n E_n = \{x_0\}$.

$x_n \to x_0 \quad \forall \epsilon > 0$, $\exists\, k \in N$ $s.t$ $d(x_n, x_0) < \epsilon \quad \forall\, n > k$.

$\forall\, n > k$, $x_n \in E_n \subseteq E_k \quad x_n \in E_k$

$\therefore \quad x_n \in E_n \quad \forall\, n$ most of the term of the sequence in E_n $\forall n$.

$\therefore$ most of the term of the sequence in $\cap_n E_n$.

$\therefore$ by proposition (4.29) either $x_0 \in \cap_n E_n$ *or* x_0 is a cluster point for $\cap_n E_n$ (intersection of closed sets is closed).

$\therefore \cap_n E_n$ is closed, hence $x_0 \in \cap_n E_n$.(proposition4.16)

Uniqueness: Suppose $\exists\, y_0 \in \cap_n E_n$ *and* $x_0 \neq y_0$.

$0 < d = d(x_0, y_0)$, $\text{diam}(E_n) \to 0$

$\forall\, \epsilon > 0$, $\exists\, l \in N$ such that $\text{diam}(E_l) < \epsilon$.

In particular when $\epsilon = d$

$x_0, y_0 \in E_l$

$d = d(x_0, y_0) < \text{diam}(E_l) < d$. $C!$.

$\therefore \quad x_0 = y_0$

Contracting mapping principle theorem(4.34):

Let (X, d) be a metric space and $T: X \to X$ be a mapping satisfies there exists $0 \le \theta < 1$ such that:

$$d(T_x, T_y) \le \theta d(x, y) \qquad \forall\, x, y \in X$$

(in this case T is called a contracting mapping), if X is complete, then there exists only one point such that $T_x = x$ (x is called a fixed point.

Proof: Since $X \neq \emptyset$ $\quad T: X \to X$

let $x_0 \in X$.

Let $x_1 = T_{x_0}$.

$x_2 = T_{x_1} = TT_{x_0} = T^2{}_{x_0}$

$x_3 = T_{x_2} = TT_{x_1} = TTT_{x_0} = T^3{}_{x_0}$

$\vdots$

$x_n = T_{x_{n-1}} = T^n{}_{x_0}$

Claim: $\langle x_n \rangle$ is a Cauchy sequence.

$\forall\, n, m$ if $m > n$

$$d(x_n, x_m) = d\left(T^n{}_{x_0}, T^m{}_{x_0}\right)$$

$$= d\left(T^n{}_{x_0}, T^{m-n}T^n{}_{x_0}\right)$$

$$= d\left(T^n{}_{x_0}, T^nT^{m-n}{}_{x_0}\right) = d\left(T^n{}_{x_0}, T^n{}_{x_{m-n}}\right)$$

$$\leq \theta^n d(x_0, x_{m-n}).$$

$$\theta^n d(x_0, x_{m-n}) \leq \theta^n\{d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{m-n-1}, x_{m-n})\}$$

$$= \theta^n\left\{d(x_0, x_1), d\left(T_{x_0}, T_{x_1}\right) + d\left(T^2{}_{x_0}, T^2{}_{x_1}\right) + \cdots + d\left(T^{m-n-1}{}_{x_0}, T^{m-n-1}{}_{x_1}\right)\right\}$$

$$\leq \theta^n\{d(x_0, x_1) + \theta d(x_0, x_1) + \theta^2 d(x_0, x_1) + \cdots + \theta^{m-n-1} d(x_0, x_1)\}$$

$d(x_n, x_m) \leq \theta^n d(x_0, x_{m-n})$

$\leq \theta^n d(x_0, x_1)\{1 + \theta + \theta^2 + \cdots + \theta^{m-n-1}\}$

By using mathematical induction $= \frac{\theta^n d(x_0, x_1)}{1-\theta}$

$\therefore \forall \epsilon > 0 \quad 0 \leq \theta < 1 \quad \exists k \in N$

Such that $\frac{\theta^k d(x_0, x_1)}{1-\theta} < \epsilon \quad \forall n > k$

$\therefore \quad \frac{\theta^n d(x_0, x_1)}{1-\theta} < \epsilon \qquad \forall n > k$

Claim: $T_x = x.$

$d(T_x, x) \leq d(T_x, x_n) + d(x_n, x)$

$= d\left(T_x, T_{x_{n-1}}\right) + d(x_n, x)$

$\leq \theta d(x, x_{n-1}) + d(x_n, x) \quad \forall n > k$

$< \theta\epsilon + \epsilon \qquad \forall n + 1 > k$

$< \epsilon(\theta + 1) \qquad \forall \epsilon > k$

$T_x = x$

Claim: uniqueness

Suppose $\exists y \quad s.t \quad T_y = y \quad y \neq x$

$0 < d(x, y) = d\left(T_x, T_y\right) \leq \theta d(x, y) \quad C!$

$\therefore d(x, y) = 0$

$\therefore x = y$

Example:

Let $f:[a,b] \to [a,b]$ be a mapping satisfies $\exists\, \theta \quad , \quad 0 \le \theta < 1$ such that

$|f(x) - f(y)| < \theta|x - y| \quad \forall\, x, y \in [a,b]$

$[a,b]$ is closed subset of a complete metric space R, then by propostion (4.30) $[a,b]$ is complete.

By theorem (4.34) f has only one fixed point.

Remark(4.35):

If $f: R \to R$ is a differentiable mapping satisfies $|f'(x)| \le \theta \quad \forall\, x \in R$ such that $0 \le \theta < 1$, then f is a contracting mapping**.**

$\left|\frac{f(y)-f(x)}{y-x}\right| < \theta$

$|f(y) - f(x)| < \theta\, |y - x| \qquad \forall\, x, y \in [a,b]$

And hence f has exactly only one fixed point.

Example:

Let $f:[-1,1] \to [-1,1]$ defined by:

$f(x) = \frac{1}{5}\, x^2 + \frac{1}{4} \sin 2x \quad , \quad \forall\, x \in [-1,1]$ such that

$f'(x) = \frac{2}{5}\, x + \frac{1}{2} \cos 2x$

$|\cos x| \le 1 \qquad , \qquad |\sin x| \le 1$

$$|f'(x)| = \left| \frac{2}{5}\, x + \frac{1}{2}\, \cos 2x \right|$$

$$\leq \frac{2}{5}\, |x| + \frac{1}{2}\, |\cos 2x|$$

$$\leq \frac{2}{5}.1 + \frac{1}{2}.1 \leq \frac{4+5}{10} = \frac{9}{10} < 1$$

∴ the mapping $f(x) = \frac{1}{5}\, x^2 + \frac{1}{4}\, \sin 2x$ has only one fixed point.

i.e the equation $f(x) = x$ has only one root.

Compact space:

Defnition(4.36):

Let (X, d) be a metric space and $S \subseteq X$ and let $\{V_\alpha\}_{\alpha \in \Lambda}$ be a family of open sets in X, we say that $\{V_\alpha\}_{\alpha \in \Lambda}$ is an open covering for S if $S \subseteq \cup_{\alpha \in \Lambda} V_\alpha$.

Note:

Every set has at least one open covering X, since $S \subseteq X = \cup_{x \in X} B_x$.

Defnition(4.37):

S is called a compact subset of X, if for any open covering for S, there exists a finite open subcovering for S.

i.e

if any open covering $\{V_\alpha\}_{\alpha\in\wedge}$ for S, $S \subseteq \cup_{\alpha\in\wedge} V_\alpha$, there exists $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that $S \subseteq \cup_{i=1}^{n} V_{\alpha_i}$.

In this case if $S = X$, then X is compact.

Defnition(4.38):

If $\{V_\alpha\}_{\alpha\in\wedge}$ is an open covering for S, we say that $\{G_\alpha\}_{\alpha\in\wedge}$, is an open sub covering from $\{V_\alpha\}_{\alpha\in\wedge}$, $\forall\ \alpha \in\wedge$, if $G_\alpha \in \{V_\alpha\}_{\alpha\in\wedge}$.

Examples:

1) Every finite set in any metric space is compact.

Let $S = \{x_1, x_2, \cdots, x_n\} \subseteq X$

Let $\{V_\alpha\}_{\alpha\in\wedge}$ is an open covering for S.

i.e

$S \subseteq \cup_{\alpha\in\wedge} V_\alpha$,

$x_1 \in S \subseteq \cup_{\alpha\in\wedge} V_\alpha$, then $\alpha_1 \in\wedge$ such that $x_1 \in V_{\alpha_1}$.

$x_2 \in S \subseteq \cup_{\alpha\in\wedge} V_\alpha$ then $\alpha_2 \in\wedge$ such that $x_2 \in V_{\alpha_2}$

$\vdots$

$x_n \in S \subseteq \cup_{\alpha\in\wedge} V_\alpha$ then $\alpha_n \in\wedge$ such that $x_n \in V_{\alpha_n}$

$\therefore\ S \subseteq \cup_{i=1}^{n} V_{\alpha_i}$

$\therefore\ \{V_{\alpha_1}, V_{\alpha_2}, \cdots, V_{\alpha_n}\}$ is an open subcovering for $\{V_\alpha\}_{\alpha\in\Lambda}$.

2) Let $S = \left\{0,1,\frac{1}{2},\frac{1}{3},\cdots,\frac{1}{n},\cdots\right\} \subseteq R$ is a compact subset of R.

Let $\{V_\alpha\}_{\alpha\in\Lambda}$ is an open covering for S. i.e $S \subseteq \cup_{\alpha\in\Lambda} V_\alpha$,

$0 \in S \subseteq \cup_{\alpha\in\Lambda} V_\alpha$, then $\exists\ \alpha_0 \in\Lambda$ such that $0 \in V_{\alpha_0} = (-r, r)$.

$\forall\, r > 0 \quad , \ \exists\, k \in N \quad s.t\ \ 0 < \frac{1}{k} < r \quad , \quad \frac{1}{k} \in V_{\alpha_0}$

$\therefore\ \ 0 < \frac{1}{n} < \frac{1}{k} < r \qquad \forall\, n > k \quad \rightarrow \quad \frac{1}{n} \in V_{\alpha_0} \quad \forall\, n > k$

$\therefore\ \ \frac{1}{n} \in V_{\alpha_0} \quad , \ \forall\, n \geq k \quad , \quad 0 \in V_{\alpha_0} \quad , \quad 1, \cdots, \frac{1}{k-1} \in S$

$1 \in V_{\alpha_1}, \frac{1}{2} \in V_{\alpha_2}, \cdots, \frac{1}{k-1} \in V_{\alpha_{k-1}}$

$\therefore\ \{V_{\alpha_1}, V_{\alpha_2}, \cdots, V_{\alpha_n}\}$ is an open subcovering for $\{V_\alpha\}_{\alpha\in\Lambda}$.

$\therefore\ S \subseteq \cup_{i=1}^{n} V_{\alpha_i}$

$\therefore\ S$ is a compact subset of R.

3) $(0,1)$ is not compact subset of R.

$\forall\, n > 0$, let $A_n = \left(\frac{1}{n}, 1.5\right)$, $(0,1) \subseteq \cup_{n\in N} A_n$, $(0,1) \subseteq \cup_{n\in N} \left(\frac{1}{n}, 1.5\right)$

$\forall\, r > 0 \quad , \ \exists\, k \in N \qquad s.t \quad 0 < \frac{1}{k} < r$

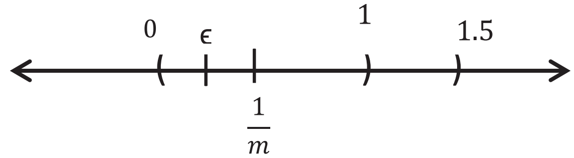
Claim: $\{A_n\}_{n\in N}$ has no finite subcovering for $(0,1)$ if there exists a finite subcovering from $\{A_n\}_{n\in N}$ for S, then:

$(0,1) \subseteq \cup_{i=1}^{m} A_i$

$(0,1) \subseteq A_1 \cup A_2 \cup \cdots \cup A_m$

$$= (1,1.5) \cup \left(\frac{1}{2}, 1.5\right) \cup \left(\frac{1}{3}, 1.5\right) \cup \cdots \cup \left(\frac{1}{m}, 1.5\right) = \left(\frac{1}{m}, 1.5\right)$$

$(0,1) \subseteq \left(\frac{1}{m}, 1.5\right)$ a contraction since $m \neq 0$.



$0 < \epsilon < \frac{1}{m}$

$\forall \epsilon > 0 \;,\; \exists\, k \in Z^{+} \quad s.t \quad 0 < \frac{1}{k} < \epsilon < \frac{1}{m}$

$\therefore (0,1) \;\; s.t \;\; \frac{1}{k} \notin \left(\frac{1}{m}, 1.5\right) \quad C!$

H.W:

$(0,1], [0,1), (-1,1),$ are not compact.

Proposition (4.40):

Let (X, d) be a compact metric space, if S is a closed subset of X, then S is compact

Proof: let $S \subseteq X$, S is a closed.

Let $\{V_\alpha\}_{\alpha \in \wedge}$ is an open covering for S. i.e $S \subseteq \cup_{\alpha \in \wedge} V_\alpha$,

$X = \cup_{\alpha \in \wedge} V_\alpha \cup (X - S)$

Since S is compact, then $\exists\, \alpha_1, \alpha_2, \cdots, \alpha_n \quad s.t \quad X = \cup_{i=1}^{n} V_{\alpha_i} \cup (X - S)$.

$S = X \cap S = \left(\cup_{i=1}^{n} V_{\alpha_i} \cup (X - S)\right) \cap S$

$= \left(\cup_{i=1}^{n} V_{\alpha_i}\right) \cap S \cup (X - S) \cap S = \left(\cup_{i=1}^{n} V_{\alpha_i}\right) \cap S$

$\therefore S \subseteq \left(\cup_{i=1}^{n} V_{\alpha_i}\right)$

Examples:

1) Let $S = \left\{0,1,\frac{1}{2},\frac{1}{3},\cdots,\frac{1}{n},\cdots\right\} \subseteq [0,1]$

$[0,1]$ is a compact subset of R.

$l(S) = 0$ is closed

2) $(0,1)$ is not compact subset of R.

$(0,1) \subseteq [0,1]$

$[0,1]$ is a compact subset of R

Note:

If S is not compact, then S is not closed.

Theorem: (4.41):

Let (X,d) be a metric space, if S is a compact subset of X, then S is closed.

Proof: Suppose that S is not closed.

i.e $\exists$ a cluster point p for S such that $p \notin S$.

$\because p$ is a cluster point for S, then any open set V such that $p \in V$, we have $V \cap S \neq \emptyset$

In particular any ball of form $B_{\frac{1}{n}}(p)$, $n \in N$, $B_{\frac{1}{n}}(p) \cap S \neq \emptyset$.

$D_{\frac{1}{n}}(p) \cap S \neq \emptyset \qquad \cdots (1)$

$\forall\, n \in N$, $D_{\frac{1}{n}}(p)$ is closed set.

Let $V_n = X - D_{\frac{1}{n}}(p)$ is an open set $\forall\, n \in N$.

Claim: $\cap_{n\in N} D_{\frac{1}{n}}(p) = \{p\}$

Suppose $p \neq q$, $q \in \cap_{n\in N} D_{\frac{1}{n}}(p)$, $0 < d(p,q)$

$\exists\, k \in Z^+ \quad s.t \quad \frac{1}{k} < d(p,q)$, $q \notin D_{\frac{1}{n}}(p)$ a contraction since $p \in \cap_{n\in N} D_{\frac{1}{n}}(p)$.

$S \subseteq X - D_{\frac{1}{n}}(p)$, $p \notin S$

$= X - \cap_{n\in N} D_{\frac{1}{n}}(p) = \cup_{n\in N}\left(X - D_{\frac{1}{n}}(p)\right) = \cup_{n\in N}(V_n)$ V_n is open

$\therefore$ V_n is an open covering for S

Since S is compact, then $\exists\, V_1, V_2, \cdots, V_m \quad s.t \quad S \subseteq \cup_{i=1}^{m} V_i = V_1 \cup V_2 \cup \cdots \cup V_m$

Since $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_m$, then $S \subseteq V_m$, then $S \cap V_m \neq \emptyset$.

$S \subseteq X - D_{\frac{1}{m}}(p) \quad then \quad S \cap D_{\frac{1}{m}}(p) \neq \emptyset \quad C! \quad with\ (1)$

Proposition: (4.42):

Let (X,d) be a metric space and S be a compact subset of X, then S is bounded.

Proof: let $x_0 \in S \quad \forall\, n \in N$.

$B_n(x_0) = \{x \in X: d(x,x_0) < n\}$

$\forall\, x \in S$, $\exists\, n \in Z^+ \quad s.t \quad d(x,x_0) < n.1.$

$\therefore$ $S \subseteq \cup_{n\in N}\left(B_n(x_0)\right)$, $\therefore$ $\{B_n(x_0)\}$ is an open covering for S

Since S is compact, then $\exists\ B_1, B_2, \cdots, B_m \quad s.t \quad S \subseteq \cup_{i=1}^{m} B_i$

Then $S \subseteq B_m$, $\therefore S$ is bounded.

Examples:

1) $(0,1)$ is not compact subset of R since not bounded and not closed.
2) Q, Q' is not compact subset of R since not bounded and not closed.
3) R is not compact since not bounded.

Hein Borel theorem(4.43):

Any bounded closed subset of R^k is compact.

Proof: let S be a bounded closed subset of R
Since S is bounded, then there exists an open interval I (ball) such that $S \subseteq I$, and hence $S \subseteq I_1$, where $I_1 = \bar{I}$.

Let $\{V_\alpha\}_{\alpha\in\Lambda}$ is an open covering for S. i.e $S \subseteq \cup_{\alpha\in\Lambda} V_\alpha$, and suppose that S can't be covered by a finite subcovering from $\{V_\alpha\}_{\alpha\in\Lambda}$ $\cdots(*)$.
Divide I_1 into two equal closed intervals $\{I_2 , I_2'\}$ at least one of the sets $I_2 \cap S$ or $I_2' \cap S$ can't be covered by a finite subcovering from $\{V_\alpha\}_{\alpha\in\Lambda}$ for otherwise, we get $S = (I_2 \cap S) \cap (I_2' \cap S)$ covered by a finite subcovering from $\{V_\alpha\}_{\alpha\in\Lambda}$ a contradiction with $(*)$.

Let $I_2 \cap S$ be the set which can't be covered by a finite subcovering from $\{V_\alpha\}_{\alpha\in\Lambda}$.

Divide I_2 into two equal closed intervals $\{I_3 , I_3'\}$ at least one of the sets $I_3 \cap S$ or $I_3' \cap S$ can't be covered by a finite subcovering from $\{V_\alpha\}_{\alpha\in\Lambda}$ say $I_3 \cap S$.

continue in this way, hence we get a sequence of closed intervals $\langle I_n \rangle$ a satisfies:

1) $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots \quad \forall n.$
2) $\forall n \in N$, I_n is a non-empty closed sets.
3) The sequence $\langle |I_n| \rangle = \langle \frac{1}{2^{n-1}} \rangle \to 0$

And $I_n \cap S \ \forall n$ can't be covered by a finite subcovering from $\{V_\alpha\}_{\alpha \in \Lambda}$ by the nested intervals theorem we get $\cap_n \ I_n = \{x_0\}$

Claim: x_0 a cluster point for S.

Let V be an open set such that $x_0 \in V$

Since $|I_n| \to 0$, then $\exists\, k \in Z^+$ such that $I_k \subseteq V$ by Archimedean

$I_k \cap S \subseteq I_k \subseteq V$, but $I_k \cap S$ is an infinite set, $V - \{x_0\} \cap S \neq \emptyset$.

$\therefore$ x_0 a cluster point for S

Since S is closed, then $x_0 \in S$.

Since $x_0 \in S \subseteq \cup_{\alpha \in \Lambda} V_\alpha$, then $\exists\, V_{\alpha_0}$ such that $x_0 \in V_{\alpha_0}$.

Then $\exists\, m \in Z^+$ such that $I_m \subseteq V_{\alpha_0}$, hence $I_m \cap S \subseteq V_{\alpha_0}$ a contradiction since $I_m \cap S$ can't be covered by a finite subcovering from $\{V_\alpha\}_{\alpha \in \Lambda}$.

Corollary(4.44):

Let $S \subseteq R^k$, then S is compact iff S is closed and bounded.

Proof: ⇒) by proposition (4.42) every compact set is bounded and by proposition (4.41) every compact set is closed.

⇐) by Heine-Borel theorem (every bounded and closed subset of R^k is compact).

Examples:

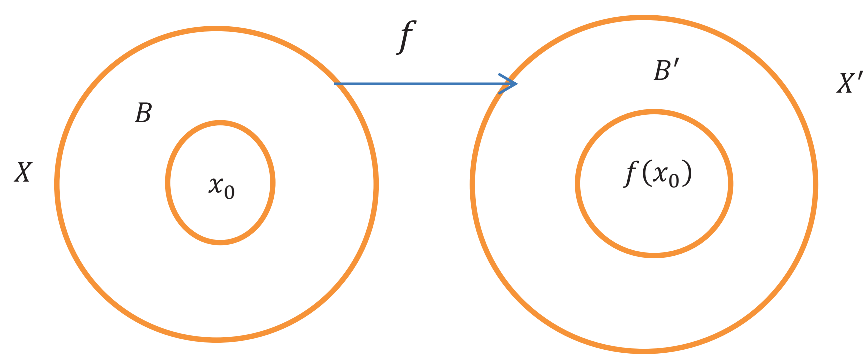
1) $\left\{0,1,\frac{1}{2},\frac{1}{3},\cdots,\frac{1}{n},\cdots\right\}$ is compact.
2) $\left\{1,\frac{1}{2},\frac{1}{3},\cdots,\frac{1}{n},\cdots\right\}$ is not compact.
3) $Q \subseteq R$ is not compact.
4) $(-1,2]$ is not compact.
5) $[a,b]$ is compact.
6) $S = \{X_1, X_2, \cdots, X_n\}$ is compact (every finite set is closed and every finite set is bounded).

Chapter (5)

Continuity:

Definition(5.1):

Let (X,d) and (X',d') be metric spaces and let $f: X \to X'$ be a function, f is said to be continuous at $x_0 \in X$ if $\forall\, \epsilon > 0$, $\exists\, \delta = \delta(x_0,\epsilon)$ such that for any $x \in X$, if $d(x,x_0) < \delta$, then $d'\big(f(x), f(x_0)\big) < \epsilon$.



i.e f is continuous at $x_0 \in X$, if for any ball in X' with center $f(x_0)$ and radius $\epsilon, B'_\epsilon\big(f(x_0)\big)$, there exists a ball $B_\delta(x_0)$ in X with center x_0 and radius δ such that $f(B) \subseteq B'$.

Note:- if f is continuous at each $x_0 \in X$, we say that f is continuous.

Proposition (5.2):-

Let $f: X \to X'$ be a function, then f is continuous at $x_0 \in X$ iff for any open set V in X' with $f(x_0) \in V$, $f^{-1}(V)$ is open in X, where $f^{-1}(V) = \{x \in X: f(x) \in V\}$.

Proof:- ⇒) let V be an open in X' such that $f(x_0) \in V$. To prove $f^{-1}(V)$ is open in X. Since $f(x_0) \in V$, then ∃ aball $B'_\epsilon\big(f(x_0)\big) \subseteq V$ but f is continuous so ∃ a ball B in X such that $f(B) \subseteq B' \subseteq V$. Hence $B \subseteq f^{-1}(V)$

⇐) let $x_0 \in X$, $f(x_0) \in X'$, $B'_\epsilon\big(f(x_0)\big)$ be a ball in X' with center $f(x_0)$ and radius ϵ. To show ∃ a ball $B(x_0)$ in X such that $f(B) \subseteq B'$.

Since B' is open in X', $f(x_0) \in B'$, then by assumption $f^{-1}(B')$ is open in X, clearly $x_0 \in f^{-1}(B')$ (since $f(x_0) \in B'$), then ∃ a ball $B(x_0)$ in X such that $B(x_0) \subseteq f^{-1}(B')$ (by definition of open set), hence $f\big(B(x_0)\big) \in B'\big(f(x_0)\big)$. Thus f is continuous at x_0.

Proposition (5.3):-

Let $f: X \to X'$ be a function, f is continuous at $x_0 \in X$ iff for any closed set E in X' with $f(x_0) \in E$, $f^{-1}(E)$ is closed in X.

Proof:- (H.W)

Hint: $f^{-1}(X' - E) = X - f^{-1}(E)$. E is closed. To prove $f^{-1}(E)$ is closed, we have to show $X - f^{-1}(E)$ is open.

Proposition (5.4):-

Let (X, d) and (X', d') be two metric spaces and let $f: X \to X'$ be a mapping, f is continuous at $x_0 \in X$ iff for each sequence $\langle x_n \rangle$ converge to $x_0 \in X$, the sequence $f\langle x_n \rangle$ converge to $f(x_0)$.

Proof:- :-⇒) Suppose f is continuous at x_0 and let $\langle x_n \rangle$ be a sequence in X. To prove $f\langle x_n \rangle$ converge to $f(x_0)$.

let V be an open set such that $f(x_0) \in V$, since f is continuous at x_0, then $f^{-1}(V)$ is open in X, clearly $x_0 \in f^{-1}(V)$. Since $x_n \to x_0$, then $f^{-1}(V)$ contain most of the terms of the sequence $\langle x_n \rangle$.**i.e** V contain most of the terms of the sequence $\langle f(x_n) \rangle$. Thus $f(x_n) \to f(x_0)$.

⇐) Suppose the result is not true **i.e** $\exists \epsilon > 0$ such that $\forall n \in N , \delta = \frac{1}{n}$, $\exists x_n \in X$ such that ,if $d(x_n, x_0) < \frac{1}{n}$, then $d'\big(f(x_n), f(x_0)\big) \geq \epsilon$ **i.e** $\exists$ a sequence $\langle x_n \rangle$ in X such that $x_n \to x_0 \in X$.(by Archimedes ($\forall \epsilon > 0$, $\exists k \in Z^+$ such that $\frac{1}{k} < \epsilon$, then $d(x_n, x_0) < \frac{1}{n} < \frac{1}{k} < \epsilon \quad \forall n > k$).but $f(x_n) \nrightarrow f(x_0)$ a contradiction, thus the result is true and f is continuous at x_0.

Examples(5-5):

5) Let $f: R \to R$ is defined by $f(x) = c$, $\forall x \in R$. Is f continuous?

Let $x_0 \in R$. To prove f is continuous at x_0. Let $\langle x_n \rangle$ be a sequence in R such that $x_n \to x_0$, we have to show $f(x_n) \to f(x_0)$. $f(x_n) = c$ and $f(x_0) = c$

$\therefore f$ is continuous everywhere since $c \to c$.

6) Let $f: R \to R$is defined by $\mathrm{f(x) = x}$, $\forall \mathrm{x} \in \mathrm{R}$. Is f continuous?

Let $x_0 \in R$. To prove f is continuous at x_0. Let $\langle x_n \rangle$ be a sequence in R such that $x_n \to x_0$, we have to show $f(x_n) \to f(x_0)$. $f(x_n) = x_n$ and $f(x_0) = x_0$, $f(x_n) = x_n \to x_0 = f(x_0)$. Thus $f(x_n) \to f(x_0)$ and f is continuous at x_0.

7) Let $f: \mathrm{R}^+ \to \mathrm{R}$ is defined by $\mathrm{f(x)} = \frac{1}{x}$, $\forall\, \mathrm{x} \in \mathrm{R}^+$. Is f continuous?

Let $x_0 \in \mathrm{R}^+$ and let $\epsilon > 0$ To prove $\exists\, \delta(\epsilon, x_0)$ such that if $|x - x_0| < \delta$, we have to show $|f(x) - f(x_0)| < \epsilon$.

$|f(x) - f(x_0)| = \left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{x - x_0}{x\, x_0}\right| = \frac{|x - x_0|}{x\, x_0} < \epsilon$?

If we take $\delta = 1$, then $|x - x_0| < 1$, but $|x_0| - |x| \leq |\, x_0 - x| < 1$, then $|x_0| - 1 < |x|$, which implies that $|x| > |x_0| - 1$.

So that $\frac{|x - x_0|}{x\, x_0} < \frac{|x - x_0|}{x_0} < \epsilon$? , then choose $\delta = \min\{1, x_0\epsilon\}$ $i.e$ $|x - x_0| < x_0\epsilon$.

Now, it is easy to show that δ satisfies this relation. In fact if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| = \left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{x - x_0}{x\, x_0}\right| = \frac{|x - x_0|}{x\, x_0} < \frac{|x - x_0|}{x_0} < \frac{\delta}{x_0} \leq \frac{x_0\epsilon}{x_0} < \epsilon$. Thus f is continuous.

Real value mapping:-

Let (X, d) be a metric space, the mapping $f: X \to R$ is called a real valued mapping.

Proposition (5.6):-

Let (X, d) be a metric space and $f, g: X \to R$ be a real valued mapping, if f and g are continuous at x_0 , then:-

1. $f \mp g$ is continuous at x_0 such that $(f \mp g)(x) = f(x) \mp g(x)$
2. $f.g$ is continuous at x_0 such that $(f.g)(x) = f(x).g(x)$
3. $\frac{f}{g}$ is continuous at x_0 , $g \neq 0$ such that $\frac{f}{g}$ $(x)=\frac{f(x)}{g(x)}$.
4. cf is continuous at x_0 such that $(cf)(x) = cf(x)$.
5. $|f|$ is continuous at x_0 such that $|f|(x) = |f(x)|$.

Example:

If $f:[-2,3] \to R$ and $g:(0,4] \to R$, then $f+g:(0,3] \to R$

Proof:- (3)

$\frac{f}{g}: X \to R$

Let $\langle x_n \rangle$ be a sequence in X such that $x_n \to x_0$, we have to show $\frac{f}{g}(x_n) \to \frac{f}{g}(x_0)$.

$x_n \to x_0$, since f and g continuous at x_0, then. $f(x_n) \to f(x_0)$ and $g(x_n) \to g(x_0)$, hence $\frac{f(x_n)}{g(x_n)} \to \frac{f(x_0)}{g(x_0)}$, then $\frac{f}{g}(x_n) \to \frac{f}{g}(x_0)$. Thus $\frac{f}{g}$ is continuous at x_0.

Proposition :

Let (X,d), (X',d') and (X'',d'') be metric spaces and $f:X \to X'$ be a continuous mapping at x_0 and $g:X' \to X''$ be a continuous mapping at $f(x_0)$, then $f \circ g: X \to X''$ is a continuous mapping at x_0.

Proof:- (H.W)

Definition (5.7):-

Let $f:X \to R$ be a real valued mapping, we say that f is bounded if there exists $M \in R$, $M > 0$ such that $|f(x)| \le M \quad \forall x \in X$.

$i.e \; -M \le f(x) \le M \quad \forall x \in X$, $f(X) = \{f(x): x \in X\}$.

Proposition (5.8):-

Let $f:X \to X'$ be a continuous mapping, if X is compact, then $f(X)$ is compact, hence $f(X)$ is bounded and closed.

Proof:

Let $\{V_\alpha\}$ be an open covering for $f(X) = Y$, $f(X) \subseteq \cup_{\alpha\in\Lambda} V_\alpha \quad \forall\alpha \in \Lambda$, V_α is open in X', since f is a continuous, then $f^{-1}(V_\alpha)$ is open in $X \quad \forall\alpha \in \Lambda$, since $f(X) \subseteq \cup_{\alpha\in\Lambda} V_\alpha \quad \cdots (1)$.

Claim: $X = \cup_{\alpha\in\Lambda} f^{-1}(V_\alpha) \ \forall\alpha \in \Lambda$.

Let $x \in X$, then $f(x) \in f(X) \subseteq \cup_{\alpha\in\Lambda} V_\alpha$ from (1), hence $\exists\, \beta \in \Lambda \ \ni\ f(x) \in V_\beta \ \ iff \ \ x \in f^{-1}(V_\beta)$, then $x \in \cup_{\alpha\in\Lambda} f^{-1}(V_\alpha)$, hence $X \subseteq \cup_{\alpha\in\Lambda} f^{-1}(V_\alpha)$ and $\cup_{\alpha\in\Lambda} f^{-1}(V_\alpha) \subseteq X$. Thus $X = \cup_{\alpha\in\Lambda} f^{-1}(V_\alpha)$. So that $\{ f^{-1}(V_\alpha)\}$ is an open covering for X.

Since X is compact, then $\exists\, \alpha_1, \alpha_2, \cdots, \alpha_n \in \Lambda$ such that $X = \cup_{i=1}^{n} f^{-1}(V_{\alpha_i})$, then $f^{-1}(V_{\alpha_i}) = \{x \in X : f(x) \in V_{\alpha_i}\}$ and $f(X) \subseteq \cup_{i=1}^{n} V_{\alpha_i}$,. Thus $f(X)$ is compact. Also since $f(X)$ is compact, then by Hein Boral theorem $f(X)$ is bounded and closed.

Remark:-

Let $f : (0, \infty) \to R$ be a real valued mapping defined by $f(x) = \frac{1}{x} \quad \forall x \in (0, \infty)$, then.

1) f is continuous mapping.
2) $(0, \infty) \subseteq R$ is not compact.
3) f is not bounded.

In fact $\forall\, M \in R$, $M > 0$, $\exists\, n \in Z^+$ such that $f(n) = \frac{1}{n} > M$.

Definition (5.9):-

Let $f : X \to X'$ be a mapping, if there exists $x_0 \in X$ such that $f(x_0) \leq f(x) \quad \forall x \in X$, then x_0 is called a minimum point, if there exists $z_0 \in X$ such that $f(x) \leq f(z_0) \quad \forall x \in X$, then z_0 is called a maximum point.

Proposition (5.10):-

Let $f : X \to R$ be a continuous mapping, if X is compact, then there exists $x_0,\ z_0 \in X$ such that $f(x_0) \leq f(x) \leq f(z_0) \quad \forall x \in X$.

$i.e$ (f has minimum and maximum point).

Proof:

By proposition (5.8) $f(X)$ is compact and hence is closed and bounded, since $f(X)$ is bounded (below, above).

Below: $\exists\ N \in R$ such that $N \leq f(x) \quad \forall x \in X$, if $N \in f(X) = \{f(x): x \in X\} \subseteq R$, then $\exists\ x_0 \in X$ such that $N = f(x_0)$, then $N = f(x_0) \leq f(x)$. Thus x_0 is the minimum point.

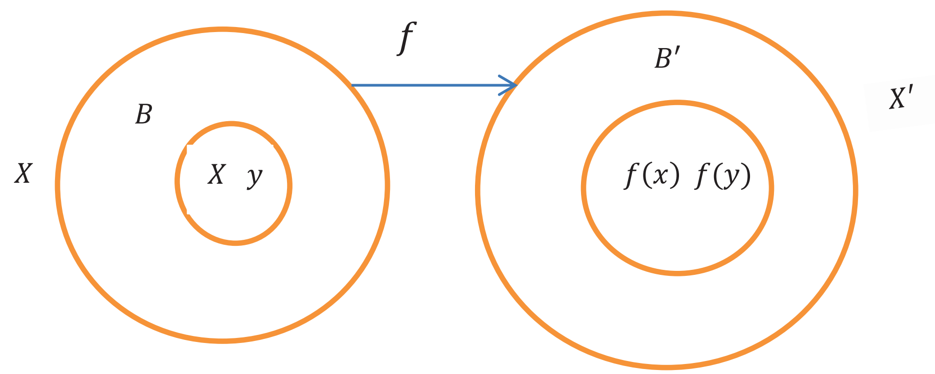
If $N \notin f(X) = Y$, then N is a cluster point for $f(X) = Y$, $N \in (-\epsilon, \epsilon)$, $(-\epsilon, \epsilon) \cap f(X) \neq \emptyset$ (Since $N = g.l.b(f(x))$, hence N is a cluster point for $f(X) = Y$). Thus $N \in f(X)$ (since $f(X)$ is closed). Then $\exists\ x_0 \in X$ such that $N = f(x_0)$, then $N = f(x_0) \leq f(x)$. Thus x_0 is the minimum point.

Above: (H.W).

Uniform Continuity

Definition(5.11):

Let (X, d) and (X', d') be metric spaces and let $f: X \to X'$ be a mapping, we say that f is uniformly continuous if $\forall\ \epsilon > 0$, $\exists\ \delta = \delta(\epsilon)$ such that, if $d(x, y) < \delta$, then $d'(f(x), f(y)) < \epsilon \quad \forall x, y \in X$.



Clearly every uniformly continuous mapping is continuous, but the convers is not true as the following example show:

Example:

Let $f: R \to R$ is defined by $f(x) = x^2$, $\forall x \in R$. Is f continuous?

Let $x_0 \in R$. To prove f is continuous at x_0. Let $\langle x_n \rangle$ be a sequence in R such that $x_n \to x_0$, we have to show $f(x_n) \to f(x_0)$. $f(x_n) = {x_n}^2 = x_n \cdot x_n \to x_0 \cdot x_0 = {x_0}^2 = f(x_0)$. Thus $f(x_n) \to f(x_0)$ and f is continuous.

The proof of continuity by using definition:
let $\epsilon > 0$, $\exists\, \delta$? $x_0 \in X$, if $|x - x_0| < \delta$ *and* $\delta < 1$, then $|x| - |x_0| < |x - x_0| < \delta < 1$, hence $|x| < 1 + |x_0|$ and $|x + x_0| < |x| + |x_0| < 1 + 2|x_0|$
$|f(x) - f(x_0)| = |x^2 - {x_0}^2| = |x - x_0|\,|x + x_0| \le |x - x_0|\,(|x| + |x_0|) < |x - x_0|(1 + 2|x_0|)$. Take $\delta = \min\left\{1 , \frac{\epsilon}{1+2|x_0|}\right\}$
Thus $|f(x) - f(x_0)| < \epsilon$ and f is continuous.
But f is not uniformly continuous.
Take $x_n = n + \frac{1}{n}$, $y_n = n$ $n \in N$ $|x_n - y_n| = \frac{1}{n}$, by Archimedean $\forall\, \delta > 0$,
$\exists\, k$ such that $\frac{1}{k} < \delta$, $|x_k - y_k| = \frac{1}{k} < \delta$.
$|f(x_k) - f(y_k)| = |{x_k}^2 - {y_k}^2| = \left|\left(k + \frac{1}{k}\right)^2 - k^2\right|$
$= \left|k^2 + 2 + \frac{1}{k^2} - k^2\right| = 2 + \frac{1}{k^2} > \epsilon = 2$. Thus f is not uniformly continuous.

<u>Notice</u> that f is uniformly continuous on $(-1, a]$, $\forall a \ge 1$.

let $\epsilon > 0$, $\exists\, \delta(\epsilon)$? , such that $\forall\, x, y \in X$, if $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$
$|f(x) - f(y)| = |x^2 - y^2| = |x - y|\,|x + y| \le |x - y|(\,|x| + |y|) \le |x - y|(\,a + a) = |x - y|(\,2a) < \epsilon$

Take $\delta = \frac{\epsilon}{2a}$ $\forall\, x, y \in (-1, a]$.

Example:

Let $f: (0, \infty) \to R$ is defined by $f(x) = \frac{1}{x}$, $\forall x \in (0, \infty)$. f is continuous but not uniformly continuous on $(0, \infty)$. Take $\delta = \min\{1 , |x_0|\epsilon\}$

Let $\epsilon = 1$ we must show that $\forall\ \delta > 0$ there exists $x, y \in (0, \infty)$ such that $|x - y| < \delta$ but $|f(x) - f(y)| > 1$. By Archimedean there exist a positive integer n such that $\frac{1}{n} < \delta$.

Let $x_n = \frac{1}{n}$, $y_n = \frac{2}{n}$ $n \in N$.
$|x_n - y_n| = \left|\frac{1}{n} - \frac{2}{n}\right| = \left|-\frac{1}{n}\right| = \frac{1}{n}$, $\forall\ \delta > 0$, $\exists\ k$ such that $\frac{1}{k} < \delta$, hence
$|x_n - y_n| < \delta$
$|f(x_k) - f(y_k)| = \left|\frac{1}{x_k} - \frac{1}{y_k}\right| = \left|k - \frac{k}{2}\right| = \frac{k}{2} > \epsilon = \frac{1}{4}$

By Archimedean there exist a positive integer n such that $\frac{n.k}{2} > 1 \Rightarrow \frac{k}{2} > \frac{1}{n} = \epsilon.$

Or let $x = \frac{1}{n}$, $y = \frac{2}{n}$, then $|x - y| = \frac{1}{n} < \delta$ but $|f(x) - f(y)| = \left|n - \frac{n}{2}\right| = \frac{n}{2} > 1.$
Thus f is not uniformly continuous.

Notice that f is uniformly continuous on $[a, \infty)$, $\forall a > 0.$

let $\epsilon > 0$, $\exists\ \delta(\epsilon)$? , such that $\forall\ x, y \in X$, if $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$
$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y-x}{xy}\right| = \frac{|y-x|}{xy} \le \frac{|y-x|}{a^2}.$

So that if $\epsilon > 0$, we can find $\delta = a^2\epsilon$. In this case if $|x - y| < \delta$, then

$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|y-x|}{xy} \le \frac{|y-x|}{a^2} < \frac{\delta}{a^2} = \epsilon$. Thus f is not uniformly continuous.

H.W:

Let $f: R^+ \to R$ is defined by $f(x) = sin\frac{1}{x}$, $\forall x \in (0, \infty)$. Prove that f is continuous but not uniformly continuous on $(0, \infty)$.

Theorem (5.12):-

Let $f: X \to R$ be a continuous mapping, if X is compact, then f is uniformly continuous.

Proof:

let $\epsilon > 0$, $\exists\, \delta(\epsilon)$? , such that $\forall\, x, y \in X$, if $d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon$, let $x \in X$, $f(x) \in R$, let $V_{f(x)} = (f(x) - \frac{\epsilon}{2}, f(x) + \frac{\epsilon}{2})$ be an open interval, since f is continuous, then $\exists\, U_{\delta(x)}$ is open in X and $f\left(U_{\delta(x)}\right) \subseteq V$ and $x \in U_{\delta(x)}$.

Let $B_{\frac{1}{3}\delta(x)}(x)$ be a ball with center x and radius $\frac{1}{3}\delta(x)$, hence $\left\{B_{\frac{1}{3}\delta(x)}(x)\right\}_{x \in X}$ is an open covering for X $i.e$ $X = \cup_x B_{\frac{1}{3}\delta(x)}(x)$, since X is compact, then $\exists\, x_1, x_2, \cdots, x_n$ such that $X = \cup_{i=1}^{n} B_{\frac{1}{3}\delta(x_i)}(x_i)$.

Choose $\delta = min\left\{\frac{1}{3}\delta(x_1), \frac{1}{3}\delta(x_2), \cdots, \frac{1}{3}\delta(x_n)\right\}$.

Claim:- δ is satisfies the condition of uniformly continuous.

$\forall\, x, y \in X$, if $d(x,y) < \delta$ $T.P$ $|f(x) - f(y)| < \epsilon$.

Since $x \in X = \cup_{i=1}^{n} B_{\frac{1}{3}\delta(x)}(x_i)$, then $\exists\, k \in N$ such that $x \in B(x_k)$ i.e $d(x, x_k) < \frac{1}{3}\delta(x_k)$.

$d(y, x_k) \leq d(y, x) + d(x, x_k) < \delta + \frac{1}{3}\delta(x_k) < \frac{1}{3}\delta(x_k) + \frac{1}{3}\delta(x_k) < \delta(x_k)$, since f is continuous, then

$|f(x) - f(y)| = |f(x) - f(x_k) + f(x_k) - f(y)| \leq |f(x) - f(x_k)| + |f(x_k) - f(y)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Corollary (5.13):-

If $f: [a, b] \to R$ is a continuous mapping, then f is uniformly continuous.

Examples:-

1. Let $f: [-2,3] \to R$ be defined by $f(x) = x^4 + 2x^3 + 3x^2 + 5$.
   Let $\langle x_n \rangle$ be a sequence such that $x_n \to x_0$.

$f(x_n) = {x_n}^4 + 2{x_n}^3 + 3{x_n}^2 + 5 = x_n.x_n.x_n.x_n + 2x_n.x_n.x_n + 3x_n.x_n + 5$

$f(x_0) = {x_0}^4 + 2{x_0}^3 + 3{x_0}^2 + 5$

Since $x_n \to x_0$, then $f(x_n) \to f(x_0)$.

2. Let $f: [1,7] \to R$ be defined by $f(x) = \frac{2}{x} \quad \forall\, x \in [1,7]$.

Let $\langle x_n \rangle$ be a sequence such that $x_n \to x_0$. $f(x_n) = \frac{2}{x_n}$, $f(x_0) = \frac{2}{x_0}$

Since $x_n \to x_0$, then $f(x_n) \to f(x_0)$, hence f is uniformly continuous.

3. Let $f: [-2,2] \to R$ be defined by $f(x) = \sin(x) \quad \forall\, x \in [-2,2]$.

Let $\langle x_n \rangle$ be a sequence such that $x_n \to x_0$. $f(x_n) = \sin(x_n)$, $f(x_0) = \sin(x_0)$

Since $x_n \to x_0$, then $f(x_n) \to f(x_0)$, hence f is uniformly continuous.

Definition (5.14):-The intermediate value property

Let $f: [a, b] \to R$ be a mapping, f is said to be satisfies the intermediate value property, if for all $x, y \in [a, b]$ and for each z between $f(a)$ and $f(b)$, then there exists s between x and y such that $f(s) = z$.

Theorem (5.15):- The intermediate value - theorem

Let $f: [a, b] \to R$ be a continuous mapping and z between $f(a)$ and $f(b)$, there exists s in $[a, b]$ such that $f(s) = z$

Proof:

Let $I = [a, b]$, since z between $f(a)$ and $f(b)$, then either $f(a) < z < f(b)$ or $f(b) < z < f(a)$

1) If $f(b) < z < f(a)$, let $m = \frac{a+b}{2}$.

If $f(m) = z$, then we are done.

If not $i.e\ f(m) \neq z$, then either $f(m) < z$, then $f(m) < z < f(a)$ or $f(m) > z$, then $f(b) < z < f(m)$ **.**

Let $[a_1, b_1] = [a, m]$ or $[a_1, b_1] = [m, b]$, then either $f(a_1) < z < f(b_1)$ or $f(b_1) < z < f(a_1)$

Let $I_1 = [a_1, b_1]$, let $m_1 = \frac{a_1+b_1}{2}$.

If $f(m_1) = z$, then we are done.

If either $f(m_1) < z$, then $f(m_1) < z < f(a_1)$ or $f(m_1) > z$, then $f(b_1) < z < f(m_1)$ **.**

Let $[a_2, b_2] = [a_1, m_1]$ or $[a_2, b_2] = [m, b_1]$, then either $f(a_2) < z < f(b_2)$ or $f(b_2) < z < f(a_2)$ **.**

Continuo in this way we get a sequence of closed intervals $\langle I_n \rangle$ such that $f(b_n) < z < f(a_n) \quad \forall\, n \quad \cdots (2)$ **.**and $|\, I_n|\ \to 0$ sinc ($\frac{1}{2^{n-1}} \to 0$), hence by nested intervals theorem of closed intervals $\cap_n\ I_n = \{s\}$

<u>Claim</u>:- $f(s) = z$

$a_n \to s$ and $b_n \to s$, Since $|\, I_n|\ \to 0$, then $\forall \epsilon > 0$, $\exists\, k \in N$, such that $|\, I_k| < \epsilon$.
$|a_k - s\,| < |\, I_k| < \epsilon \quad \forall\, n > k$ and $|b_k - s\,| < |\, I_k| < \epsilon \quad \forall\, n > k$, since f is continuous, then $f(a_n) \to f(s)$ and $f(b_n) \to f(s)$ and by (2) $f(b_n) < z < f(a_n) \quad \forall\, n$. Thus $f(s) = z$.

If $f(s) \neq z$, then either $f(s) < z$ or $f(s) > z$.

If $f(s) < z < f(a_n)\ \to\ f(s) \quad C!$.

If $f(s) > z > f(b_n)\ \to\ f(s) \quad C!$.

2) $\quad f(a) < z < f(b)$ **<u>(H.W)</u>**.

Examples:-

1) Let $f:[a,b] \to R$ be a continuous mapping, if $\forall\, x \in [a,b]$, $f(-x) = -f(x)$, then f has at least one real root.

Proof:

If $f(x) > 0 \Longrightarrow f(-x) = -f(x) < 0$

$f(-x) = -f(x) < f(x)$

$-f(x) < 0 < f(x) \Longrightarrow$

Thus By intermediate value theorem $\exists\, s$ such that $-x < s < x$ where $f(s) = 0$

If $f(x) < 0$

Then $-f(x) > 0 \Longrightarrow f(-x) > 0$

$f(x) < 0 < f(-x)$

Then $\exists\, s$ such that $x < s < -x \quad and \quad f(s) = 0$

2) If $f(x) = x^3 + 3x$ odd and continuous.

$\therefore$ by satisficing theorem (5.15)

If $f(x) > 0 \Longrightarrow f(-x) = -f(x) < 0$

$f(-x) < 0 < f(x) \Longrightarrow \exists\; s$ such that $f(s) = 0$

If $f(x) < 0 \Longrightarrow -f(x) > 0 \Longrightarrow f(-x) > 0$

$f(x) < 0 < f(-x)$

Hence $\exists\, s$ such that $f(s) = 0$

3) If $p(x)$ is even, then p may have <u>no</u> real root.

Example:-

$x^2 + 1 = 0$

$x = \mp i$

Prewar Theorem (5.16):-

Let $f: D^n \to D^n$ be a continuous mapping, then f has at least one fixed point where D^n disk in R.

Example:

Let $f:[a,b] \to [a,b]$ be a continuous mapping, then f has at least one fixed point.

Sol:

Let $g(x) = f(x) - x$ $\qquad$ g is continuous mapping on $[a,b]$.

$g(a) = f(a) - a \geq 0 \qquad a \leq f(a) \leq b.$
$g(b) = f(b) - b \leq 0 \qquad a \leq f(a) \leq b.$
$g(b) < 0 < g(a)$
By theorem (5.15) $\exists\, x \in [a, b]$ such that $g(x) = 0$
$f(x) - x = 0$
$f(x) = x$
$\therefore$ f has at least one fixed point.

Chapter (6)

Sequences and series of functions:

Definition (6.1):

Let $D \subseteq R$. Define $F(D) = \{\, f: D \to R: f \text{ is a mapping} \}$, the sequence $\langle f_n \rangle$, $n \in N$ is called a sequence of function where $\forall\, n \in N,\ f_n \in F(D)$, $f_n: D \to R$.

Definition (6.2):**(Point wise convergence and uniform convergence)**

Let $\langle f_n \rangle$ be a sequence of function on D, we say that $\langle f_n \rangle$ converges to a function f on D, if $\forall\, \epsilon > 0 \ \ and \ \ \forall\, x \in D$, $\exists\, k \in Z^+$, $k = k(\epsilon, x)$ such that $|f_n(x) - f(x)| < \epsilon$.

$\forall\, n > k$.

In this case, we say that $\langle f_n \rangle$ converges point wise to a function f for short, we write $f_n \overset{p.w}{\longrightarrow} f$

$i.e\ \lim_{n \to \infty} f_n(x) = f(x) \quad \forall\ x \in D \qquad or \qquad f_n(x) \to f(x) \qquad \forall\ x \in D$

And $\langle f_n \rangle$ converges uniformly to a function f on D, if $\forall\, \epsilon > 0$, $\exists\, k \in Z^+$, $k = k(\epsilon)$ such that $|f_n(x) - f(x)| < \epsilon \quad \forall\, n > k$, $\forall\ x \in D$, for short we write $f_n \overset{u.}{\to} f$

Examples (6-3):

8) $\forall\, n \in N$, let $f_n : R \to R$ be defined by $f_n(x) = \frac{x}{n}$, $\forall x \in R$. Is $\langle f_n \rangle$ converges point wise to $f = 0$?

$\langle f_n \rangle = \langle \frac{x}{n} \rangle$, $\forall x \in R$. $\langle f_n \rangle = x, \frac{x}{2}, \frac{x}{3}, \cdots, \frac{x}{n}$.

1) $\lim_{n\to\infty}\langle f_n(x) \rangle = \lim_{n\to\infty} \frac{x}{n} = 0$, then a sequence $f_n \to 0$.
2) Let $\epsilon > 0$

   $|f_n(x) - 0| = \left|\frac{x}{n} - 0\right| = \left|\frac{x}{n}\right| = \frac{|x|}{n}$, by Archimedean property $\exists\, k \in Z^+$ $s.t$ $|x| < k\epsilon$, then $\frac{|x|}{k} < \epsilon$

   $\therefore$ $|f_n(x) - 0| = \left|\frac{x}{n}\right| = \frac{|x|}{n} < \frac{|x|}{k} < \epsilon$ $\forall\, n > k$.

   Thus $f_n(x) = \frac{x}{n} \overset{p.w}{\longrightarrow} 0$

   But $f_n(x) = \frac{x}{n}$ does not converge to 0 uniformly.

   Since if $\forall\, x \in R$, $\exists\, k_0 = k_0(\epsilon)$, $\frac{|x|}{k_0} < \epsilon$, then $|x| < k_0\epsilon$ $i.e$ $k_0\epsilon < x < k_0\epsilon$.

   Which is contradiction, since R is <u>not</u> bounded.

To show the sequence $\langle \frac{x}{n} \rangle$ is converges uniformly to a function $f = 0$ $\forall\, x \in (0, a]$

$\forall n$ $\frac{x}{n} : (0, a] \to R$.

$|f_n(x) - 0| = \left|\frac{x}{n} - 0\right| = \left|\frac{x}{n}\right| = \frac{|x|}{n} \le \frac{a}{n}$ $\forall\, x \in (0, a]$, by Archimedean property on ϵ,a $\exists\, k \in Z^+$ $s.t$ $a < k\epsilon$, then $\frac{a}{k} < \epsilon$

$\therefore$ $\left|\frac{x}{n}\right| \le \frac{a}{n} < \frac{a}{k} < \epsilon$ $\forall\, n > k$. Thus $f_n(x) = \frac{x}{n} \overset{u.}{\to} 0$ on $(0, a]$.

9) $\forall\, n \in N$, let $f_n : [0,1] \to R$ be defined by $f_n(x) = x^n$, $where$ $0 \le x \le 1$. Is $\langle f_n \rangle$ converges point wise to $f = 0$?

$\langle x^n \rangle$ is decreasing sequence and bounded below by zero, so it is converges sequence. In fact if $\epsilon > 0$, $\exists\, k \in Z^+$ $s.t$ $x^k < \epsilon$, which implies that $x^n < \epsilon$ $\forall\, n > k$, therefore If $x = 0$, then $f_n = x^n \to 0$.

If $x = 1$, then $f_n = x^n = 1, 1, 1, \cdots, 1 \to 1$ 0 ,then $f_n = x^n \to 1$.

Thus $f_n \xrightarrow{p.w} f$ $\quad when \quad f(x)=\begin{cases}0 & 0\le x<1\\ 1 & x=1\end{cases}$

But f_n does not converge uniformly.

Is $\exists\, k \in Z^+ \;\; s.t \;\; |f_n(x)-f(x)|<\epsilon \quad \forall\, n>k \quad \forall x\in[0,1]$?

Specially is $\exists\, k \quad s.t \quad |x^n|<\epsilon \quad \forall\, n>k \quad \forall x\in[0,1]$?

if $x_n = 2^{-\frac{1}{n}}$, then $|f_n(x)-0| = (x^n)^n = \left(2^{-\frac{1}{n}}\right)^n = \frac{1}{2} > \frac{1}{4} = \epsilon$.

Thus $\langle f_n\rangle$ does not converge uniformly on $[0,1]$.

To show the sequence $\langle x^n\rangle$ is converges uniformly to a function $f=0 \quad \forall\, x\in[0,a]$

$\forall\, n \quad x^n : [0,a]\to R \quad \forall x\in[0,1] \quad 0<a<1$.

$|f_n(x)-f(x)| = |x^n-0| = |x^n| < \epsilon$? , by Archimedean property on ϵ,

$\exists\, k\in Z^+ \;\; s.t \quad k<\epsilon$, then $a^k<\epsilon$

$\therefore\; |x^n| \le x^n < a^n < a^k < \epsilon \quad \forall\, n>k \quad x\in[0,a]$.

Thus $f_n(x) = x^n \xrightarrow{u.} 0$ on $[0,a]$.

10) $\forall\, n\in N$, let $f_n : R\to R$ be defined by $f_n(x) = \frac{nx}{1+n^2x^2}$, $\forall x\in R$. Is $\langle f_n\rangle$ converges point wise to $f=0$?

$\langle f_n\rangle = \langle \frac{nx}{1+n^2x^2}\rangle$, $\forall x\in R$.

Let $\epsilon>0$

$|f_n(x)-0| = \left|\frac{nx}{1+n^2x^2}\right| < \frac{n|x|}{n^2x^2} = \frac{1}{n|x|}$, by Archimedean property on $|x|\epsilon , 1$, $\exists\, k = k(\epsilon,x) \quad s.t \quad \frac{1}{k} < |x|\epsilon$, then $\frac{1}{k|x|}<\epsilon$

$\therefore\; |f_n(x)-0| = \left|\frac{nx}{1+n^2x^2}\right| < \frac{1}{n|x|} < \frac{1}{k|x|} < \epsilon \quad \forall\, n>k$. Thus $f_n \xrightarrow{p.w} 0$

But $f_n(x) = \frac{x}{n}$ does not converge to 0 uniformly.

Since if if $x_n = \frac{1}{n}$, then $|f_n(x)-0| = \left|\frac{n\frac{1}{n}}{1+n^2\left(\frac{1}{n}\right)^2}\right| = \frac{1}{2} > \frac{1}{4} = \epsilon$.

Thus $\langle f_n\rangle$ does not converge uniformly on $[0,1]$.

To show the sequence $\langle\frac{nx}{1+n^2x^2}\rangle$ is converges uniformly to a function $f=0 \quad \forall\, x\in(a,\infty)$

$\forall\, n \quad \frac{nx}{1+n^2x^2} : (a,\infty)\to R$.

$|f_n(x) - f(x)| = \left|\frac{nx}{1+n^2x^2} - 0\right| = \left|\frac{nx}{1+n^2x^2}\right| < \left|\frac{nx}{n^2x^2}\right| = \frac{1}{n|x|} < \frac{1}{na} < \epsilon\ ?$ $\quad \forall\, x \in (a\,.,\infty)$, by Archimedean property on $a\epsilon\,, 1$, $\exists\, k \in Z^+ \quad s.t \quad 1 < ka\,\epsilon$, then $\frac{1}{ka} < \epsilon$

$\therefore \left|\frac{nx}{1+n^2x^2}\right| < \frac{1}{na} < \frac{1}{ka} < \epsilon \qquad \forall\, n > k \ ,\ k = k(\epsilon) \ ,\ x \in (a,\infty)$.

Thus $\langle f_n \rangle$ converges uniformly on (a,∞).

Proposition (6.4):-

Let $\langle f_n \rangle$ be a sequence of function such that $\langle f_n \rangle$ convergence point wise to a function f on D, and $\langle T_n \rangle = Sup_{x\in D}|f_n(x) - f(x)|$, then $\langle f_n \rangle$ converge uniformly to $f\ iff\ \langle T_n \rangle \to 0$.

$T_1 = Sup_{x\in D}|f_1(x) - f(x)|$
$T_2 = Sup_{x\in D}|f_2(x) - f(x)|$
$T_3 = Sup_{x\in D}|f_3(x) - f(x)|$
$\vdots$

Example:

$\forall\, n \in N$, let $f_n\colon [0,1] \to R$ be defined by $f_n(x) = \frac{x}{nx+1}$, $\forall x \in R$. Show that whether $\langle f_n \rangle$ convergence uniformly or not.?

$T_n = Sup_{x\in D}|f_n(x) - f(x)| = Sup_{x\in D}\left|\frac{x}{nx+1} - 0\right| = Sup_{x\in[0,1]}\left|\frac{x}{nx+1}\right|$ and by proposition (5.10) $Sup_{x\in[0,1]}\left|\frac{x}{nx+1}\right| = max_{x\in[0,1]}\left|\frac{x}{nx+1}\right| = \frac{1}{n+1}$.

$f_n(x) = \frac{x}{nx+1}$

$f'_n(x) = \frac{(nx+1)-nx}{(nx+1)^2} = \frac{1}{(nx+1)^2} > 0$.

Then $\forall\, n \in N\ \ f_n(x)$ is increasing function, hence $\langle T_n \rangle = \langle \frac{1}{n+1} \rangle$ so that $T_n \to 0$. Thus $\frac{x}{nx+1} \xrightarrow{u.} 0$. By (6.4).

The following propositions give some properties of uniformly converges.

Proposition (6.5):-

Let $\langle f_n \rangle$ be a sequence of mapping on D, if $\forall\, n \in N \quad f_n$ is bounded on D and $\langle f_n \rangle$ converges uniformly to f on D, then f is bounded on D.

Proof:

To proof $\exists\, M' > 0$, $M' \in R \quad s.t \quad |f(x)| \leq M' \quad \forall\, x \in D$

Since $f_n \overset{u.}{\rightarrow} f$ **i.e** $\forall\, \epsilon > 0$, $\exists\, k \in Z^+$, $k = k(\epsilon)$ such that $|f_n(x) - f(x)| < \epsilon < 1$

$\forall\, n > k$, $\forall\; x \in D$.

Since f_n is bounded $\forall\, n$, then $\exists \;\; 0 < M \in R$ such that $|f_n(x)| \leq M$, $\forall\; x \in D$, hence $|f_{k+1}(x)| \leq M$.

$|f(x)| = |f(x) - f_{k+1}(x) + f_{k+1}(x)|$

$\leq |f(x) - f_{k+1}(x)| + |f_{k+1}(x)| \leq 1 + M \leq M' \quad \forall\, x \in D$.

Thus $|f(x)| \leq M' \qquad \forall\, x \in D$.

Proposition (6.6):-

Let $\langle f_n \rangle$ be a sequence of continuous mapping on D such that $\langle f_n \rangle$ converges uniformly to f on D, then f is continuous.

Proof:

To proof f is continuous at a point x_0, it's enough to show that for each sequence $\langle x_n \rangle$ converge to x_0 on D, the sequence $f\langle x_n \rangle$ converge to $f(x_0)$.

Let $x_0 \in D$ and $\langle x_m \rangle$ be a sequence on D such that $x_m \rightarrow x_0$ T.P. $f(x_m) \rightarrow f(x_0)$.

Since f_n is continuous and $x_m \rightarrow x_0$, then $f_n(x_m) \rightarrow f_n(x_0)$, $i.e \;\; \exists\, k_1 \in Z^+$ such that $|f_n(x_m) - f_n(x_0)| < \frac{\epsilon}{3}$

$\forall\, m > k_1$.

Take $\epsilon > 0$, since $f_n \overset{u.}{\rightarrow} f$, $\exists\, k_2 \in Z^+$, $k = k(\epsilon)$ such that $|f_n(x) - f(x)| < \frac{\epsilon}{3}$

$\forall\, n > k_2$, $\forall\; x \in D$.

$|f(x_m) - f(x_0)| = |f(x_m) - f_n(x_m) + f_n(x_m) - f_n(x_0) + f_n(x_0) - f(x_0)| \le$
$|f_n(x_m) - f(x_m)| + |f_n(x_m) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$,
take $k = max\{k_1, k_2\}$. Thus $f(x_m) \to f(x_0)$.

Remark:

The above Proposition is not true if $f_n \overset{p.w}{\longrightarrow} f$ for example

$f_n = x^n = x, x^2, x^3, \cdots, x^n, \cdots$, then $f_n = x^n \to 1$.

Thus $f_n = x^n \overset{p.w}{\longrightarrow} f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$

But $f = 1$ is not continuous.

Theorem (6.7):-

Let $\langle f_n \rangle$ be a sequence of continuous mapping on D that converges to f on D, if either $f_{n+1}(x) \le f_n(x) \quad \forall\ x \in D \quad \forall\, n \in N$ or $f_{n+1}(x) \ge f_n(x) \quad \forall\ x \in D \quad \forall\, n \in N$ and D is compact, then $\langle f_n \rangle$ converges uniformly to f on D.

Proof:

Case (1):- When $f_{n+1}(x) \le f_n(x) \quad \forall\ x \in D \quad \forall\, n \in N$, to proof $f_n \overset{u.}{\to} f$.
Let $g_n = f_n \overset{u.}{\to} f$ we will prove that $g_n \overset{u.}{\to} 0$.
$\forall\, n \in N \quad g_n$ is continuous on D.
$g_{n+1}(x) = f_{n+1}(x) - f(x) \le f_n(x) - f(x) = g_n(x)$, hence $g_{n+1} \le g_n$.

Since $f_n \overset{pw.}{\longrightarrow} f$, then $g_n = f_n - f \overset{pw.}{\longrightarrow} 0$.

Thus $\forall\, \epsilon > 0$, $\exists\, k \in Z^+$, $k = k(x)$ s.t $|g_{k(x)}(x)| < \epsilon \quad , \quad \forall\ x \in D \quad \cdots (1)$.

$\forall\, n \in N \quad g_n$ is continuous $\forall\ x \in D$, $\exists\, \delta = \delta(x, \epsilon)$ $s.t$ $|g_{k(x)}(x) - g_{k(x)}(y)| < \epsilon$ whenever $|x - y| < \delta$.

(i.e $\forall$ ball $I_{f(x)}$ in R, $\exists$ a ball J_x with center x in D such that $x \in J_x$ and $g_{k(x)}(J_x) \subseteq I_{f(x)}$.

So that $\{J_x\}_{x \in D}$ is an open covering for D, ($D \subseteq \cup_{x \in D} J_x$).

Since D is compact, $\exists\ x_1, x_2, x_3, \cdots, x_n$ such that $D \subseteq \cup_{i=1}^{n} J_x$, take
$k = \max\{k(x_1), k(x_2), k(x_3), \cdots, k(x_n)\}\ s.t\ \ g_n(x) < |g_{k(x)}(x)| < \epsilon$.

Thus $g_n \overset{u.}{\rightarrow} 0$.

Case (2):- When $f_{n+1}(x) \geq f_n(x) \quad \forall\ x \in D \quad \forall\, n \in N$ (increasing) **(H.W)**.

Definition (6.8):

Let $\langle f_n \rangle$ be a sequence of mapping on D, we say that $\langle f_n \rangle$ is uniformly bounded sequence if there exists a real number $M > 0$ such that $|f_n(x)| \leq M \quad \forall\, n,\ \forall\ x \in D$.

i.e $|f_1(x)| \leq M \quad \forall\, n\ ,\ \forall\ x \in D$

$|f_2(x)| \leq M \quad \forall\, n\ ,\ \forall\ x \in D$

$\vdots$

Example:

$\forall\, n \in N$, let $f_n: [0,3] \to R$ be defined by $f_n(x) = \frac{x}{n}\ ,\ \forall x \in [0,3]$. Show that $\langle f_n \rangle$ uniformly bounded

$\langle f_n(x) \rangle = x, \frac{x}{2}, \frac{x}{3}, \cdots, \frac{x}{n}, \cdots$
$|f_n(x)| \leq 3 \quad \forall\, n\ ,\ \forall x \in [0,3]$.

Definition (6.9):

Let $\langle f_n \rangle$ be a sequence of mapping on D, we say that $\langle f_n \rangle$ is a bounded converging to f on D if:-

1) $\langle f_n \rangle$ converges to f on D. $f_n \overset{p.w}{\longrightarrow} f$
2) $\langle f_n \rangle$ uniformly bounded sequence.

Example:

$\forall\, n \in N$, let $f_n: [0\,,1] \to R$ be defined by $f_n(x) = x^n$, $\forall x \in [0\,,1]$. Show that $\langle f_n \rangle$ uniformly bounded

1) $f_n \overset{p.w}{\longrightarrow} f$ where $f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$

2) $|f_n(x)| = |x^n| \le 1$, $\forall x \in [0\,,1]$, $\forall\, n$.

$\langle x^n \rangle$ is uniformly bounded sequence. Thus $f_n \overset{b.c}{\to} f$.

Theorem (6.10):

Let $\langle f_n \rangle$ be a sequence of mapping on D that converges uniformly to f on D, if $\forall\, n \in N$, f_n is bounded, then $\langle f_n \rangle$ is a bounded converges to f on D.

Proof:

To proof $\exists\ M > 0$, $M \in R$ $s.t$ $|f_n(x)| \le M$ $\forall\, n$ $\forall\, x \in D$.

Since $f_n \overset{u.}{\to} f$ by proposition (6.5) f is bounded on D **i.e** $\exists\ M_1 > 0$ $s.t$ $|f(x)| \le M_1$ $\forall\, x \in D$ $\forall\, n > k$.

Also $f_n \overset{u.}{\to} f$ i e $\forall\, \epsilon > 0$ $\exists\, k = k(\epsilon)$ such that $|f_n(x) - f(x)| \le \epsilon < 1$, $\forall\ x \in D$ $\forall\, n > k$.

$|f_n(x)| = |f_n(x) - f(x) + f(x)| \le |f_n(x) - f(x)| + |f(x)| \le 1 + M_1$ $\forall\, n > k$.

Take $M = \max\{|f_1(x)|\,, |f_2(x)|\,, |f_3(x)|\,, \cdots\,, |f_k(x)|\,, 1 + M_1\}$.

Uniformly converges $\to$ bounded converges $\to$ point wise converges.

But the converse in general is not true.

Example:

$\forall\, n \in N$, let $f_n: [0\,,1] \to R$ be defined by $f_n(x) = x^n$, $\forall x \in [0\,,1]$. Is $\langle f_n \rangle$ uniformly converges?

1) $f_n \overset{p.w}{\longrightarrow} f$ where $f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$ and $f_n \overset{b.c}{\to} f$

2) $f_n \nrightarrow f$ uniformly .

Thus $\langle x^n \rangle$ is not uniformly converges sequence.

Example:

$\forall\, n \in N$, let $f_n: (0,1] \to R$ be defined by $f_n(x) = \frac{1}{nx}$, $\forall x \in (0,1]$. Is $\langle f_n \rangle$ uniformly bounded

$\frac{1}{nx} \xrightarrow{p.w} 0$?

$\forall\, \epsilon > 0$, $\left| \frac{1}{nx} - 0 \right| = \left| \frac{1}{nx} \right| = \frac{1}{nx}$, $\exists\, k = k(\epsilon, x)$ s.t $\frac{1}{n} < \frac{1}{k} < \epsilon x \quad \forall\, n > k$.

Then $|f_n(x) - f(x)| = \frac{1}{nx} < \epsilon$, $\forall\, n > k$.

$\langle \frac{1}{nx} \rangle = \frac{1}{x}, \frac{1}{2x}, \cdots, \frac{1}{nx}, \cdots$ is not uniformly bounded

If $\langle \frac{1}{nx} \rangle$ is uniformly bounded , then $\exists\; M > 0 \quad s.t \quad |f_n(x)| = \left| \frac{1}{nx} \right| \leq M \quad \forall\, n$ $\forall\, x \in (0,1] \qquad \cdots (1).$

By Archimedes on $\frac{1}{nx}, M \qquad \exists\, k = k(\epsilon, x) \quad s.t \quad \frac{k}{nx} > M$

If $n \geq k$, then $\frac{n}{nx} \geq \frac{k}{nx} > M$, then $f_1(x) = \frac{1}{x} > M \quad C!$ with (1).

If $n < k$?

Hence $\langle f_n(x) \rangle$ is not uniformly bounded

Proposition (6.11):

Let $\langle f_n \rangle$ be a bounded convergence sequence to f on D, then $\langle f \rangle$ is bounded.

Proof:

To proof $\exists\; M > 0$, $M \in R \quad s.t \quad |f(x)| \leq M \quad \forall\, x \in D.$

Since $f_n \xrightarrow{b.c} f$

1) $f_n \xrightarrow{p.w} f$
2) $\langle f_n \rangle$ is uniformly bounded.

i e $\forall \epsilon > 0 \quad \forall x \in D \quad \exists k = k(\epsilon, x)$ such that $|f_n(x) - f(x)| < \epsilon < 1 \quad , \quad \forall n > k$. In particular $|f_{k+1}(x) - f(x)| < 1 \quad , \forall n > k$. Since $\langle f_n \rangle$ is bounded, then $\exists \ M > 0 \quad s.t \quad |f_n(x)| \leq M \qquad \forall x \in D \qquad \forall n \in N$.

$$|f(x)| = |f(x) - f_{k+1}(x) + f_{k+1}(x)| \leq |f_{k+1}(x) - f(x)| + |f_{k+1}(x)|$$

$$\leq 1 + M \qquad \forall x \in D.$$

$$\therefore \quad |f(x)| \leq 1 + M = M' \qquad \forall x \in D.$$

Seris of mapping (6.12)

Let $\langle f_n \rangle$ be a sequence of real valued mapping where on D , $(D = R)$, the sum $\sum_{n=1}^{\infty} f_n$ is called the series of mappings.

$$\sum_{n=1}^{\infty} f_n = f_1 + f_2 + f_3 + \cdots + f_n + \cdots$$

$$S_1(x) = f_1(x)$$

$$S_2(x) = f_1(x) + f_2(x)$$

$$S_3(x) = f_1(x) + f_2(x) + f_3(x)$$

$$\vdots$$

$$S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x) = \sum_{i=1}^{n} f_i.$$

$$\vdots$$

$\langle S_n(x) \rangle$ is called the sequence of partial sums of $\sum_n f_n$

If $\langle S_n(x) \rangle$ converges uniformly to a function $f(x)$ on D , then $\sum_n f_n = f$ and the convergence is uniformly on D

If $\langle S_n(x) \rangle$ converges point wise to a function $f(x)$ In this case, $\sum_n f_n = f$ and the convergence point wise on D

Example:

$\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + \cdots + x^{n-1} + \cdots$. Geometric series.

When $x = 0$, then $\sum_{n=1}^{\infty} x^{n-1} = 1$.

$(1-x)\, S_n(x) = 1 - x^n$, then $S_n(x) = \frac{1-x^n}{1-x}$ $when\ x \neq 1$.

If $|x| < 1$, $-1 < x < 1$, then $x^n \to 0$. thus $S_n(x) \to \frac{1}{1-x}$

If $|x| \geq 1$, then $S_n(x)$ is diverges series since $\langle x^n \rangle$ not bounded, hence diverges.

Thus $\sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$ only on $(-1,1)$.

Power seris (6.13)

The power series is of the form:-

$$\sum_{n=1}^{\infty} a_{n-1}(x-a)^{n-1} = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_{n-1}(x-a)^{n-1} + \cdots$$

When $a = 0$, then

$$\sum_{n=1}^{\infty} a_{n-1}(x-a)^{n-1} = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + \cdots$$

When $x = 0$, then

$$\sum_{n=1}^{\infty} a_{n-1}(x-a)^{n-1} = a_0$$

Thus $\sum_{n=1}^{\infty} a_{n-1}(x-a)^{n-1}$ converges when $x = 0$.

Example:

$\sum_{n=1}^{\infty}(n-1)!\ x^{n-1} = 1 + x + 2!\, x^2 + \cdots + \cdots$.

Since $\langle (n-1)!\ x^{n-1} \rangle$ is not bounded $\forall\, x$, then $\langle (n-1)!\ x^{n-1} \rangle$ is diverges sequence, hence $\sum_{n=1}^{\infty}(n-1)!\ x^{n-1}$ is diverges series.

Thus the series $\sum_{n=1}^{\infty}(n-1)!\ x^{n-1}$ is converges only when $x = 0$.

Theorem (6.14):

Let $\sum_{n=1}^{\infty} a_{n-1}x^{n-1}$ be a power series, if $\sum_{n=1}^{\infty} a_{n-1}x^{n-1}$ converges at $x_0 \neq 0$, then $\sum_{n=1}^{\infty} a_{n-1}x^{n-1}$ converges at each x_1 such that $|x_1| < |x_0|$.

Proof:

$\sum_{n=1}^{\infty} a_{n-1}x_1^{n-1} = a_0 + a_1x_1 + a_2x_1^2 + \cdots + a_{n-1}x_1^{n-1} + \cdots$

$\sum_{n=1}^{\infty}|a_{n-1}x_1^{n-1}| = \sum_{n=1}^{\infty}|a_{n-1}x_0^{n-1}| \left|\frac{x_1}{x_0}\right|^{n-1} \quad x_0 \neq 0$

Since $\sum_{n=1}^{\infty}|a_{n-1}x_0^{n-1}|$ is a converges series, then by proposition (3.5) the sequence $\langle|a_{n-1}x_0^{n-1}|\rangle$ convergence to zero, hence $\langle|a_{n-1}x_0^{n-1}|\rangle$ is bounded sequence i.e $\exists M > 0$ $such\ that$ $|a_{n-1}x_0^{n-1}| \leq M \quad \forall n \in N$.

$\sum_{n=1}^{\infty}|a_{n-1}x_1^{n-1}| = \sum_{n=1}^{\infty}|a_{n-1}x_0^{n-1}| \left|\frac{x_1}{x_0}\right|^{n-1} \leq \sum_{n=1}^{\infty} M \left|\frac{x_1}{x_0}\right|^{n-1} \quad x_0 \neq 0$ is a geometric series, hence $\sum_{n=1}^{\infty} M \left|\frac{x_1}{x_0}\right|^{n-1}$ converges when $\left|\frac{x_1}{x_0}\right| < 1$ but $|x_1| < |x_0|$. Hence $\left|\frac{x_1}{x_0}\right| < 1$.

Remark (6.15):

Let $\sum_{n=1}^{\infty} a_{n-1}x^{n-1}$ be a power series

1) $\sum_{n=1}^{\infty} a_{n-1}x^{n-1}$ converges only at $x = 0$.
2) $\sum_{n=1}^{\infty} a_{n-1}x^{n-1}$ absolutely converges on R.
3) There exists $r > 0$ such that $\sum_{n=1}^{\infty} a_{n-1}x^{n-1}$ absolutely converges for each x with $|x| < r$ in this case r is called the radius of convergence of the series and $(-r, r)$ is called an interval of convergence.

Theorem (6.16):

Let $\sum_{n=1}^{\infty} a_{n-1}x^{n-1}$ be a power series with $a_{n-1} \neq 0 \quad \forall n$.

1) If the sequence $\langle\left|\frac{a_n}{a_{n-1}}\right|\rangle$ converges to p , then $r = \frac{1}{p}$ when $p \neq 0$, and if $p = 0$, then $r = \infty$. Thus $(-\infty, \infty)$ is a convergence interval.
2) If the sequence $\langle\left|\frac{a_n}{a_{n-1}}\right|\rangle$ is not bounded, then $\sum_{n=1}^{\infty} a_{n-1}x^{n-1}$ converges only when $x = 0$.

Examples (6.17):

1. $\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$, $a_{n-1} = \frac{1}{(n-1)!}$

$\langle \left| \frac{a_n}{a_{n-1}} \right| \rangle = \langle \left| \frac{\frac{1}{n!}}{\frac{1}{(n-1)!}} \right| \rangle = \langle \frac{(n-1)!}{n!} \rangle = \langle \frac{1}{n} \rangle \to 0$

Hence $p = 0$, $r = \infty$, then $\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$ Converges $\forall\, x \in R$. Thus $(-\infty, \infty)$ is a convergence interval.

2. $\sum_{n=1}^{\infty} \frac{x^{n-1}}{2^{n-1}}$, $a_{n-1} = \frac{1}{2^{n-1}}$

$\langle \left| \frac{a_n}{a_{n-1}} \right| \rangle = \langle \left| \frac{2^{n-1}}{2^n} \right| \rangle = \langle \frac{1}{2} \rangle \to \frac{1}{2}$

Hence $p = \frac{1}{2}$, $r = 2$, then $\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$ Converges $\forall\, x \in R$. Thus $(-2,2)$ is a convergence interval.

3. $\sum_{n=1}^{\infty} (n-1)!\, x^{n-1}$, $a_{n-1} = (n-1)!$

$\langle \left| \frac{a_n}{a_{n-1}} \right| \rangle = \langle \left| \frac{n!}{(n-1)!} \right| \rangle = \langle n \rangle$ not bounded.

Hence $\sum_{n=1}^{\infty} (n-1)!\, x^{n-1}$ Converges only at $x = 0$.

(H.W): $\sum_{n=1}^{\infty} \left(\frac{x}{r}\right)^{n-1}$.

Chapter (7)

Riemann integration:

Definition (7.1):

Let $f:[a,b] \to R$ be a bounded mapping , and $\pi = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ π is called a Riemann partition, put $J_i = [x_{i-1}, x_i]$, since f is bounded, then f has sup. and inf. Let $M = \sup\{f(x): x \in J\}$, $m = \inf\{f(x): x \in J\}$

$M_i = \sup\{f(x): x \in J_i\}$, $m_i = \inf\{f(x): x \in J_i\}$.

Clearly:- $m \le m_i \le M_i \le M \quad \forall\ i = 1,2,\cdots,n$.

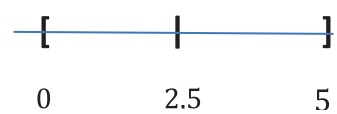
$\bar{R}(f,\pi) = \sum_{i=1}^{n} M_i|J_i|$ is called Riemann upper sum, and $\underline{R}(f,\pi) = \sum_{i=1}^{n} m_i|J_i|$ is called Riemann lower sum.

Clearly:- $\underline{R}(f,\pi) \le \bar{R}(f,\pi)$, (since $m_i \le M_i \quad \forall\ i = 1,2,\cdots,n$).

Definition (7.2):

A partition π' on $[a,b]$ is called refirement for π if every x_i in π is in π'.

Example:- $[a,b] = [0,5]$.



Proposition (7.3):-

If π' is a refirement for π, then $\bar{R}(f,\pi') \le \bar{R}(f,\pi)$, and $\underline{R}(f,\pi') \ge \underline{R}(f,\pi)$

Proof: (H.W)

Proposition (7.4):-

For any partitions π_1, π_2 on $[a,b]$ we have $\underline{R}(f,\pi_1) \le \bar{R}(f,\pi_2)$.

Proof:

Let $\pi = \pi_1 \cup \pi_2$, clearly π is a refirement for π_1 and π_2. Thus

$\underline{R}(f,\pi_1) \le \underline{R}(f,\pi) \le \bar{R}(f,\pi) \le \bar{R}(f,\pi_2)$.

Let $\bar{R}(f) = \{\bar{R}(f,\pi): \pi \; is \; any \text{ partition on } J\}$ bound below.
$\underline{R}(f) = \{\underline{R}(f,\pi): \pi \; is \; any \text{ partition on } J\}$ bound above.
By proposition (7.4), we have each element in $\underline{R}(f)$ is a lower bound for $\bar{R}(f)$ and each element in $\bar{R}(f)$ is an upper bound for $\underline{R}(f)$.
And by completeness of R, $\bar{R}(f)$ has a greatest lower bound and $\underline{R}(f)$ has a least upper bound.
Now, let $R\overline{\int} f = \inf\left(\bar{R}(f)\right)$ which is called Riemann upper integral and
$R\underline{\int} f = \sup\left(\underline{R}(f)\right)$ which is called Riemann lower integral.
Clearly that:- $R\underline{\int} f \leq R\overline{\int} f$.
If $R\underline{\int} f = R\overline{\int} f$, then f is called Riemann integrable. O.W we say that f is not Riemann integrable.

Examples (7.5):-

1) Let $f: [a,b] \to R$ be defined by $f(x) = c \quad \forall\, c \in R$. is f Riemann integrable?
Let $\pi_n = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ be a partition on $J_i = [x_{i-1}, x_i]$.

$M_i = \sup \{f(x): x \in J_i\}$, $m_i = \inf \{f(x): x \in J_i\}$

$\bar{R}(f,\pi) = \sum_{i=1}^{n} M_i|J_i| = c|J_1| + c|J_2| + \cdots + c|J_n|$

$= c(|J_1| + |J_2| + \cdots + |J_n|) = c|J| = c(b-a)$

$\underline{R}(f,\pi) = \sum_{i=1}^{n} m_i|J_i| = c|J_1| + c|J_2| + \cdots + c|J_n|$

$= c(|J_1| + |J_2| + \cdots + |J_n|) = c|J| = c(b-a)$

$\bar{R}(f) = \{ c(b-a): forany \text{ partition } \pi \text{ on } J \}$.

$\underline{R}(f) = \{c(b-a): forany \text{ partition } \pi \text{ on } J \}$

$R\overline{\int} f = \inf\left(\bar{R}(f)\right) = c(b-a)$
$R\underline{\int} f = \sup\left(\underline{R}(f)\right) = c(b-a)$
$\therefore \; R\underline{\int} f = R\overline{\int} f$. Thus f is Riemann integrable.

2) - Let $f:[0,1]\to R$ be defined by $f(x)=2x \quad \forall\, x\in[0,1]$. is f Riemann integrable?

Let $\pi_n=\left\{0=\frac{(0)(3)}{n},\frac{(1)(3)}{n},\frac{(2)(3)}{n},\cdots,\frac{(n)(3)}{n}\ =\ 3\right\}$ be a partition on $J_i=[x_{i-1},x_i]$.

$M_i=\sup\,\{f(x):x\in J_i\}\ ,\ \ m_i=\inf\,\{f(x):x\in J_i\}$

$\bar{R}(f,\pi)=\sum_{i=1}^{n}M_i|J_i|=M_1|J_1|+M_2|J_2|+\cdots+M_{n-1}|J_{n-1}|+M_n|J_n|$

$J_1=\left[0,\frac{3}{n}\right],J_2=\left[\frac{3}{n},\frac{6}{n}\right],J_3=\left[\frac{6}{n},\frac{9}{n}\right],\cdots,J_n=\left[\frac{3(n-1)}{n},3\right]$

$$=\frac{2.3}{n}.\frac{3}{n}+\frac{12}{n}.\frac{3}{n}+\frac{18}{n}.\frac{3}{n}+\frac{24}{n}.\frac{3}{n}+\cdots+\frac{6n}{n}.\frac{3}{n}$$

$$=\frac{6}{n}.\frac{3}{n}(1++2+3+\cdots+n)=\frac{18}{n^2}.\left(\frac{1}{2}\right)(n)(n+1)$$

$$=\frac{9}{n}\cdot(n+1)=9+\frac{9}{n}$$

$\underline{R}(f,\pi)=\sum_{i=1}^{n}m_i|J_i|\ =m_1|J_1|+m_2|J_2|+\cdots+m_{n-1}|J_{n-1}|+m_n|J_n|$

$J_1=\left[0,\frac{3}{n}\right],J_2=\left[\frac{3}{n},\frac{6}{n}\right],J_3=\left[\frac{6}{n},\frac{9}{n}\right],\cdots,J_n=\left[\frac{3(n-1)}{n},3\right]$

$$=0.\frac{3}{n}+\frac{6}{n}.\frac{3}{n}+\frac{12}{n}.\frac{3}{n}+\frac{18}{n}.\frac{3}{n}+\cdots+\frac{2(n-1)}{n}.\frac{3}{n}$$

$$=\frac{6}{n}.\frac{3}{n}[1++2+3+\cdots+(n-1)]$$

$$=\frac{18}{n^2}.\left(\frac{1}{2}\right)(n)(n-1)=\frac{9}{n}\cdot(n-1)=9-\frac{9}{n}$$

$\bar{R}(f)=\left\{9+\frac{9}{n}:n\in N\right\}$.

$\underline{R}(f)=\left\{9-\frac{9}{n}\ :n\in N\right\}$

$R\overline{\int}\ f=\inf\big(\bar{R}(f)\big)=9$

$R\underline{\int} f=\sup\left(\underline{R}(f)\right)=9$

$\therefore\ \ R\underline{\int} f=R\overline{\int}\ f$. Thus f is Riemann integrable.

3) $f:[1,5] \to R$ such that $f(x) = \frac{1}{2}x$. is f Riemann integrable? (H.W).

4) Let $f:[a,b] \to R$ be defined by $f(x) = \begin{cases} 2 & x \in Q \cap [a,b] \\ 3 & x \in Q' \cap [a,b] \end{cases}$. is f Riemann integrable?

Let $\pi_n = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ be a partition on $J_i = [x_{i-1}, x_i]$.

$M_i = \sup\{f(x): x \in J_i\}$, $m_i = \inf\{f(x): x \in J_i\}$

$\bar{R}(f,\pi) = \sum_{i=1}^{n} M_i|J_i| = 3|J_1| + 3|J_2| + \cdots + 3|J_n|$

$= 3(|J_1| + |J_2| + \cdots + |J_n|) = 3|J| = 3(b-a)$

$\underline{R}(f,\pi) = \sum_{i=1}^{n} m_i|J_i| = 2|J_1| + 2|J_2| + \cdots + 2|J_n|$

$= 2(|J_1| + |J_2| + \cdots + |J_n|) = 2|J| = 2(b-a)$

$\bar{R}(f) = \{3(b-a): forany$ partition π on $J\}$.

$\underline{R}(f) = \{2(b-a): forany$ partition π on $J\}$

$R\overline{\int} f = \inf(\bar{R}(f)) = 3(b-a)$

$R\underline{\int} f = \sup\left(\underline{R}(f)\right) = 2(b-a)$

$\therefore$ $R\underline{\int} f \neq R\overline{\int} f$. Thus f is not Riemann integrable

Q1/ Is there exists a discontinuous mapping in finite infinite of point and Riemann integrable?
Q2/ Is every continuous mapping and Riemann integrable?
Q3/ Is there exists a relation between points of continuity and Riemann integrable?

There exist discontinuous mappings in a point and Riemann integrable.

5) Let $f:[-3,4] \to R$ be defined by $f(x) = \begin{cases} 5 & x \geq 0 \\ 1 & x < 0 \end{cases}$. Is f Riemann integrable? f is not continuous at 0.

Let $\pi_n = \left[-3, \frac{-1}{n}\right] \cup \left[\frac{-1}{n}, \frac{1}{n}\right] \cup \left[\frac{1}{n}, 4\right]$ be a partition on $J_i = [x_{i-1}, x_i]$.

$J_{n_1} = \left[-3, \frac{-1}{n}\right]$, $J_{n_2} = \left[\frac{-1}{n}, \frac{1}{n}\right]$, $J_{n_3} = \left[\frac{1}{n}, 4\right]$

$M_1 = \sup\ \{f(x): x \in J_{n_1}\} = \sup\ \left\{f(x): x \in \left[-3, \frac{-1}{n}\right]\right\} = \sup\ \{1\} = 1\ .$

$M_3 = \sup\ \{f(x): x \in J_{n_2}\} = \sup\ \left\{f(x): x \in \left[\frac{-1}{n}, \frac{1}{n}\right]\right\} = \sup\ \{1{,}5\} = 5\ .$

$M_3 = \sup\ \{f(x): x \in J_{n_3}\} = \sup\ \left\{f(x): x \in \left[\frac{1}{n}, 4\right]\right\} = \sup\ \{5\} = 5\ .$

$\bar{R}(f, \pi_n) = \sum_{i=1}^{3} M_i |J_{n_i}| = M_1 |J_{n_1}| + M_2 |\ J_{n_2}| + M_3 |J_{n_3}|$

$$= 1.(3 - \frac{1}{n}) + 5.\frac{2}{n} + 5.(4 - \frac{1}{n})$$

$$= 3 - \frac{1}{n} + \frac{10}{n} + 20 - \frac{5}{n} = 23 + \frac{4}{n}$$

$m_1 = inf\ \{f(x): x \in J_{n_1}\} = inf\ \left\{f(x): x \in \left[-3, \frac{-1}{n}\right]\right\} = inf\ \{1\} = 1.$

$m_2 = inf\ \{f(x): x \in\ J_{n_2}\} = inf\ \left\{f(x): x \in \left[\frac{-1}{n}, \frac{1}{n}\right]\right\} = inf\ \{1{,}5\} = 1$

$m_3 = inf\ \{f(x): x \in\ J_{n_3}\} = inf\ \left\{f(x): x \in \left[\frac{1}{n}, 4\right]\right\} = inf\ \{5\} = 5$

$\underline{R}(f, \pi_n) = \sum_{i=1}^{3} m_i |J_{n_i}|\ = m_1 |J_{n_1}| + m_2 |J_{n_2}| + m_3 |J_{n_3}|$

$$= 1.(3 - \frac{1}{n}) + 1.\frac{2}{n} + 5.(4 - \frac{1}{n})$$

$$= 3 - \frac{1}{n} + \frac{2}{n} + 20 - \frac{5}{n} = 23 - \frac{4}{n}$$

$\bar{R}(f) = \{\bar{R}(f, \pi_n): for any$ partition π_n on $J\ \} = \left\{\ 23 + \frac{4}{n} : n \in N\ \right\}$.

$\underline{R}(f) = \{\ \underline{R}(f, \pi_n): for any$ partition π_n on $J\ \}\ = \left\{\ 23 - \frac{4}{n}\ : n \in N\ \ \right\}$

$R\overline{\int}\ f = \inf(\bar{R}(f)) = 23$

$R\underline{\int}\ f = \sup\left(\underline{R}(f)\right) = 23$

$\therefore\ \ R\underline{\int}\ f = R\overline{\int}\ \ f$. Thus f is Riemann integrable.

6) Let $f: [-2,2] \to R$ be defined by $f(x) = \begin{cases} 3 & -2 \le x < -1 \\ 2 & -1 \le x < 0 \\ 5 & 0 < x \le 2 \end{cases}$. is f Riemann integrable? (H.W)

$\pi_n = \left[-2, -1-\frac{1}{n}\right] \cup \left[-1-\frac{1}{n}, -1+\frac{1}{n}\right] \cup \left[-1+\frac{1}{n}, \frac{-1}{n}\right] \cup \left[\frac{-1}{n}, \frac{1}{n}\right] \cup \left[\frac{1}{n}, 2\right]$

To answer question tow

Lemma (7.6):

Let $f: [a, b] \to R$ be a bounded function f is Riemann integrable iff for each $\epsilon > 0$, there exists a partition π_0 on $[a, b]$ such that $\bar{R}(f, \pi_0) - \underline{R}(f, \pi_0) < \epsilon$.

Proof:

$\Rightarrow$) Let $\epsilon > 0$, since f is Riemann integrable, then $R\underline{\int} f = R\overline{\int} f$

$R\overline{\int} f = inf\left(\bar{R}(f)\right) = inf\ \{\bar{R}(f, \pi_1): forany$ partition π_1 on $J\}$.

i.e there exists a partition π_1 on J such that $\bar{R}(f, \pi_1) - R\overline{\int} f < \frac{\epsilon}{2}$ $\cdots (1)$.

$$\bar{R}(f, \pi_1) < \frac{\epsilon}{2} + R\overline{\int} f$$

$R\underline{\int} f = sup\left(\underline{R}(f)\right) = sup\ \{\underline{R}(f, \pi_2): forany$ partition π_2 on $J\}$.

i.e there exists a partition π_2 on J such that $\underline{R}(f, \pi_2) - R\underline{\int} f < \frac{\epsilon}{2}$ $\cdots (2)$.

$$-\underline{R}(f, \pi_2) < \frac{\epsilon}{2} - R\underline{\int} f$$

Let $\pi_0 = \pi_1 \cup \pi_2$ clearly π_0 is a refirement to each π_1 and π_2.

$\bar{R}(f, \pi_0) - \underline{R}(f, \pi_0) < \bar{R}(f, \pi_1) - \underline{R}(f, \pi_2)$

By proposition (7.3) $< \frac{\epsilon}{2} + R\overline{\int} f - R\underline{\int} f + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ (since f is integrable)

$\Leftarrow$) Let $\epsilon > 0$ and there exists a partition π_0 on J such that $\bar{R}(f, \pi_0) - \underline{R}(f, \pi_0) < \epsilon$.

$R\underline{\int} f - R\overline{\int} f < \bar{R}(f,\pi_0) - \underline{R}(f,\pi_0) < \epsilon$

$\therefore \quad R\underline{\int} f = R\overline{\int} f$. Thus f is Riemann integrable

Proposition (7.7):

Let $f\colon [a,b] \to R$ be a bounded function, if f is continuous, then f is Riemann integrable.

Proof:

By previous proposition (5.10) f has a minimum and a maximum points and also by proposition (5.12) f is uniformly continuous.

Divided J into n equal closed intervals each of length $\frac{b-a}{n}$, $J_i = [x_{i-1}, x_i]$ $\forall\, i = 1,2,\cdots,n$

$\forall\ i = 1,2,\cdots,n$ f is uniformly continuous on J_i.

Let $\epsilon > 0$, $\exists\, \delta(\epsilon)$: if $|x_i - x_{i-1}| < \delta$, then $|f(x_i) - f(x_{i-1})| < \frac{\epsilon}{b-a}$.

$$\bar{R}(f,\pi_n) - \underline{R}(f,\pi_n) = \sum_{i=1}^{n} M_i|J_i| - \sum_{i=1}^{n} m_i|J_i| = \sum_{i=1}^{n}(M_i - m_i)|J_i|$$

$$M_i = \max\{f(x)\colon x \in J_i\} = f(x_i)\ ,\ m_i = min\ \{f(x)\colon x \in J_i\} = f(x_{i-1})$$

$$= \sum_{i=1}^{n}|f(x_i) - f(x_{i-1})||J_i| \le \sum_{i=1}^{n}\frac{\epsilon}{b-a}\ |J_i| = \frac{\epsilon}{b-a}\cdot\frac{b-a}{n} = \frac{n\epsilon}{n} = \epsilon$$

Thus f is Riemann integrable.by (7.6)

Monotonic function and Riemann integrable:

Definition (7.8):

Let $f\colon [a,b] \to R$ be a function f is called a non-decreasing (increasing) if $\forall\, x, y \in [a,b]$ if $x < y$, then $f(x) \le f(y)$ ($f(x) < f(y)$) and f is said to be a non-increasing (decreasing) if $\forall\, x, y \in [a,b]$ if $x < y$, then $f(x) \ge f(y)$ ($f(x) > f(y)$).

Examples:

1. $f(x) = \sin(x) \quad \forall\, x \in R$, f is continuous function but not monotonic.
2. $f(x) = |x| \quad on \ [-2,2]$ is monotonic function but not continuous.

Remarks (7.9):

1) Let $f: [a, b] \to R$ be a monotonic function, then f is bounded.

If f is non-decreasing, then $\forall\, x \in [a, b]$, $f(a) \leq f(x) \leq f(b)$, $a < x < b$.

If f is non-increasing, then $\forall\, x \in [a, b]$, $f(b) \leq f(x) \leq f(a)$, $a < x < b$.

2) Let $f: [a, b] \to R$ be a monotonic function and f is non-decreasing, then $-f$ is non increasing and if f is non-increasing, then $-f$ is non-decreasing.

Theorem (7.10):

Let $f: [a, b] \to R$ be a monotonic function, then f is Riemann integrable.

Proof: (H.W)

Definition (7.11):

Let $S \subseteq R$, S is called a negligible set (zero set) if for each $\epsilon > 0$, there exists a countable collection of open intervals $\{I_n\}$ such that.

1. $S \subseteq \cup_n I_n$
2. $\sum_n |I_n| < \epsilon$.

Remarks examples (7.12):

1) Every finite set is a negligible set.

Let $S = \{x_1, x_2, \cdots, x_n\} \subseteq R$.

Let $\epsilon > 0$, $I_k = \left(x_k - \frac{\epsilon}{4n}, x_k + \frac{\epsilon}{4n}\right)$

1. $S \subseteq \cup_{i=1}^{k} I_i$
2. $\sum_{i=1}^{n} |I_i| = \sum_{i=1}^{n} \frac{\epsilon}{2n} = \frac{n\epsilon}{2n} = \frac{\epsilon}{2} < \epsilon$.

2) In general every countable (finite or infinite) set is a negligible set.

Let $S = \{x_1, x_2, \cdots, x_n, \cdots\} \subseteq R$.

Let $\epsilon > 0,\ I_k = \left(x_k - \frac{\epsilon}{2^{k+2}}, x_k + \frac{\epsilon}{2^{k+2}}\right)$

1) $S \subseteq \cup_{k \subseteq w} I_k$

2) $\sum_k |I_k| = \sum_k \frac{2\epsilon}{2^{k+2}} = \sum_k \frac{\epsilon}{2^{k+1}} = \sum_k \frac{\epsilon}{4}.(\frac{1}{2})^{k-1} = \frac{\frac{\epsilon}{4}}{1-\frac{1}{2}} = \frac{\epsilon}{4} \cdot 2 = \frac{\epsilon}{2} < \epsilon$.

In particular Q (the set of rational numbers) is a zero set.

3) If S_1 is a negligible set and $S_2 \subseteq S_1$, then S_2 is a negligible set.

Proof:

Since S_1 is a negligible set, then $\forall \epsilon > 0$, $\exists \{I_n\}$, $n \in N$ of open intervals such that

1. $S_1 \subseteq \cup_n I_n$
2. $\sum_n |I_n| < \epsilon$.

Since $S_2 \subseteq S_1 \subseteq \cup_n I_n$, and $\sum_n |I_n| < \epsilon$ $|I_n| < \epsilon$, hence we are done.

4) The union of a countable number of negligible sets is again negligible set.

Proof:

Let $\{S_k\}$ .be a countable collection number of a negligible set. T.P. $\cup_k S_k$ is a negligible set.

$\forall \epsilon > 0$, $\exists \left\{ I_n^{(k)} \right\}$ a countable collection of open intervals such that 1) $S_k \subseteq \cup_n I_n^{(k)}$,

2) $\sum_n |I_n| < \frac{\epsilon}{2^{k+2}}$.

1) $\cup_k S_k \subseteq \cup_k \cup_n I_n^{(k)}$

2) $\sum_k \left|\cup_n I_n^{(k)}\right| = \sum_k \sum_n \left|I_n^{(k)}\right| \leq \sum_k \frac{\epsilon}{4}.(\frac{1}{2})^{k-1} = \frac{\frac{\epsilon}{4}}{1-\frac{1}{2}} = \frac{\epsilon}{2} < \epsilon$.

5) Every interval in R is not a negligible set.

Proof:

Every open covering for $|I| = \epsilon$, any intervals I is of length equal or greater than $|I|$ and hence when $\epsilon = \frac{1}{2} |I|$.

The condition (2) is not hold.

6) Q' is not a negligible set.

$R = Q \cup Q'$.

R is not a negligible set by (5) and Q is a negligible set by (2)
If Q' is a zero set, then $R = Q \cup Q'$. is a zero set C!

Theorem (7.13): (Lebesgue theorem in Riemann integration)

Let $f: [a, b] \to R$ be a bounded function, then f is Riemann integrable if and only if the set of discontinuous points $[D(f)]$ of f on $[a, b]$ is a negligible set.

Example: Every empty set is a zero set.

Let $\epsilon > 0$, $\exists \left(\frac{-\epsilon}{3}, \frac{\epsilon}{3}\right) = I_\epsilon$

1) $\emptyset \subseteq I_\epsilon$
2) $|I_\epsilon| = \frac{2\epsilon}{3} < \epsilon$.

Corollary (7.14):

Let $f: [a, b] \to R$ be a monotonic function, then the set of discontinuous points of f on $[a, b]$, $(D(f))$ is a zero set.

Proof:

By remark (7.9) f is bounded and by theorem (7.10) f is Riemann integrable, then by (7.13) $D(f)$ is a zero set.

Corollary (7.15):

Let $f: [a, b] \to R$ be a bounded Riemann integrable function and let $g: [c, d] \to R$ be a bounded function, if $[c, d] \subseteq [a, b]$, then g is Riemann integrable.

Proof:

Since $f: [a, b] \to R$ is bounded and Riemann integrable, then by (7.13)the set $D(f)$ of f on $[a, b]$ is a zero set and every subset of a zero set is also is a zero set, hence $[c, d]$ is a zero set and again by (7.13), then g is Riemann integrable

Proposition (7.16):

Let $f: [a,b] \to R$ be a bounded function and let $c \in [a,b]$, if f is Riemann integrable on $[a,b]$, then f is Riemann integrable on $[a,c]$ and $[c,b]$. Moreover.

$$R\int_a^b f = R\int_a^c f + R\int_c^b f$$

Proof:

Let π_1 and π_2 be partitions on $[a,c]$ and $[c,b]$ respectively.

$\pi = \pi_1 \cup \pi_2$.

$$\underline{R}(f,\pi_1) + \underline{R}(f,\pi_2) = \underline{R}(f,\pi) \qquad \cdots (1)\ .$$

$$\bar{R}(f,\pi_1) + \bar{R}(f,\pi_2) = \bar{R}(f,\pi) \qquad \cdots (2)$$

Notice that f is Riemann integrable on $[a,c]$ and $[c,b]$.

By corollary (7.15)

$$\underline{R}(f,\pi) = \underline{R}(f,\pi_1) + \underline{R}(f,\pi_2) < \int_a^c f + \int_c^b f \ < \ \underline{R}(f,\pi_1) + \frac{\epsilon}{2} + \underline{R}(f,\pi_2) + \frac{\epsilon}{2}$$

$$= \underline{R}(f,\pi) + \epsilon$$

$$\underline{R}(f,\pi) < \int_a^c f + \int_c^b f \ < \ \underline{R}(f,\pi) + \epsilon \qquad \cdots (*)$$

And

$$\underline{R}(f,\pi) < \int_a^b f \ < \ \underline{R}(f,\pi) + \epsilon \qquad \cdots (**)$$

From $(*)$ and $(**)$ we get $\int_a^b f - \left(\int_a^c f + \int_c^b f\right) < \epsilon$

Thus $\int_a^b f = \int_a^c f + \int_c^b f$

Remark (7.17):

Let $RI[a,b] = \{ f: [a,b] \to R \ \ni \ f \text{ bounded Riemann integrable function}\}$, then

$(RI[a,b], +, \cdot)$ is a vector space.
Let $f, g \in RI[a,b]$, $f:[a,b] \to R$, $g:[a,b] \to R$, $f+g:[a,b] \to R$,.
then $f+g \in RI[a,b]$
$D(f+g) = D(f) \cup D(g)$, $D(f+g)$ is a zero set.
$\forall\ c \in R$, let $f \in RI[a,b]$, $D(f)$ is a zero set. $D(c \cdot f) \subseteq D(f)$.
$D(f)$ is a zero set, then $D(c \cdot f)$ is a zero set.
Thus $c \cdot f$ is a Riemann integrable, then $c \cdot f \in RI[a,b]$.
Now, define $R\int : RI[a,b] \to R$
$R\int f$ =(Number)
$R\int\ is\ a\ linear\ transformation\ \ i.e$
1) $R\int(f+g) = R\int f + R\int g$
2) $R\int(c.f) = c\,R\int f \qquad \forall\, c \in R\ ,\ \forall\, f, g \in RI[a,b]$

Proposition (7.18):

Let $f:[a,b] \to R$ be a bounded function, if f is Riemann integrable and $f(x) \geq 0\ \forall\, x \in [a,b]$, then $R\int_a^b f \geq 0$.

Proof:

Let π be a partition on $[a,b]$.
$\pi = \{a = x_0,\ x_1,\ x_2,\ \cdots,\ x_n = b\}$.
$J_i = [x_{i-1}, x_i]$, $M_i = \sup\{f(x): x \in J_i\}$, $m_i = inf\ \{f(x): x \in J_i\}$

$\bar{R}(f,\pi) = \sum_{i=1}^{n} M_i|J_i| \geq 0$, since $f(x) \geq 0$

$\bar{R}(f,\pi) = \sum_{i=1}^{n} M_i|J_i| = M_1|J_1| + M_2|J_2| + \cdots + M_n|J_n|$

$\bar{R}(f) = \{\bar{R}(f,\pi): for any \text{ partition } \pi \text{ on } [a,b]\} \geq 0$

$R\overline{\int} f = inf\ (\bar{R}(f)) \geq 0$

Since f is Riemann integrable $0 \leq \overline{\int} f = \underline{\int} f = \int_a^b f$

Corollary (7.19):

Let $f, g: [a, b] \to R$ be bounded functions, if f and g are Riemann integrable and $f(x) \geq g(x) \quad \forall\, x \in [a, b]$, then $\int_a^b f \geq \int_a^b g$.

Proof:

Let $h(x) = f(x) - g(x) \geq 0 \quad \forall\, x \in [a, b]$

Since f is bounded, then $\exists\ M_1 \in R \quad s.t \quad |f(x)| \leq M_1 \quad \forall\, x \in [a, b]$.

Since g is bounded, then $\exists\ M_2 \in R \quad s.t \quad |g(x)| \leq M_2 \quad \forall\, x \in [a, b]$.

$|h(x)| = |f(x) - g(x)| \leq |f(x)| + |-g(x)| \leq |f(x)| + |g(x)| \leq M_1 + M_2 \quad \forall\, x \in [a, b]$

Then $h(x)$ is bounded and $h(x)$ is Riemann integrable $[f, g \in RI[a, b]]$ and by proposition (7.18) $\int_a^b h \geq 0$, then $\int_a^b h = \int_a^b f - g = \int_a^b f + \int_a^b -g = \int_a^b f - \int_a^b g \geq 0$. Thus $\int_a^b f \geq \int_a^b g$.

Corollary (7.20):

If $f \in RI[a, b]$, then $|f| \in RI[a, b]$ and $\left|\int_a^b f\right| \leq \int_a^b |f|$.

Proof:

1. Since $Dom|f| \subseteq Dom\, f,\ g \circ f = \{x \in Dom\, f : f(x) \in Dom\, g\}$

Since $f \in RI[a, b]$, then by Lebesgue theorem $D_{[a,b]}(f)$ is a negligible set. Hence $D_{[a,b]}|f|$ is a negligible set and then by Lebesgue theorem $|f| \in RI[a, b]$.

2. Since $-|f(x)| \leq f(x) \leq |f(x)| \quad \forall\, x \in [a, b]$, then by corollary (7.19) $\int_a^b -|f| \leq \int_a^b f \leq \int_a^b |f|$, thus $\left|\int_a^b f\right| \leq \int_a^b |f|$.

Remark :

The convers in general is not true i.e if $|f| \in RI[a, b]$, then needn't be $f \in RI[a, b]$.

Example:

$$f(x) = \begin{cases} 2 & x \in Q \cap [a, b] \\ -2 & x \in Q' \cap [a, b] \end{cases}.$$

$|f| = 2 \quad \forall\, x \in [a, b]$ is Riemann integrable but f is not Riemann integrable

Remark :

Clearly that $\int_a^b 0 = 0$, but if $\int_a^b f = 0$ is $f = 0$?

In general No

Examples:

1) Let $f(x) = [-1,1] \to R$. be defined by $f(x) = x \quad \forall\, x \in [a,b]$

$\int_{-1}^{1} x = 0$ but $f \neq 0$

2) Let $f(x) = [-2,3] \to R$. be defined by $f(x) = \begin{cases} 0 & x \neq 2 \\ 4 & x = 2 \end{cases}$.

$\int_{-2}^{3} f = 0$ but $f \neq 0$

Proposition (7.21):

Let $f \in RI[a,b]$ and $f(x)$ is continuous function, $f(x) \geq 0 \;\forall\, x \in [a,b]$ and $\int_a^b f(x) = 0$, then $f = 0$.

Proof:

Suppose that the result is not true (i.e) $\exists\, x_0 \in [a,b]\;\; s.t\;\; f(x_0) > 0$.

Let $V = \left(f(x_0) - \frac{\epsilon}{4}, f(x_0) + \frac{\epsilon}{4}\right)$ be a ball in R, since f is continuous on $[a,b]$, then $\exists$ a ball U in $[a,b]$ such that $f(U) \subseteq V$

Let $E = \bar{U}$ closed interval, since $f(x_0) > 0$, then $f(x) > 0 \;\; \forall\, x \in E$.

E is closed and bounded, then by Hien-Borel theorem E is compact, hence $f: E \to R$ is continuous on a compact space, hence f has minimum and maximum points.

$m = \min\{f(x): x \in E\}$ from (*) since $f(x) > 0$

$$0 = \int_a^b f > \int_E f \;\geq m|E| \quad C!$$

$\pi_n = E_1 \cup E_2 \cup \cdots \cup E_n$

$\underline{R}(f,\pi) = m_1|E_1| + m_2|E_2| + \cdots + m_n|E_n|$

$> m|E_1| + m|E_2| + \cdots + m|E_n|$

$> m(E_1 + E_2 + \cdots + E_n)$

$> m|E| > 0$

Definition (7.22):

Let $\langle f_n \rangle$ be a sequence of real valued functions on $[a,b]$, we say that $\langle f_n \rangle$ converges (point wise) to f on $[a,b]$ if $\forall\, \epsilon > 0$, $\forall\, x \in [a,b]$, $\exists\, k = k(\epsilon, x)$ such that $|f_n(x) - f(x)| < \epsilon \quad \forall\, n > k$.

And we say that $\langle f_n \rangle$ converges uniformly to f on $[a,b]$ if $\forall\, \epsilon > 0$, $\exists\, k = k(\epsilon)$ such that $|f_n(x) - f(x)| < \epsilon \quad \forall\, n > k \quad \forall\, x \in [a,b]$.

Q_1: If $\langle f_n \rangle$ is a sequence of real valued bounded function on $[a,b]$ that converges point wise to f on $[a,b]$ and $\forall\, n \in N$ the sequence $\langle f_n \rangle$ is Riemann integrable on $[a,b]$. Is f Riemann integrable?

Answer: No in general as the following example show:

Example:

Let $[a,b] \subseteq R$, let $\{r_1, r_2, \cdots, r_n\}$ be the set of rational numbers in $[a,b]$

$\forall\, n \in N$, $f_n : [a,b] \to R$ be defined by $f_n(x) = \begin{cases} 2 & x \in \{r_1, r_2, \cdots, r_n\} \\ -2 & x \notin \{r_1, r_2, \cdots, r_n\} \end{cases}$.

$$f_1(x) = \begin{cases} 2 & x \in \{r_1\} \\ -2 & x \notin \{r_1\} \end{cases}$$

$$f_2(x) = \begin{cases} 2 & x \in \{r_1, r_2\} \\ -2 & x \notin \{r_1, r_2\} \end{cases}$$

$$f_3(x) = \begin{cases} 2 & x \in \{r_1, r_2, r_3\} \\ -2 & x \notin \{r_1, r_2, r_3\} \end{cases}$$

⋮

$D_{[a,b]} f_n = \{r_1, r_2, \cdots, r_n\}$ is a negligible set $\forall\, n$. Hence $f_n \in RI[a,b]$ by Lebesgue.

Claim: $f_n(x) \xrightarrow{p.w} f(x)$, where $f(x) = \begin{cases} 2 & x \in [a,b] \cap Q \\ -2 & x \in [a,b] \cap Q' \end{cases}$

$f_n(r_1) \overset{?}{\to} f(r_1)$,

$f_n(r_2) \overset{?}{\rightarrow} f(r_2),$

$\vdots$

$f_n(r_n) \overset{?}{\rightarrow} f(r_n),$

$f_n(r_1) = f_1(r_1), f_2(r_1), \cdots, f_n(r_1) = 2,2,\cdots,2 \rightarrow 2 = f(r_1)$

$f_n(r_2) = f_1(r_2), f_2(r_2), \cdots, f_n(r_2) = 2,2,\cdots,2 \rightarrow 2 = f(r_2)$

$f_n(r_3) = f_1(r_3), f_2(r_3), \cdots, f_n(r_3) = 2,2,\cdots,2 \rightarrow 2 = f(r_3)$

If $x \notin \{r_1, r_2, \cdots, r_n\}$ $\forall\, n$, then $f_n(x) = -2 \rightarrow -2 \in [a,b] \cap Q'$

Thus $f_n(x) \overset{p.w}{\longrightarrow} f(x)$ and $f \notin RI[a,b]$.

H.W:

$$f_n(x) = \begin{cases} n - n^2 x & 0 < x < \frac{1}{n} \\ 0 & o.w \end{cases}$$ converges point wise to 0.

Q_2: If $\langle f_n \rangle$ is a sequence of real valued bounded functions on $[a,b]$ that converges point wise to f on $[a,b]$, $\forall\, n \in N$ if the sequence $\langle f_n \rangle$ is Riemann integrable on $[a,b]$ and f Riemann integrable. Is $\lim \int f_n = \int \lim f_n$?

Answer: No. in general as the following example show:

Example:

$\forall\, n \in N$, $f_n : [a,b] \rightarrow R$ be defined by $f_n(x) = \begin{cases} n^2 x & 0 \leq x \leq \frac{1}{n} \\ -n^2 x + 2n & \frac{1}{2} \leq x \leq \frac{2}{n} \\ 0 & \frac{2}{n} \leq x \leq 1 \end{cases}$.

$f_1(x) = x \qquad 0 \leq x < 1$

$$f_2(x) = \begin{cases} 4x & 0 \leq x \leq \frac{1}{2} \\ -4x + 4 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$f_3(x) = \begin{cases} 9x & 0 \leq x \leq \frac{1}{3} \\ -9x + 4 & \frac{1}{3} \leq x \leq \frac{2}{3} \\ 0 & \frac{2}{3} \leq x \leq 1 \end{cases}$$

$$f_4(x) = \begin{cases} 16x & 0 \leq x \leq \frac{1}{4} \\ -16x + 8 & \frac{1}{4} \leq x \leq \frac{1}{2} \\ 0 & o.w \end{cases}$$

$\int f_n = 1$. Hence $f_n \in RI[a, b]$

$$\int_0^1 f_n = \int_0^{\frac{1}{n}} n^2 x + \int_{\frac{1}{n}}^{\frac{2}{n}} -n^2 x + 2n + \int_{\frac{2}{n}}^{1} 0$$

$$= n^2 \int_0^{\frac{1}{n}} x - n^2 \int_{\frac{1}{n}}^{\frac{2}{n}} -n^2 x + \int_{\frac{1}{n}}^{\frac{2}{n}} 2n$$

$$f_n(x) \xrightarrow{p.w} 0$$

$\int_0^1 0 = 0$, hence in general $\lim \int f_n \neq \int \lim f_n$

$$\int f_n = 1 \quad \nrightarrow \int 0 = 0$$

Note: If the converges is uniformly the answer for two questions are yes.as in the following theorem:

Theorem (7.23):

Let $\langle f_n \rangle$ be a sequence of bounded functions on $[a, b]$ that converges uniformly to f on $[a, b]$ and if $\forall\, n \in N$, $f_n \in RI[a, b]$, then $f \in RI[a, b]$.

Moreover $\langle \int f_n \rangle$ converges to $\int f$ **i.e** $\lim \int f_n = \int \lim f_n$.

Proof:

Since $f_n \overset{u.}{\to} f$ and $\langle f_n \rangle$ is bounded $\forall\, n$, then f is bounded by proposition (6.5).

Let $D(f_n) =$ the set of discontinuous points of $f_n \quad \forall\, n$ on $[a, b]$.

Since $\forall\, n \in N$, $f_n \in RI[a, b]$, then $\forall\, n \in N$, $D(f_n)$ is a negligible set.

Let $D = \cup_n D(f_n)$ is a negligible set, then $\forall\, n \in N$, f_n is continuous on $[a, b] - D$.

Since $f_n \overset{u.}{\to} f$ and $\langle f_n \rangle$ is continuous on $[a, b] - D$, then by proposition (6.6) f is continuous on $[a, b] - D$, then $D(f) \subseteq D$ where $D(f) =$ the set of discontinuous points of f, then , $D(f)$ is a negligible set and hence $f \in RI[a, b]$.

$$\left| \int f_n - \int f \right| = \left| \int (f_n - f) \right| \le \int | f_n - f |$$

Then $\exists\, k(\epsilon)$ such that $|\int f_n - \int f| < \int_a^b \frac{\epsilon}{b-a} = \frac{\epsilon}{b-a} \cdot (b - a) = \epsilon \quad \forall\, n > k \quad , \forall\, x$.

Chapter (8)

Differentiation:

Definition (8.1):

Let $f: (a, b) \to R$ be a function, we say that f is differentiable at $x_0 \in (a, b)$ if for any sequence $\langle x_n \rangle$ in (a, b) such that $x_n \neq x_0 \quad \forall\, n$ and $x_n \to x_0$, there exists a real number $\alpha = f'(x_0)$ such that the sequence

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} \to \propto$$

i.e:- $\forall\, \langle x_n \rangle$ in (a, b) such that $x_n \neq x_0 \quad \forall\, n$, $x_n \to x_0$, $\exists\, \alpha = f'(x_0)$ such that

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} \to \propto$$

α is called the derivative of f at x_0, also is denoted by $\alpha = f'(x_0) = \frac{df}{dx}|_{x_0}$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

Otherwise f is not differentiable at x_0.

Remark (8.2):

If f is differentiable at each $x_0 \in (a, b)$, then we say that f is differentiable.

Theorem (8.3):-

Let $f: (a, b) \to R$ be a function then f is differentiable at $x_0 \in (a, b)$ iff there exists a real number α and a continuous function $\omega: (a, b) \to R$ with $\omega(x_0) = 0$ satisfies $f(x) = f(x_0) + [(x - x_0)\alpha + (x - x_0)\,\omega(x)]$

Proof:

$\Rightarrow$) Since f is differentiable at $x_0 \in (a, b)$, then $\forall$ $\langle x_n \rangle$ in (a, b) such that $x_n \neq x_0$ $\forall\, n$, $x_n \to x_0$, $\exists$ $\alpha \in R$ such that $\frac{f(x_n) - f(x_0)}{x_n - x_0} \to \propto$ $\cdots (*)$

Define $\omega(x): (a, b) \to R$ as follows

$$\omega(x) = \begin{cases} \dfrac{f(x) - f(x_0)}{x - x_0} - \propto & if \quad x \neq x_0 \\ 0 & if \quad x = x_0 \end{cases}$$

Claim: ω is continuous

Let $x_n \to x_0 \in (a, b)$ T.P $\omega\,(x_n) \to \omega(x_0) = 0$.

$\omega\ (x_n) = \frac{f(x_n)-f(x_0)}{x_n - x_0} - \propto\ \rightarrow\ 0 = \omega(x_0)$ by $(*)$

Hence ω is continuous.

$$x \neq x_0 \quad , \quad \omega(x) = \frac{f(x)-f(x_0)}{x-x_0} - \propto = \frac{f(x)-f(x_0)-\alpha(x-x_0)}{x-x_0}$$

$(x-x_0)\ \omega(x) = f(x)-f(x_0)-\alpha(x-x_0).$

Thus

$$f(x) = f(x_0) + [(x-x_0)\alpha + (x-x_0)\ \omega(x)]$$

$\Leftarrow$) T.P f is differentiable at x_0 we have $\exists\ \alpha \in R$,$\exists\ \omega: (a,b) \rightarrow R$ continuous a $\omega(x_0) = 0$

$$f(x) = f(x_0) + [(x-x_0)\alpha + (x-x_0)\ \omega(x)]$$

Let $\langle x_n \rangle$ be a sequence in (a,b) such that $x_n \neq x_0\ \ \forall\, n$ and $x_n \rightarrow x_0$,

$$\frac{f(x_n)-f(x_0)}{x_n-x_0} \overset{?}{\rightarrow} \propto$$

Since $x_n \rightarrow x_0$ and ω is continuous at x_0, then

$$\omega\ (x_n) \rightarrow \omega(x_0) = 0 \qquad \cdots (**)$$

$(x-x_0)\ \omega(x) = f(x)-f(x_0)-\alpha(x-x_0)$

$\omega(x) = \frac{f(x)-f(x_0)}{x-x_0} - \propto\ ,\ \ \omega(x_0) = 0$

$\omega\ (x_n) = \frac{f(x_n)-f(x_0)}{x_n-x_0} - \propto \rightarrow 0 = \omega(x_0)$ from $(**)$

Thus

$\frac{f(x_n)-f(x_0)}{x_n-x_0}\ \rightarrow\ \propto$. And f is differentiable at x_0

Proposition (8.4):-

Let $f: (a,b) \rightarrow R$ be a function if f is differentiable at $x_0 \in (a,b)$, then f is a continuous at x_0 .

Proof:

Since f is differentiable at $x_0 \in (a,b)$, then $\exists$ a real number α and a continuous function $\omega: (a,b) \to R$ with $\omega(x_0) = 0$ satisfies

$$f(x) = f(x_0) + (x - x_0)\alpha + (x - x_0) \cdot \omega(x)$$

Since $f(x_0)$, $(x - x_0)$ and $\omega(x)$ are continuous functions.

Then f is a continuous at x_0 .

Remark :-

The convers of the above proposition in general is not true as the following example shows.

Examples:-

7) Let $f: (-1,1) \to R$ be defined by $f(x) = |x| = \begin{cases} x & x \geq 0 \quad in \ [0,1) \\ -x & x < 0 \quad in \ (-1,0) \end{cases}$.

f is a continuous at 0.

Let $x_n \to 0$, $f(x_n) = |x_n| \overset{?}{\to} f(0) = 0$

But f is not differentiable at 0 i.e $\exists$ $\langle x_n \rangle$ in (a,b) such that $x_n \neq x_0$ $\forall n$, $x_n \to x_0$, $\exists$ α such that $\frac{f(x_n) - f(x_0)}{x_n - x_0} \nrightarrow \propto$

$\frac{1}{n} \to 0$, $\frac{1}{n} \neq 0$ $\forall$ n $\frac{f\left(\frac{1}{n}\right) - f(0)}{\frac{1}{n} - 0} = \frac{\frac{1}{n} - 0}{\frac{1}{n} - 0} = 1$

$\frac{-1}{n} \to 0$, $\frac{-1}{n} \neq 0$ $\forall$ n $\frac{f\left(\frac{-1}{n}\right) - f(0)}{\frac{-1}{n} - 0} = \frac{\frac{1}{n} - 0}{\frac{-1}{n} - 0} = -1$

Thus $\propto$ is not unique and f is not differentiable at x_0

Now, we have some examples about differentiation:

8) Let $f: (a,b) \to R$ be defined by $f(x) = c$ $\forall x \in (a,b)$ $c \in R$.
f is differentiable

Let $x_0 \in (a,b)$ and let $\langle x_n \rangle \in (a,b)$, $x_n \to x_0$, $x_n \neq x_0$ $\forall n$,

$$\frac{f(x_n)-f(x_0)}{x_n-x_0}=\frac{c-c}{x_n-x_0}=0$$

9) - Let $f:(a,b)\to R$ be defined by $f(x)=x \quad \forall\, x\in(a,b)$.
f is differentiable

Let $x_0\in(a,b)$ and let $\langle x_n\rangle\in(a,b)$, $x_n\to x_0$, $x_n\neq x_0 \quad \forall n$,

$$\frac{f(x_n)-f(x_0)}{x_n-x_0}=\frac{x_n-x_0}{x_n-x_0}=1$$

10) Let $f:(a,b)\to R$ be defined by $f(x)=x^2$. is f differentiable?
Let $x_0\in(a,b)$ and let $\langle x_n\rangle\in(a,b)$, $x_n\to x_0$, $x_n\neq x_0 \quad \forall n$,

$$\frac{f(x_n)-f(x_0)}{x_n-x_0}=\frac{{x_n}^2-{x_0}^2}{x_n-x_0}=\frac{(x_n-x_0)\cdot(x_n+x_0)}{(x_n-x_0)}=x_n+x_0 \;\to x_0+x_0=2x_0$$

Then f is differentiable

11) Let $f:(a,b)\to R$ be defined by $f(x)=\begin{cases}-2 & x\in Q\cap(a,b)\\ 3 & x\in Q'\cap(a,b)\end{cases}$.

f is not differentiable **(H.W).**

Proposition (8.5):

Let $f,g:(a,b)\to R$ be differentiable functions at x_0, then:

1. $f\pm g$ is differentiable at x_0 and $(f\pm g)'(x_0)=f'(x_0)\pm g'(x_0)$.
2. $f\cdot g$ is differentiable at x_0 and $(f\cdot g)'(x_0)=f(x_0)\cdot g'(x_0)+f'(x_0)\cdot g(x_0)$.
3. $\forall\, c\in R \quad c\cdot f$ is differentiable at x_0 and $(c\cdot f)'(x_0)=cf'(x_0)$.
4. $\frac{f}{g}$ is differentiable at x_0 , $g(x_0)\neq 0$ and $\left(\frac{f}{g}\right)'(x_0)=\frac{g(x_0)f'(x_0)-f(x_0)\cdot g'(x_0)}{(g(x_0))^2}$.

Proof:(4)

Let $\langle x_n\rangle\in(a,b)$, $x_n\to x_0$, $x_n\neq x_0 \quad \forall n$,

$$\frac{\left(\frac{f}{g}\right)\ (x_n) - \left(\frac{f}{g}\right)\ (x_0)}{x_n - x_0} \overset{?}{\to} \alpha$$

$$\frac{\frac{f(x_n)}{g(x_n)} - \frac{f(x_0)}{g(x_0)}}{x_n - x_0} = \frac{f(x_n)g(x_0) - f(x_0)g(x_n)}{(x_n - x_0)g(x_n)g(x_0)}$$

$$= \frac{f(x_n)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x_n)}{(x_n - x_0)g(x_n)g(x_0)}$$

$$= \left[\frac{f(x_n) - f(x_0)}{(x_n - x_0)g(x_n)g(x_0)}\right] \cdot g(x_0) - \left[\frac{g(x_n) - g(x_0)}{(x_n - x_0)g(x_n)g(x_0)}\right] \cdot f(x_0)$$

Since $x_n \to x_0$, , $x_n \neq x_0 \quad \forall\, n$, f *and* g are differentiable function at x_0, then $\exists\ \alpha_1 = f'(x_0)$ and $\exists\ \alpha_2 = g'(x_0)$ such that.

$$\frac{f(x_n) - f(x_0)}{(x_n - x_0)} \to \alpha_1 \qquad and \qquad \frac{g(x_n) - g(x_0)}{(x_n - x_0)} \to \alpha_2$$

And $f(x_n) \to f(x_0)$, $g(x_n) \to g(x_0)$ continuous

$$\left[\frac{f(x_n) - f(x_0)}{(x_n - x_0)g(x_n)g(x_0)} \cdot g(x_0)\right] - \left[f(x_0) \cdot \frac{g(x_n) - g(x_0)}{(x_n - x_0)g(x_n)g(x_0)}\right]$$

$$\frac{f'(x_0) \cdot g(x_0)}{(g(x_0))^2} - \frac{f(x_0) \cdot g'(x_0)}{\left(g(x_0)\right)^2} = \frac{g(x_0)f'(x_0) - f(x_0) \cdot g'(x_0)}{(g(x_0))^2}$$

Proposition (8.6): (Chain Rule)

Let $f : I \to R$ be a differentiable function at x_0 and $g : J \to R$ be a differentiable function at $f(x_0)$, then $g \circ f$ is differentiable at x_0 and $(\, g \circ f)'(x_0) \ = f'(x_0) \cdot g'\big(f(x_0)\big)$; I, J are open intervals.

Proof:

Since f is differentiable at x_0 $\exists\ \alpha_1 = f'(x_0)$ and a continuous function $\omega_1 : I \to R$ with $\omega_1(x_0) = 0$, ω_1 satisfies:

$$f(x) = f(x_0) + (x - x_0)\alpha_1 + (x - x_0)\,\omega_1(x) \qquad \cdots \ (1)$$

Since g is differentiable at $f(x_0) = y_0$ $\quad f(x) = y$, $\exists\, \alpha_2 = g'(y_0) = g'\big(f(x_0)\big)$ and a continuous function $\omega_2 : J \to R$ with $\omega_2(y_0) = 0$, ω_2 satisfies:

$$g(y) = g(y_0) + (y - y_0)\alpha_2 + (y - y_0)\,\omega_2(y) \qquad \cdots \ (2)$$
$$g\big(f(x)\big) - g\big(f(x_0)\big) = \big(f(x) - f(x_0)\big)[\alpha_2 + \omega_2(y)]$$
$$g\big(f(x)\big) - g\big(f(x_0)\big) = (x - x_0) \cdot [\alpha_1 + \omega_1(x)] \cdot [\alpha_2 + \omega_2(y)]$$
$$\frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{x - x_0} = [\alpha_1 + \omega_1(x)] \cdot \big[\alpha_2 + \omega_2\big(f(x)\big)\big]$$
$$= \alpha_1 \cdot \alpha_2$$
$$= f'(x_0) \cdot g'\big(f(x_0)\big)\,.$$

Examples:-

1) Let $f : (a, b) \to R$ be a function defined by $f(x) = x^n \quad \forall\ x \in (a, b)$, $n \in Z$,. $f'(x_0) = n\, {x_0}^{n-1}$. **(H.W)**

2) $h(x) = (x^2 + 2x)^8 \quad \forall\ x \in (a, b)$, $f(x) = x^2 + 2x \quad \forall\ x \in (a, b)$,

$g(x) = x^8 \quad \forall\ x \in (a, b)$, $h(x) = (g \circ f)(x_0)$

$$(g \circ f)'(x_0) = f'(x_0) \cdot g'\big(f(x_0)\big) = (2x_0 + 2)\,.\,8\,(x_0^2 + 2x_0)^7$$
$$= 8\,(x_0^2 + 2x_0)^7 . (2x_0 + 2)$$

Definition (8.7):

Let $f : (a, b) \to R$ be a function, let $x_0 \in (a, b)$, we say that f is <u>increasing at x_0</u>, if there exists an open interval V , $(x_0 \in V)$ such that $\forall\ x \in V$ if $x < x_0$, then $f(x) < f(x_0)$, and if $x > x_0$, then $f(x) > f(x_0)$.

And f is <u>decreasing at x_0</u>, if there exists an open interval V , $(x_0 \in V)$ such that $\forall\, x \in V$ if $x_0 < x$, then $f(x_0) > f(x)$, and if $x_0 > x$, then $f(x_0) < f(x)$.

If f is increasing at x_0 , $\forall\, x_0 \in (a,b)$, then f is increasing function and if f is decreasing at x_0 , $\forall\, x_0 \in (a,b)$, then f is decreasing.

Theorem (8.8):

Let $f:(a,b) \to R$ be a differentiable function at x_0, if $f'(x_0) > 0$, then f is increasing at x_0 and if $f'(x_0) < 0$, then f is decreasing at x_0. Hence if $f'(x_0) \neq 0$, then there exists an open interval V , $(x_0 \in V)$ such that f is 1-1 and on V.

Proof:

The inverse function theorem (8.9):

Let $f: I \to R$ be a differentiable function at x_0, if $f'(x_0) \neq 0$, then there exists an open interval V and an inverse function g of f where $g: f(V) \to V$ and g is differentiable at $f(x_0)$. Moreover $g'(f(x_0)) = \frac{1}{f'(x_0)}$; I is open intervals.

Proof:

Since $f'(x_0) \neq 0$, then by theorem (8.8) there exists an open interval V, $x_0 \in V$ and $f: V \to R$ is 1-1.

$f: V \to f(V)$ is 1-1 and onto since if $y \in f(V)$, $y = f(x)$; $x \in V$. Hence f has inverse say g. $g: f(V) \to V$, $V \xrightarrow{f} f(V) \xrightarrow{g} V$, $g \circ f = I_V$ $f(V) \xrightarrow{g} V \xrightarrow{f} f(V)$, $f \circ g = I_{f(V)}$.

$(g \circ f)(x) = x \quad \forall\, x \in V$, $(g \circ f)'(x) = 1$, chain rule $g'\big(f(x)\big).f'(x) = 1$ at x_0 , then $g'\big(f(x_0)\big).f'(x_0) = 1$, then $g'(f(x_0)) = \frac{1}{f'(x_0)}$, where $f'(x_0) \neq 0$ [given].

Since f is differentiable at x_0, then $\exists\, \omega_1: I \to R$ continuous and $\omega_1(x_0) = 0$ satisfies:

$$f(x) - f(x_0) = (x - x_0) \cdot [f'(x_0) + \omega_1(x)] \qquad \cdots (1)$$

$$g\big(f(x)\big) - g\big(f(x_0)\big) = \big(f(x) - f(x_0)\big)\left[\frac{1}{f'(x_0)} + \omega_2\big(f(x)\big)\right]$$

$$= (x - x_0) \cdot \big(f'(x_0) + \omega_1(x)\big) \cdot \left[\frac{1}{f'(x_0)} + \omega_2\big(f(x)\big)\right]$$

$$(g \circ f)(x) - (g \circ f)(x_0) = (x - x_0) \cdot \big(f'(x_0) + \omega_1(x)\big) \cdot \left[\frac{1}{f'(x_0)} + \omega_2\big(f(x)\big)\right]$$

$$\frac{x-x_0}{x-x_0} = \left(f'(x_0) + \omega_1(x)\right) \cdot \left[\frac{1}{f'(x_0)} + \omega_2\big(f(x)\big)\right]$$

$$1 = \left(f'(x_0) + \omega_1(x)\right) \cdot \left[\frac{1}{f'(x_0)} + \omega_2\big(f(x)\big)\right]$$

$$\frac{1}{f'(x_0) + \omega_1(x)} - \frac{1}{f'(x_0)} = \omega_2\big(f(x)\big)$$

Thus $\omega_2\big(f(x)\big)$ is continuous on $f(x)$. And

$$\omega_2\big(f(x_0)\big) = \frac{1}{f'(x_0) + \omega_1(x_0)} - \frac{1}{f'(x_0)} = 0 \quad \text{, since } \omega_1(x_0) = 0$$

Definition (8.10):

Let $f: S \to R$ be a function $S \subseteq R$, we say that $x_0 \in S$ is a local maximum point, if there exists an open interval $V \ni x_0$, such that $\forall\, x \in V,\ f(x) \le f(x_0)$, and we say that $z_0 \in S$ is a local minimum point , if there exists an open interval $U \ni z_0$ and $f(z_0) \le f(x) \quad \forall\, x \in U$.

Proposition (8.11):

Let $f: I \to R$ be a differentiable function, if x_0 is either a local minimum point or a local maximum point, then $f'(x_0) = 0$

Proof:

If $f'(x_0) \neq 0$, then either $f'(x_0) < 0$, then f is decreasing at x_0, or $f'(x_0) > 0$, then f is increasing at x_0 in each case x_0 is not local minimum and not local maximum a contradiction, hence $f'(x_0) = 0$

Remark (8.12):-

In general the convers of the above proposition is not true as the following example shows:-

Example:-

Let $f: (-2,2) \to R$ be defined by $f(x) = x^3$.

$$f'(0) = 0$$

Since f is increasing at x_0, then x_0 is not local minimum and not local maximum

Roll's theorem (8.13):

Let $f:[a,b] \to R$ be a differentiable function on (a,b) and continuous on $[a,b]$, if $f(a) = f(b)$, then there exists $c \in (a,b)$, $a < c < b$ such that $f'(c) = 0$.

Proof:

If f is constant, then $f'(x) = 0 \quad \forall\, x \in (a,b)$

If f is not constant

Since f is continuous on $[a,b]$ (compact set), then f has maximum and minimum values say x_0 , y_0.

i.e $\exists\ x_0, y_0 \in [a,b]$ such that $f(x_0) \leq f(x) \leq f(y_0) \quad \forall\, x \in [a,b]$

Clearly $f(x_0) \neq f(y_0)$ since f is not constant.

x_0, y_0 maximum and minimum points, then x_0 is local minimum, then $f'(x_0) = 0$ by (8.11), put $x_0 = c$

Clearly $x_0 \neq a,b$ and $y_0 \neq a,b$, since if $x_0 = a$ or b , then $f(x_0) = f(a) = f(b) = f(y_0)$

Or $y_0 = a$ or b , then $f(x_0) = f(a) = f(b) = f(y_0)$

Then f is constant C!

Say $x_0 \neq a$ or b , then $f'(x_0) = 0$, put $x_0 = c$.

And $y_0 \neq a$ or b , then $f'(y_0) = 0$, put $y_0 = c$.

Mean Value Theorem (8.14):

Let $f:[a,b] \to R$ be a differentiable function on (a,b) and continuous on $[a,b]$, then there exists $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof:

Define $g:[a,b] \to R$ by:-

$$g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

g is differentiable function on (a,b) and continuous on $[a,b]$

$g(a) = f(a) - f(a) - \frac{f(b)-f(a)}{b-a}(a-a) = 0$

$g(b) = f(b) - f(a) - \frac{f(b)-f(a)}{b-a}(b-a) = 0$

So that $g(a) = g(b)$, then by Roll's theorem $\exists\ c \in (a,b)$ such that $g'(c) = 0$.

Then $0 = g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$. Thus

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Chapter (9)

Measure Theory:

The length of open bounded intervals:

Step 1. If $I = (a,b) = \{x \in R: a < x < b\}$ is an open bounded interval, then the length of I is denoted by $\Delta(I)$ or $\Delta\big((a,b)\big)$ and defined by:

$$\Delta(I) = \begin{cases} b-a & if \quad I = (a,b) \\ 0 & if \quad I = \emptyset \end{cases}$$

Where $\emptyset$ is the empty set

1. $\Delta(I) = \Delta\big((a,b)\big) = b - a \geq 0$

$$= dim(I) = Sup\{d(x,y): x,y \in (a,b)\}$$

2. If I, J are two open intervals with $I \subseteq J$, then $\Delta(I) \leq \Delta(J)$.
3. If I, J are two open intervals, then $\Delta(I \cup J) \leq \Delta(I) + \Delta(J)$ and $\Delta(I \cup J) + \Delta(I \cap J) = \Delta(I) + \Delta(J)$, if $I \cap J = \emptyset$, then $\Delta(I \cup J) = \Delta(I) + \Delta(J)$.
   In general if $I_1, I_2, \cdots, I_n$ are open intervals, then $\Delta(\cup_{i=1}^n I_i) \leq \sum_{i=1}^n \Delta(I_i)$
   If $I_1, I_2, \cdots, I_n$ are disjoint, then $\Delta(\cup_{i=1}^n I_i) = \sum_{i=1}^n \Delta(I_i)$.
4. If $\{I_n\}_{n\in N}$ is a countable number of open intervals, then $\Delta(\cup_n I_n) \leq \sum_n \Delta(I_n)$ and if $\{I_n\}_{n\in N}$ are disjoint $I_j \cap I_k = \emptyset \quad \forall\, j,k$, then $\Delta(\cup_n I_n) = \sum_n \Delta(I_n)$.

5. $\Delta(I+t) = \Delta(I) \quad \forall\, t \in R$. Where $I+t = \{x+t : x \in I\}$.

$$\Delta(I+t) = Sup\{|(x+t)-(y+t))| : x+t, y+t \in (I+t)\} = Sup\{|x-y| : x, y \in I\}$$
$$= \Delta(I)$$

The length of open bounded sets:

Step 2. If G is any bounded open subset of R

Lemma (9.1):-

Every open bounded subset of R can be written as a union of a countable number of disjoint unique open intervals and this representation is unique.

i.e $\exists \{I_n\}_{n\in N}$, I_n are open intervals $\forall\, n \quad I_j \cap I_k = \emptyset \quad \forall\, j, k$, $G = \cup_n I_n$. Hence by lemma (9.1) if G is an open subset (interval) of R, then , $G = \cup_n I_n$

Let $\mu(G) = \Delta(G) = \Delta(\cup_n I_n) = \sum_n \Delta(I_n)$ (disjoin).

Is $\sum_n \Delta(I_n)$ exits?

Let $S_1 = \Delta(I_1)$

$S_2 = \Delta(I_1) + \Delta(I_2)$

$\vdots$

$S_n = \Delta(I_1) + \cdots + \Delta(I_n) = \sum_{i=1}^{n} \Delta(I_i)$

Let $\langle S_n \rangle$ be the sequence of partial sum of $\sum_n \Delta(I_n)$.

$S_1 = \Delta(I_1) < S_2 = \Delta(I_1) + \Delta(I_2) < \cdots < S_n = \Delta(I_1) + \cdots + \Delta(I_n)$

Then $S_1 < S_2 < \cdots < S_n$. Thus $\langle S_n \rangle$ is an increasing sequence

Since $G = \cup_n I_n$, G is bounded, then $\cup_n I_n$ is bounded, hence $\exists$ an open interval (ball) I ; $G \subseteq I$, then $\cup_n I_n \subseteq I$.

$\Delta(\cup_n I_n) = Sup\,\{|x-y| : x, y \in \cup_n I_n\} = \sum_n \Delta(I_n) = L$

$S_1 = \Delta(I_1) \leq L$.

$\forall\, n$, $S_n \leq \sum_n \Delta(I_n) = L$, then S_n is bounded, hence $\langle S_n \rangle$ is bounded and monotonic sequence.

Thus $\sum_n \Delta(I_n)$ converges $(\sum_n \Delta(I_n) = L)$ and $\mu(G) = \sum_n \Delta(I_n)$ exists.

Note :-

1 $\mu(G) = \Delta(G) \geq 0$, $(\Delta(G) = \sum_n \Delta(I_n))$, $\mu(\emptyset) = \Delta(G) = 0$

2 If G_1, G_2 are open bounded subset of R with $G_1 \subseteq G_2$, then $\mu(G_1) \leq \mu(G_2)$.

By (9.1) [$G_1 = \cup_n I_n \subseteq G_2 = \cup_m I_m$, then $\sum_n \Delta(I_n) \leq \sum_m \Delta(I_m)$. Thus $\mu(G_1) \leq \mu(G_2)$].

3 If G_1, G_2 are open bounded subset of R, then $\mu(G_1 \cup G_2) \leq \mu(G_1) + \mu(G_2)$.

$G_1 = \cup_n I_n$, $G_2 = \cup_m I_m$, $G_1 \cup G_2 = \cup_{nm} (\cup_n I_n \cup_m I_m) = \cup_{nm} K_{n,m}$

$\mu(G_1 \cup G_2) \leq \sum_{nm} \Delta(K_{n,m}) = \sum_n \Delta(I_n) + \sum_m \Delta(I_m) = \mu(G_1) + \mu(G_2)$

In general if $G_1, G_2, \cdots, G_n$ are open bounded intervals, then $\mu(\cup_{i=1}^n G_i) \leq \sum_{i=1}^n \mu(G_i)$

If $G_1, G_2, \cdots, G_n$ are disjoint, then $\mu(\cup_{i=1}^n G_i) = \sum_{i=1}^n \mu(G_i)$.

4 If $\{G_n\}_{n\in N}$ is a countable collection of open bounded subset of R, then

$\mu(\cup_n G_n) \leq \sum_n \mu(G_n)$ and if $\{G_n\}_{n\in N}$ are disjoint $G_j \cap G_k = \emptyset \quad \forall j, k$, then

$\mu(\cup_n G_n) = \sum_n \mu(G_n)$.

5 If G is open bounded set, then $\Delta(G + t) = \Delta(G) \quad \forall t \in R$

$\mu(G + t) = \mu(G) \quad \forall t \in R$, where $G + t = \{x + t : x \in G\}$.

$\mu(G + t) = \Delta(G + t) = \cup_n (I_n + 1) = \sum_n \Delta(I_n + t) = \sum_n \Delta(I_n) = \mu(G)$

Step 3. If S is any bounded subset of R, let

$$A = \{ G : S \subseteq G \text{ is open and bounded} \} \neq \emptyset$$

S is bounded, then a ball I (open interval) open and bounded such that $S \subseteq I$.

$B = \{\mu(G) = \Delta(G) \geq 0 : S \subseteq G\}$ bounded below, since R is complete,

Let $\mu^*(S) = \inf\{\mu(G) = \Delta(G) : S \subseteq G , G \text{ is open and bounded}\}$

$\mu^*(S)$ is called the outer measure of S. (for short we write $\mu^*(S)$ by $\mu(S)$).

Examples:-

12) If $S = \emptyset$

$\mu^*(S) = \inf\{\mu(G) = \Delta(G) : \emptyset \subseteq G , G \text{ is open and bounded}\}$.

$$= \inf \left\{ \Delta(\, G_\epsilon) : \emptyset \subseteq \left(\frac{-\epsilon}{2}, \frac{\epsilon}{2}\right) = G_\epsilon \;,\; \forall\, \epsilon \right\}$$

$$= \inf \{\; \epsilon : \quad \forall\, \epsilon > 0\} = 0$$

13) - If $S = \{x\}$

$$\mu^*\,(S) = \inf \{\, \mu(G) = \Delta(\, G) : \{x\} \subseteq G \;,\; G\; is \text{ open and bounded}\}.$$

$$= \inf \left\{ \Delta(\, G_\epsilon) : \{x\} \subseteq \left(x - \frac{-\epsilon}{2}, x + \frac{\epsilon}{2}\right) = G_\epsilon \;,\; \forall\, \epsilon \right\} \text{, } G_\epsilon \text{ is open and bounded.}$$

$$= \inf \{\; \epsilon : \quad 0 < \epsilon < 1\} = 0$$

14) If $S = \{x_1, x_2, \cdots, x_n\}$ $\;I_i = \left(x_i - \frac{\epsilon}{2n} \;,\; x_i + \frac{\epsilon}{2n}\right)$

$$\mu^*\,(S) = \inf \{\, \mu(G) = \Delta(\, G) : \; S \subseteq G = \cup_{i=}^{n} I_i \;\; \}$$

$$= \inf \left\{ \textstyle\sum_{i=1}^{n} \Delta(\, I_i) : \sum_{i=1}^{n} \frac{\epsilon}{n} = \frac{n\epsilon}{n} = \epsilon \right\} = 0\,.$$

15) If $S = \{x_1, x_2, \cdots, x_n\,, \cdots\}$ is an infinite countable subset of R, $I_i = \left(x_i - \frac{\epsilon}{2^{i+2}} \;,\; x_i + \frac{\epsilon}{2^{i+2}}\right)$, $G = \cup_{i\in N}\; I_i$

$$\mu^*\,(S) = \inf \{\, \mu(G) = \Delta(\, G) = \textstyle\sum_{i\in N} \Delta(\, I_i) : \; S \subseteq G = \cup_{i\in N}\; I_i \;\}$$

$$= \inf \left\{ \textstyle\sum_{i\in N} \Delta(\, I_i) = \sum_{i\in N} \frac{\epsilon}{2^{i+2}} = \sum_n \frac{\epsilon}{4}\left(\frac{1}{2}\right)^{i-1} \right\} = \inf \left\{ \frac{\frac{\epsilon}{4}}{1-\frac{1}{2}} \right\} = \inf \left\{ \frac{\epsilon}{2} < \epsilon \right\} = 0.$$

16) If $S = [a, b)$

$$\mu^*\,(S) = \inf \left\{ \mu(G) = \Delta(\, G) = b - a + \frac{\epsilon}{2} \;: [a, b) \subseteq \left(a - \frac{\epsilon}{2}, b\right) is \text{ open and bounded} \right\}$$

$$= b - a$$

17) If $S = [a, b]$ **(H.W)**

18) If S is a bounded zero set , then $\mu^*(S) = 0$ and conversely if $\mu^*(S) = 0$, then S is a zero set

Proof:

Let S be a bounded zero set

i.e $\forall \epsilon > 0$, $\exists \{ I_n\}_{n\in N}$ of open interval such that:

1. $S \subseteq \cup_n I_n$.
2. $\sum_n |I_n| < \epsilon$.

T.P $\mu^*(S) = \inf \{ \mu(G) = \Delta(G) : S \subseteq G , G\ is$ open and bonded$\} = 0$

Take $G = \cup_n I_n$.

$\mu^*(S) = \inf \{ \Delta(G) = \Delta(\cup_n I_n) \leq \sum_n \Delta(I_n) < \epsilon \} = 0$

Conversely, let $\mu^*(S) = 0$. T.P S is a zero set.

Let $\epsilon > 0$, then $\exists G$ open and bounded subset of R such that $\Delta(G) < \epsilon$.

$0 = \mu^*(S) = \inf \{\mu(G) = \Delta(G) : S \subseteq G , G\ is$ open and bonded$\}$

Since G is open, then by lemma G is a union of open balls (intervals in R).

Thus $G = \cup_n I_n$, $\{ I_n\}_{n\in N}$ of open intervals in R.

1. $S \subseteq G = \cup_n I_n$.
2. Since $\mu(G) = \Delta(G) < \epsilon$, then $\mu(G) = \Delta(\cup_n I_n) \leq \sum_n |I_n| < \epsilon$.

Bounded measurable sets:

Definition (9.2):

Let S be a bounded subset of R, we say that S is a measurable set, if $\forall \epsilon > 0$, $\exists$ an open bounded subset G of R such that $S \subseteq G$ and $\mu^*(G - S) < \epsilon$.

Note: If S is a measurable set, we put $\mu(S) = \mu^*(S)$

Examples:

1) If $S = (a, b]$. is S measurable set?

Let $\epsilon > 0$,

take $G = \left(a\,,\ b + \frac{\epsilon}{2}\right)$, $(a, b] \subseteq G = \left(a\,, b + \frac{\epsilon}{2}\right)$

$G - (a, b] = \left(b\,, b + \frac{\epsilon}{2}\right)$

$\mu^*\,(G - (a, b]) = \mu^*\left(\left(b\,, b + \frac{\epsilon}{2}\right)\right) = \Delta\left(b\,, b + \frac{\epsilon}{2}\right) = \frac{\epsilon}{2} < \epsilon.$

Hence $S = (a, b]$ is measurable.

2) If $S = \{x_1, x_2, \cdots, x_n\} \subseteq R$. is S measurable set?

Let $\epsilon > 0$ $I_i = \left(x_i - \frac{\epsilon}{4n}\,,\, x_i + \frac{\epsilon}{4n}\right)$, $i = 1,2,\cdots,n$. $S \subseteq G = \cup_{i=}^{n} I_i$

$G - S = \cup_{i=}^{n} I_i - \{x_1, x_2, \cdots, x_n\}$

$$= \cup_{i=1}^{n}\left(\left(x_i - \frac{\epsilon}{4n}\,,\, x_i\right) \cup \left(x_i\,,\, x_i + \frac{\epsilon}{4n}\right)\right)$$

$$\mu^*\,(G - S) = \mu^*\left(\cup_{i=1}^{n}\left(\left(x_i - \frac{\epsilon}{4n}\,,\, x_i\right) \cup \left(x_i\,,\, x_i + \frac{\epsilon}{4n}\right)\right)\right).$$

$$= \sum_{i=1}^{n}\left(\Delta\left(x_i - \frac{\epsilon}{4n}\,,\, x_i\right) + \Delta\left(x_i\,,\, x_i + \frac{\epsilon}{4n}\right)\right)$$

$$= \sum_{i=1}^{n}\frac{\epsilon}{4n} + \frac{\epsilon}{4n} = \sum_{i=1}^{n}\frac{\epsilon}{2n} = \frac{n\epsilon}{2n} = \frac{\epsilon}{2} < \epsilon$$

3) If $S = [a, b]$ bounded. is S measurable set?

$\mu^*\,([a, b]) = \inf\left\{\mu(G) = \Delta(G)\ : [a, b] \subseteq G = \left(a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2}\right)\ is \text{ open and bounded}\right\}$

$$= \inf\left\{\Delta\left(\left(a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2}\right)\right)\ :\ \right\} = \inf\{b - a + \epsilon\} = b - a\,.$$

4) If $S = [a, b)$, (a, b) . is S measurable set? **<u>(H.W)</u>**

Proposition (9.3):

Let S be a bounded measurable set, then

1 $\mu(S) \geq 0$, $\mu(\emptyset) = 0$.

2 If S_1 , S_2 are measurable sets with $S_1 \subseteq S_2$, then $\mu(S_1) \leq \mu(S_2)$.

3 If S_1 , S_2 are measurable sets, then $\mu(S_1 \cup S_2) + \mu(S_1 \cap S_2) = \mu(S_1) + \mu(S_2)$, then $\mu(S_1 \cup S_2) \leq \mu(S_1) + \mu(S_2)$,

If S_1 , S_2 are measurable disjoint sets, then $\mu(S_1 \cup S_2) = \mu(S_1) + \mu(S_2)$.

4 If $\{S_n\}$ is a collection of measurable bounded sets, if $S = \cup_n S_n$ is bounded, then S is measurable set and $\mu(S) = \mu(\cup_n S_n) \leq \sum_n \mu(S_n)$.

If $\{S_n\}$ disjoint, then $\mu(S) = \mu(\cup_n S_n) = \sum_n \mu(S_n)$

5 If S is bounded measurable set and $t \in R$, then $\mu(S + t) = \mu(S)$.

Proposition (9.4):

Let S be a bounded subset of R. S is a measurable set iff $\forall \epsilon > 0$, $\exists$ an open bounded subset of R, G such that (1) $S \subseteq G$, (2) $\mu(G - S) < \epsilon$.

Examples:

1) Every open bounded set is a measurable set.

Let $\epsilon > 0$, take $G = S$, $G \subseteq G$, $\mu(G - G) = \mu(\emptyset) = 0 < \epsilon$.

2) Every bounded interval is a measurable set.

$S = [a, b]$, let $\epsilon > 0$, take $G_n = \left(a - \frac{1}{n} , b + \frac{1}{n}\right)$,

1) $[a, b] \subseteq G$, 2) $\mu\left(\left(a - \frac{1}{n} , b + \frac{1}{n}\right) - [a, b]\right) = \mu\left(\left(a - \frac{1}{n} , a\right) \cup \left(b , b + \frac{1}{n}\right)\right)$

$= \mu\left(\left(a - \frac{1}{n} , a\right)\right) + \mu\left(\left(b , b + \frac{1}{n}\right)\right) = \frac{2}{n} < \epsilon$.

By Arch median $\forall \epsilon > 0$, $\exists k$ $s.t$ $\frac{2}{k} < \epsilon$

3) Every bounded countable (finite or infinite) subset of R is a measurable set

Let $S = \{x_1, x_2, \cdots, x_n, \cdots\} \subseteq R$, let $G_i = \left(x_i - \frac{\epsilon}{2^{i+2}} , x_i + \frac{\epsilon}{2^{i+2}}\right)$, $S \subseteq \cup_{i \in N} G_i$

$$\mu^* (G - S) = \mu^* (\cup_{i \in N} G_i - S) = \mu^* \left(\cup_{i \in N} (G_i - \{x_i\})\right) \leq \sum_{i \in N} \mu^* (G) < \epsilon$$

Theorem (9.5):

let S be a bounded measurable subset of R, then.

1) $\mu(S) \geq 0$, $\mu(\emptyset) = 0$.

2) If S_1 , S_2 are bounded measurable sets such that $S_1 \subseteq S_2$, if S_2 is measurable, then S_1 is measurable and $\mu(S_1) \leq \mu(S_2)$

3) If $S_1, S_2, \cdots, S_n$ are bounded measurable sets, with $\cup_{i=1}^{n} S_i$ is bounded, then $\cup_{i=1}^{n} S_i$ is measurable and $\mu(\cup_{i=1}^{n} S_i) \leq \sum_{i=1}^{n} \mu(S_i)$.

If $S_i \cap S_j = \emptyset \quad \forall\ i \neq j$, then $\mu(\cup_{i=1}^{n} S_i) = \sum_{i=1}^{n} \mu(S_i)$

4) If $S_1, S_2, \cdots, S_n, \cdots$ are bounded measurable subsets of R with $\cup_n S_n$ is bounded, then $\cup_n S_n$ is measurable sets and $\mu(\cup_n S_n) \leq \sum_n \mu(S_n)$.

If $\{S_n\}$ disjoint, then $\mu(\cup_n S_n) = \sum_n \mu(S_n)$

5) If S is bounded measurable subsets of R, then for any $t \in R$, $S + t$ is a measurable set and $\mu(S) = \mu(S + t)$.

Chapter (10)

Lebesgue Theory of Integration:

Definition (10.1): Lebesgue Partition

Let S be a bounded measurable subset of R and $\{S_i\}_{i=1}^{n}$ be a finite collection of subset of S. $\{S_i\}_{i=1}^{n}$ is a Lebesgue partition on S if satisfies:

1) $\cup_{i=1}^{n} S_i = S$

2) S_i are measurable sets $\forall\ i = 1,2, \cdots n$

3) $\forall\ i \neq j$, $S_i \cap S_j$ is a zero set.

Notes:

1) If $P = \{S_i\}_{i=1}^{n}$ and $P' = \{S_j\}_{j=1}^{m}$, we say P' is a refinement to P if $\forall\ j \in N$, $S_j \in P$

2) If $P=\{S_i\}_{i=1}^{n}$ and $P'=\{S_j\}_{j=1}^{m}$ are Lebesgue partitions on S, then $L=\{S_i\cap S_j\ :\ i=1,2,\cdots n\ ,\ j=1,2,\cdots m\}$ is a Lebesgue partition on S.

Definition (10.2): Lebesgue Integral

Let S be a bounded measurable set and $f:S\to R$ be a bounded function.) If $P=\{S_i\}_{i=1}^{n}$ is a partition on S , put

$M_i=\sup\ \{f(x):x\in S_i\}\ \ ,\ \ m_i=\inf\ \{f(x):x\in S_i\}\qquad i=1,2,\cdots n\ .$

And put $M=\sup\{f(x):x\in S\}\ \ ,\ \ m=\inf\ \{f(x):x\in S\}$

Clearly:- $m\le m_i\le M_i\le M\qquad \forall\ i=1,2,\cdots,n\ .$

Now define

$\bar{L}(f,P)=\sum_{i=1}^{n}M_i\ \mu(S_i)$ is called Lebesgue upper sum for Lebesgue partition P.

And

$\underline{L}(f,P)=\sum_{i=1}^{n}m_i\ \mu(S_i)$ is called Lebesgue lower sum Lebesgue partition P.

Clearly:- $m_i\ \mu(S_i)\le\ \underline{L}(f,P)\le\bar{L}(f,P)\le M_i\ \mu(S_i)\ ,\ \ \forall\ i=1,2,\cdots,n.$

Remarks :

(1) If P_2 is a refinement to P_1, then.

$$\bar{L}(f,\ P_2)\le\bar{L}(f,\ P_1)$$

$$\underline{L}(f,\ P_2)\ge\underline{L}(f,\ P_1)$$

(2) For any two partitions P_1 and P_2 on S.

$$\underline{L}(f,\ P_1)\le\ \bar{L}(f,\ P_2)\qquad\cdots(1)$$

$\bar{L}(f)=\{\ \bar{L}(f,P): P\ is\ any\ \text{partition on}\ S\}\ \subseteq R$
$\underline{L}(f)=\{\ \underline{L}(f,P): P\ is\ any\ \text{partition on}\ S\ \}\subseteq R.$
From (1) $\bar{L}(f)$ is bound below and $\underline{L}(f)$ is bound above, by completeness of R

Put $L\overline{\int} f = \inf(\overline{L}(f))$ which is called Lebesgue upper integral and

$L\underline{\int} f = \sup\left(\underline{L}(f)\right)$ which is called Lebesgue lower integral.

Clearly that:- $L\underline{\int} f \leq L\overline{\int} f$. If $L\overline{\int} f = L\underline{\int} f$, then f is called Lebesgue integrable and we write $L\overline{\int} f = L\underline{\int} f = \int_S f$.

Remarks (10.3):

Every Riemann partition is a Lebesgue partition.

Proof:

Let $[a,b]$ be a closed interval. It's clear that $[a,b]$ measurable.

Let $\pi_n = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ be a partition on $J_i = [x_{i-1}, x_i]$

1. $\cup_{i=1}^{n} J_i = [a,b]$
2. J_i is a measurable sets $\forall\ i$
3. $\forall\ i \neq k$, $J_i \cap J_k = \emptyset$ or only one element which is a zero set.

Hence we have the following result.

Proposition (10.4):-

If $f: [a,b] \to R$ is a bounded function and f is Riemann integrable, then f is Lebesgue integrable

Proof:

By remark (10.3) every Riemann pa[illegible]n is a Lebesgue partition

$\overline{R}(f) = \{\overline{R}(f,\pi): \pi\ is\ any$ partition on $[a,b]\}$
$\underline{R}(f) = \{\underline{R}(f,\pi): \pi\ is\ any$ partition on $[a,b]\}$
$\overline{L}(f) = \{\overline{L}(f,P): P\ is\ any$ partition on $[a,b]\}$

$\underline{L}(f) = \{\underline{L}(f,P): P\ is\ any$ partition on $[a,b]\}$

It's clear that $\overline{R}(f) \subseteq \overline{L}(f)$ and $\underline{R}(f) \subseteq \underline{L}(f)$

Then

$sup\left(\underline{R}(f)\right) \leq sup\left(\underline{L}(f)\right)$

$inf\left(\bar{R}(f)\right) \geq inf\left(\bar{L}(f)\right)$

This means that

$R\underline{\int} f \leq L\underline{\int} f \leq L\overline{\int}\ f\ \leq R\overline{\int}\ f$

Since f is Riemann integrable, then $R\underline{\int} f = R\overline{\int}\ f$ and hence $L\underline{\int} f = L\overline{\int}\ f$.

Remark :

The converse of remark (10.4) is not true in general as the following example shows.

Example :

Let $f: [a, b] \to R$ be defined by $f(x) = \begin{cases} 1 & x \in Q \cap [a, b] \\ 5 & x \in Q' \cap [a, b] \end{cases}$

$D(f) = [a, b]$ is not a zero set, then by Lebesgue theorem $f \notin RI[a, b]$. Thus f is not Riemann integrable

Let $S = [a, b]$, $P = \{S_1, S_2\}$ where $S_1 = Q \cap [a, b]$ and $S_2 = Q' \cap [a, b]$.

Claim: P is a Lebesgue partition

1) $S_1 \cup S_2 = [a, b] = S$
2) $\mu(S_1) = 0$, $S_1 \cup S_2 = [a, b]$, $S_1 \cap S_2 = \emptyset$ (disjoint)

   $\mu(S_1 \cup S_2) = \mu(S_1) + \mu(S_2)$

   $\mu([a, b]) = 0 + \mu(S_2) = b - a$.

   $\mu(S_2) = b - a$.
3) $S_1 \cap S_2 = \emptyset$ is a zero set.

$M_i = \sup\ \{f(x): x \in S_i\}$, $m_i = \inf\ \{f(x): x \in S_i\}$

$\bar{L}(f,P) = \sum_{i=1}^{2} M_i\mu(S_i) = M_1\mu(S_1) + M_2\mu(S_2)$

$= 1(0) + 5(b-a) = 5(b-a)$

$\underline{L}(f,P) = \sum_{i=1}^{2} m_i\mu(S_i) \ = m_1\mu(S_1) + m_2\mu(S_2)$

$= 1(0) + 5(b-a) = 5(b-a)$

$\bar{L}(f) = \{\ 5(b-a): forany$ partition $P \ \}$.

$\underline{L}(f) = \{5(b-a): forany$ partition $P \ \}$

$L\overline{\int}\ f = \inf\big(\bar{L}(f)\big) = 5(b-a)$

$L\underline{\int} f = \sup\left(\underline{L}(f)\right) = 5(b-a)$

$\therefore \ L\underline{\int} f = L\overline{\int}\ f$. Thus f is Lebesgue integrable.

Proposition (10.5):-

Let S be a measurable bounded set and $f: S \to R$ be a bounded function, then f is Lebesgue integrable iff for each $\epsilon > 0$, there exists a Lebesgue partition P_0 such that $\bar{L}(f,P_0) - \underline{L}(f,P_0) < \epsilon$.

Proof:

Compare with **Lemma (7.6)** in chapter (7).

Some properties of lebesgue integral:

Remarks :

19) Let S be a measurable bounded set and $f: S \to R$ be a function defined by $f(x) = a$, $\forall x \in S$, $a \in R$, then f is Lebesgue integrable and $\int_S f = a\,\mu(S)$

Proof:

Let $P = \{S_i\}_{i=1}^{n}$ be any partition on S.

$M_i = \sup\ \{f(x): x \in S_i\}$, $m_i = \inf\ \{f(x): x \in S_i\}$

$\overline{L}(f,P) = \sum_{i=1}^{n} M_i \, \mu(S_i) = a\mu(S_1) + a\mu(S_2) + \cdots + a\mu(S_n)$

$= a\big(\mu(S_1) + \mu(S_2) + \cdots + \mu(S_n)\big) = a\mu(S)$?

$\cup_{i=1}^{n} S_i = S$ and $S_i \cap S_j =$ zero set, then $\mu(S) = \mu(\cup_{i=1}^{n} S_i) + \mu(\cap_{i=1}^{n} S_i) = \sum_{i=1}^{n} \mu(S_i)$
$\mu(S) = \mu(\cup_{i=1}^{n} S_i) = \sum_{i=1}^{n} \mu(S_i) = a\mu(S)$
Thus $\overline{L}(f,P) = a\mu(S)$.

$\underline{L}(f,P) = \sum_{i=1}^{n} m_i \mu(S_i) = a\mu(S_1) + a\mu(S_2) + \cdots + a\mu(S_n)$

$= a\big(\mu(S_1) + \mu(S_2) + \cdots + \mu(S_n)\big) = a\,\mu(S)$

$\overline{L}(f) = \{\, a\,\mu(S): forany$ partition $P \,\}$.

$\underline{L}(f) = \{a\,\mu(S): forany$ partition $P\ \}$

$L\overline{\int} f = \inf\big(\overline{L}(f)\big) = a\,\mu(S)$
$L\underline{\int} f = \sup\Big(\underline{L}(f)\Big) = a\,\mu(S)$
$\therefore\ L\overline{\int} f = L\underline{\int} f = a\,\mu(S).$
Thus f is Lebesgue integrable.

20) Let S is a bounded measurable set and $f: S \to R$ be a bounded Lebesgue integrable function, if $a \le f(x) \le b$, $\forall\, x \in S$, then $a\mu(S) \le \int_S f \le b\mu(S)$.

Proof:

Let $P = \{S_i\}_{i=1}^{n}$ be a Lebesgue partition on S, $M_i = \sup\ \{f(x): x \in S_i\}$

$\overline{L}(f,P) = \sum_{i=1}^{n} M_i \, \mu(S_i) = M_1\mu(S_1) + M_2\mu(S_2) + \cdots + M_n\mu(S_n)$

$\le b\mu(S_1) + b\mu(S_2) + \cdots + b\mu(S_n)$

$= b\big(\mu(S_1) + \mu(S_2) + \cdots + \mu(S_n)\big) = b\,\mu(S)$

$$\int_S f = L\overline{\int} f = \inf\big\{\, \overline{L}(f) = \{\, \overline{L}(f,P): forany \text{ partition } P \,\}\big\}$$

$$\le b\,\mu(S)$$

21) If S is a zero set and $f: S \to R$ is a bounded function, then f is Lebesgue integrable and $\int_S f = 0$.

Proof:

Let $P = \{S_i\}_{i=1}^{n}$ be a Lebesgue partition on S.

1) $\cup_{i=1}^{n} S_i = S$.

2) S_i are measurable sets $\forall\, i$

3) $\forall\, i \neq j$, $S_i \cap S_j$ is a zero set.

Since S is a zero set, then each S_i is a zero set and $\mu(S_i) = 0$, $\forall\, i$.

$M_i = \sup\,\{f(x): x \in S_i\}$, $m_i = \inf\,\{f(x): x \in S_i\}$, then

$$\overline{L}(f, P) = \sum_{i=1}^{n} M_i\, \mu(S_i) = M_1\mu(S_1) + M_2\mu(S_2) + \cdots + M_n\mu(S_n)$$

$$= M_1 \cdot 0 + M_2 \cdot 0 + \cdots + M_n \cdot 0 = 0$$

Similarly $\underline{L}(f, P) = 0$.

Then $L\underline{\int} f = L\overline{\int} f = 0$, hence f is Lebesgue integrable and $\int_S f = 0$.

22) If S is a bounded measurable set and $f: S \to R$ is a bounded Lebesgue integrable function, $f(x) \geq 0$, $\forall\, x \in S$, then $\int_S f \geq 0$.

Proof:

From (2) $0 \leq f(x)$, $\forall\, x$, then $0 \cdot \mu(S) \leq \int_S f\,(x)$, $\forall\, x$. Thus

$0 \leq \int_S f$

Proposition (10.6):-

Let $f: S \to R$ be bounded function, S be a measurable bounded set if A , B are subsets of S such that $S = A \cup B$ and $A \cup B = \emptyset$ and f is Lebesgue integrable, then

$$\int_S f = \int_A f + \int_B f$$

Proposition (10.7):

Let S be a measurable bounded set and $f, g: S \to R$ be bounded Lebesgue integrable functions, then.

3) $\int_S (f+g) = \int_S f + \int_S g$

4) $\int_S (c.f) = c \int_S f \qquad \forall\, c \in R$

Corollary (10.8):

If S is a measurable bounded set and $f, g: S \to R$ are bounded Lebesgue integrable functions such that $f(x) \le g(x) \quad \forall\, x \in S$, then $\int_S f \le \int_S g$.

Proof:

Let $h(x) = g(x) - f(x) \ge 0 \quad \forall\, x \in S$

By (4) $0 \le \int_S h = \int (g-f) = \int_S g + \int_S -f = \int_S g - \int_S f$.

Then

$\int_S f \le \int_S g$.

Corollary (10.9):

If S is a measurable bounded set and $f: S \to R$ is a bounded Lebesgue integrable function ,

then $|f|$ is Lebesgue integrable and $\left|\int_S f\right| \le \int_S |f|$.

Proof: (H.W)

Definition (10.10):

Let $f, g: S \to R$ be functions, if there exists a zero set $S_0 \subset S$ such that $f(x) = g(x)$ $\forall\, x \notin S_0$ $[\forall\, x \in (S - S_0)]$, then we say that $f = g$ almost everywhere $(a.e)$

Proposition (10.11):

Let S is be bounded measurable set and $f, g: S \to R$ be bounded functions, if f is Lebesgue integrable and $f = g \quad a.e$, then g is Lebesgue integrable and $\int_S f = \int_S g$.

Example:

Let $f: [-2,2] \to R$ be defined by $f(x) = \begin{cases} 2 & x \in Q \cap [a,b] - \{0\} \\ -1 & x \in Q' \cap [a,b] - \sqrt{2} \\ 4 & x = \{0, \sqrt{2}\} \end{cases}$.

$g(x) = -1 \quad \forall\, x \in [-2,2]$, $f(x) = g(x) \quad \forall\, x \in S_0 = Q' \cap [a,b] - \sqrt{2}$

$f = g \quad a.e \quad \forall\, x \notin Q \cap [a,b] \cup \{0, \sqrt{2}\}$

$\int_S f = \int_S g = -1\,(4) = -4$

Measurable functions and integrable functions

Definition(Measurable functions) (10.12):

Let $S \subseteq R$, $f: S \to R$ be bounded function, f is said to be a measurable function if for each open set G in R, $f^{-1}(G) \subseteq S$ is a measurable set.

Remarks:

1) If $f: S \to R$ is a measurable function, then S is a measurable set.
   Since R is open, then $f^{-1}(R) = S$ is a measurable set.

2) If S is a measurable set and f is a continuous function, then f is a measurable function.
   Proof:
   Let G be any open set, since f is a continuous function, then $f^{-1}(G) \subseteq S$, $f^{-1}(G)$ is open set and S is a measurable set, hence $f^{-1}(G)$ is a measurable set.

Proposition (10.13):

If $S \subseteq R$ and $f: S \to R$ is a function, then the following are equivalent:

1) f is a measurable function.

2) For each closed set $E \subseteq R$, then $f^{-1}(E)$ is a measurable set.
3) For each $[\, a, b), [\, a, b), (a, b) \subseteq R$, then $f^{-1}\big([\, a, b)\big), f^{-1}\big([\, a, b)\big), f^{-1}\big((a, b)\big)$ are measurable sets
3) For each $a \subseteq R$, then $f^{-1}\big((a, \infty)\big), f^{-1}\big((-\infty, a)\big)$ are measurable sets.
4) For each $a \subseteq R$, then $f^{-1}\big([\, a, \infty)\big), f^{-1}\big((-\infty, a]\big)$ are measurable sets.

Corollary(10.14):

Let $f: S \to R$ is a function:

1) f is a measurable function iff for each $a \subseteq R$, then $f^{-1}\big((a, \infty)\big)$ is a measurable set.
2) f is a measurable function iff for each $a \subseteq R$, then $f^{-1}\big([\, a, \infty)\big)$ is a measurable set.
3) f is a measurable function iff for each $a \subseteq R$, then $f^{-1}\big((-\infty, a)\big)$ is a measurable set.
4) f is a measurable function iff for each $a \subseteq R$, then $f^{-1}\big((-\infty, a]\big)$ is a measurable set.

Example:

Let $f: [a, b] \to R$ be defined by $f(x) = \begin{cases} -2 & x \in Q \cap [a, b] \\ 1 & x \in Q' \cap [a, b] \end{cases}$.

Sol: let $G \subseteq R$, G is open

$$f^{-1}(G) = \begin{cases} [a, b] & 1, -2 \in G \\ Q' \cap [a, b] & 1 \in G, -2 \notin G \\ Q \cap [a, b] & 1 \notin G, -2 \in G \end{cases}$$

$f^{-1}(G) = \{x \in [a, b]: f(x) \in G\}$

$Q \cap [a, b]$ is bounded countable set, the $\mu\,(Q \cap [a, b]) = 0$

$[a, b] = (Q \cap [a, b]) \cup (Q' \cap [a, b])$ [disjoint]

$\mu[a, b] = \mu(Q \cap [a, b]) + \mu(Q' \cap [a, b])$

$b - a = 0 + \mu(Q' \cap [a, b])$

In each case $f^{-1}(G)$ is a measurable set, hence f is a measurable function.

Remark:

If $f: [a, b] \to R$ is a monotonic function, then f is a measurable function? (why)

Proposition (10.15):

If $S \subseteq R$ and $f: S \to R$ is a measurable function and $g: R \to R$ are continuous function, then $g \circ f$ is a measurable function.

Proof:

Let G be any open set in R, since g is a continuous function, then $g^{-1}(G)$ is open set in R. $(g \circ f)^{-1}(G) = f^{-1}\left(g^{-1}(G)\right)$, since $f^{-1}(G)$ is open set in R and f is a measurable function, hence $f^{-1}\left(g^{-1}(G)\right)$ is a measurable set.

Bounded variation functions

Definition(10.16):

Let $f: [a, b] \to R$ be a function and Let $\pi_n = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ be a partition on $[a, b]$, $J_i = [x_{i-1}, x_i]$ $\quad i = 1,2, \cdots, n$

Let

$$V(f, \pi_n) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \geq 0$$

Let $V(f) = \{V(f, \pi_n): \pi_n \text{ is any partition on } [a, b]\} \subseteq R \geq 0$, then $V(f)$ is bound below.
If $V(f)$ is bound above, then $V(f)$ has least upper bound.
Put $V = Sup\left(V(f)\right)$.
V is called the variation of f on $[a, b]$

f is called the bounded variation function.

Otherwise if $V(f)$ is bound above, then f is not bounded variation function.

Remark (10.17):

If $f: [a, b] \to R$ is a bounded variation function, then f is bounded.

Proof:

To proof $\exists\ M > 0$, $M \in R$ $\quad s.t \quad |f(x)| \leq M \quad \forall\, x \in [a, b]$.

Let $\pi_n = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ be a partition on $[a, b]$, $J_i = [x_{i-1}, x_i]$

$i = 1,2,\cdots,n$

$$V\,(f,\pi_n) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \geq 0$$

$V(f) = \{\, V\,(f,\pi_n) \colon \pi_n \; is \; any$ partition on $[a,b]\} \subseteq R \geq 0$.

$V = Sup\big(V(f)\big)$ exists, then $|f(x) \leq| \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq V \quad \forall\, x \in [a,b]$

Take $M = V$.

Thus $|f(x)| \leq M \quad \forall\, x \in [a,b]$

Remark (10.18):

If $f \colon [a,b] \to R$ is a bounded monotonic function, then f is a bounded variation function.

Remark (10.19):

If $f, g \colon [a,b] \to R$ are bounded variation functions, then $f + g$ is a bounded variation function.

Proof:

Let $\pi_n = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ be a partition on $[a,b]$, $J_i = [x_{i-1}, x_i]$

$i = 1,2,\cdots,n$

$V\,(f + g\,,\pi_n) = \sum_{i=1}^{n} |(\,f + g)\,(x_i) - (f + g)\,(x_{i-1})| \geq 0$.

$= \sum_{i=1}^{n} |f(x_i) - f(x_{i-1}) + g(x_i) - g(x_{i-1})|$

$\leq \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|$

$V(f + g) = \{\, V\,(f + g\,,\pi_n) \colon \pi_n \; is \; any$ partition on $[a,b]\}$

$\leq \{\, V\,(f,\pi_n) + V\,(g,\pi_n) \colon \pi_n \; is \; any$ partition on $[a,b]\}$

$= \{\, V\,(f,\pi_n) \colon \pi_n \; is \; any$ partition on $[a,b]\} \cup \{V\,(g,\pi_n) \colon \pi_n \; is \; any$ partition on $[a,b]\}$

$= V(f) \cup V(g)$

$Sup\,\big(V(f + g)\big) \leq Sup\big(V(f)\big) + Sup\big(V(g)\big)$

Thus $V(f + g) \leq V(f) + V(g)$

Remark (10.20):

If $f:[a,b] \to R$ is a bounded variation function, then cf is a bounded variation function.

Proof: **(H.W)**