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قسم الكيمياء

صباحي+مسائي

Chapter (1)

Integration

It is the reverse process of differentiation.
OR It is the inverse of derivative.
i.e Given $\frac{dy}{dx} = f'(x)$ to find $y = f(x)$.

Indefinite integration

If $u(x)$ and $v(x)$ denote differentiable functions of the independent variable (x) and a, n and C are constants, then we have the following basic rules:

1. $\int du = u + C$, $\int dx = x + C$ and $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ $(n \neq -1)$
2. $\int a\, du = a \int du$, $\int a\, dx = a \int dx$
3. $\int (du \pm dv) = \int du \pm \int dv$
4. $\int u^n du = \frac{u^{n+1}}{n+1} + C$ $(n \neq -1)$

Note $\boxed{\frac{d}{dx} \int f(x)\, dx = f(x)}$ $\Rightarrow$ eg: $\frac{d}{dx} \int 3x^2 dx = 3x^2$

examples

1- $\int 3x^5 dx = 3 \int x^5 dx = 3 \cdot \frac{x^6}{6} + C = \frac{x^6}{2} + C$

2. $\int x^{-1.4} dx = \frac{x^{-1.4+1}}{-1.4+1} + C = \frac{x^{-0.4}}{-0.4} + C$

3. $\int \frac{dx}{\sqrt{x}} = \int x^{-1/2} dx = \frac{x^{-1/2+1}}{-1/2+1} + C = \frac{x^{1/2}}{1/2} + C = 2\sqrt{x} + C$

4. $\int (x^2+5)^5\, 2x\, dx = \frac{(x^2+5)^6}{6} + C$

5- $\int \frac{x^2}{\sqrt{2x^3+5}} dx = \frac{1}{6} \int (2x^3+5)^{-1/2}\, 6x^2 dx = \frac{1}{6} \frac{(2x^3+5)^{-1/2+1}}{-1/2+1} + C$

6. $\int \frac{x^3 + 4x^2 + 5}{x^2} dx = \int \frac{x^3}{x^2} dx + \int \frac{4x^2}{x^2} dx + 5 \int \frac{1}{x^2} dx = \left[\frac{x^2}{2} + 4x + 5\left(\frac{x^{-2+1}}{-2+1}\right)\right] + C$

7- $\int (4x^2+3)(2x-1)\, dx = \int (8x^3 - 4x^2 + 6x - 3) dx = \left[8\frac{x^4}{4} - \frac{4x^3}{3} + 6\frac{x^2}{2} - 3x\right] + C$

8- $\int \sqrt{2x+1}\, dx = \int (2x+1)^{1/2} dx = \frac{1}{2} \int (2x+1)^{1/2}\, 2\, dx = \frac{1}{2} \frac{(2x+1)^{1/2+1}}{1/2+1} + C$

$= \frac{1}{2} \frac{(2x+1)^{3/2}}{3/2} + C = \frac{1}{3} (2x+1)^{3/2} + C$

9. $\int \frac{\sqrt{1+\sqrt{x}}}{2\sqrt{x}}\,dx = \int (1+\sqrt{x})^{1/2}\frac{dx}{2\sqrt{x}} = \int u^{1/2}\,du = \frac{(1+\sqrt{x})^{1/2+1}}{1/2+1} + C$

10. Find a function $f(x)$, such that $f'(x) = \sqrt{3x+1}$ and $f(1)=5$.

$f'(x) = \frac{df(x)}{dx} = \sqrt{3x+1}$

$f(x) = \int d\,f(x) = \int \sqrt{3x+1}\,dx = \int (3x+1)^{1/2}dx = \frac{1}{3}\int (3x+1)^{1/2}\,3\,dx$

$\therefore f(x) = \frac{1}{3}\frac{(3x+1)^{3/2}}{3/2} + C = \frac{2}{9}(3x+1)^{3/2} + C$

$f(1) = \frac{2}{9}(3*1+1)^{3/2} + C$

$5 = \frac{2}{9}(4)^{3/2} + C \Rightarrow C = \frac{29}{9}$

$\therefore f(x) = \frac{2}{9}(3x+1)^{3/2} + \frac{29}{9}$

11. Find $f(x)$ whose graph passes through $(0, 4/3)$ and whose derivative $x\sqrt{1-x^2}$. [i.e. find $y = f(x)$]

$\frac{dy}{dx} = f'(x) = x\sqrt{1-x^2} \Rightarrow \therefore y = \int dy = \int x\sqrt{1-x^2}\,dx$

$\Rightarrow y = \int \sqrt{1-x^2}\,x\,dx \Rightarrow y = \frac{1}{-2}\int (1-x^2)^{1/2}(-2x\,dx)$

$\therefore f(x) = y = -\frac{1}{2}\frac{(1-x^2)^{3/2}}{3/2} + C \Rightarrow f(x) = -\frac{(1-x^2)^{3/2}}{3} + C$

$\because$ the point $(0, 4/3)$ passes through the curve

$\therefore 4/3 = -\frac{(1-(0)^2)^{3/2}}{3} + C \Rightarrow \frac{4}{3} = -\frac{1}{3} + C \Rightarrow \frac{4}{3} + \frac{1}{3} = C$

$\therefore C = 5/3$

$\therefore f(x) = -\frac{(1-x^2)^{3/2}}{3} + 5/3$

12. Solve the following differential equations

a) $\frac{dy}{dx} = x^2+1 \Rightarrow \int dy = \int (x^2+1)\,dx \Rightarrow y = \int x^2dx + \int dx = \left(\frac{x^3}{3}+x\right) + C$

b) $\frac{dy}{dx} = \frac{1}{x^2} + x$, $(x>0)$ $\Rightarrow y = \int dy = \int \frac{1}{x^2}dx + \int x\,dx$

$\Rightarrow y = \int x^{-2}dx + \int x\,dx \Rightarrow y = \left[\frac{x^{-2+1}}{-2+1} + \frac{x^2}{2}\right] + C = \left[\frac{x^{-1}}{-1} + \frac{x^2}{2}\right] + C$

c) $\frac{dy}{dx} = \frac{x}{y} \Rightarrow \int y\,dy = \int x\,dx = \frac{y^2}{2} + C_1 = \frac{x^2}{2} + C_2$

$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + (C_2 - C_1) \Rightarrow y^2 = x^2 + 2(C_2 - C_1) \Rightarrow y^2 = x^2 + C$

Solved Problems

1. Solve the following differential equations:

a) $\frac{dy}{dx} = \sqrt{xy}$, $(x>0, y>0)$

$\frac{dy}{y^{1/2}} = x^{1/2}dx \Rightarrow \int y^{-1/2}dy = \int x^{1/2}dx \Rightarrow \frac{y^{-1/2+1}}{-1/2+1} + c_1 = \frac{x^{3/2}}{3/2} + c_2$

$\Rightarrow \left[\frac{y^{1/2}}{1/2} = \frac{x^{3/2}}{3/2} + c_2 - c_1\right] * 3/2 \Rightarrow 3y^{1/2} = x^{3/2} + 3/2(c_2 - c_1)$

$\Rightarrow \therefore 3y^{1/2} = x^{3/2} + c$, $(c = 3/2(c_2 - c_1))$

b) $\frac{dy}{dx} = \sqrt[3]{\frac{y}{x}}$ $(x>0, y>0)$ $\Rightarrow \frac{dy}{dx} = \frac{\sqrt[3]{y}}{\sqrt[3]{x}} \Rightarrow \frac{dy}{\sqrt[3]{y}} = \frac{dx}{\sqrt[3]{x}} \Rightarrow \int y^{-1/3}dy = \int x^{-1/3}dx$

$\Rightarrow \left[\frac{y^{2/3}}{2/3} + c_1 = \frac{x^{2/3}}{2/3} + c_2\right] * 2/3 \Rightarrow y^{2/3} = x^{2/3} + 2/3(c_2 - c_1)$

$\Rightarrow \therefore y^{2/3} = x^{2/3} + c$, $c = 2/3(c_2 - c_1)$

c) $\frac{dy}{dx} = 2xy^2$, $(y>0) \Rightarrow \frac{dy}{y^2} = 2x\,dx \Rightarrow \int y^{-2}dy = \int 2x\,dx$

$\Rightarrow \frac{y^{-1}}{-1} + c_1 = 2\frac{x^2}{2} + c_2 \Rightarrow -\frac{1}{y} = x^2 + (c_2 - c_1) \Rightarrow -\frac{1}{y} = x^2 + c$

d) $\frac{ds}{dt} = 3t^2 + 4t - 6 \Rightarrow \int ds = \int (3t^2 + 4t - 6)dt \Rightarrow s = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + c$

$\Rightarrow \therefore s = t^3 + 2t^2 - 6t + c$

e) $\frac{dr}{dz} = (2z+1)^3 \Rightarrow \int dr = \int (2z+1)^3 dz \Rightarrow r = \frac{1}{2}\int (2z+1)^3 2dz$

$\Rightarrow r = \frac{1}{2}\frac{(2z+1)^4}{4} + c \Rightarrow \therefore r = \frac{(2z+1)^4}{8} + c$

f) $\frac{du}{dv} = 2u^2(4v^3 + 4v^{-3})$, $v>0$, $u>0$

$\frac{du}{2u^2} = (4v^3 + 4v^{-3})dv \Rightarrow \frac{1}{2}\int u^{-2}du = 4\int (v^3 + v^{-3})dv$

$\Rightarrow \left[\frac{1}{2}\frac{u^{-1}}{-1} + c_1 = 4\left(\frac{v^4}{4} + \frac{v^{-2}}{-2}\right) + c_2\right] * 2 \Rightarrow -u^{-1} = 2v^4 - 4v^{-2} + 2(c_2 - c_1)$

$\Rightarrow -u^{-1} = 2v^4 - 4v^{-2} + c$ $\qquad c = 2(c_2 - c_1)$

g) $\frac{dx}{dt} = 8\sqrt{x}$, $x > 0$ $\Rightarrow \frac{dx}{\sqrt{x}} = 8\,dt \Rightarrow \int x^{-1/2}dx = 8\int dt$

$\frac{x^{1/2}}{1/2} + C_1 = 8t + C_2 \Rightarrow \left[2x^{1/2} = 8t + (C_2 - C_1)\right] * \frac{1}{2}$

$\Rightarrow x^{1/2} = 4t + C$, $C = \frac{1}{2}(C_2 - C_1)$

h) $\frac{dy}{dt} = (2t + t^{-1})^2$, $t > 0 \Rightarrow dy = (2t + t^{-1})^2 dt$

$\Rightarrow dy = (4t^2 + 4tt^{-1} + t^{-2})dt \Rightarrow \int dy = \int (4t^2 + 4 + t^{-2})dt$

$\Rightarrow y = \frac{4}{3}t^3 + 4t + \frac{t^{-1}}{-1} + C \Rightarrow y = \frac{4}{3}t^3 + 4t - t^{-1} + C$

2. Evaluate the following intervals:

a) $\int (x^2 - \sqrt{x})dx = \int x^2 dx - \int \sqrt{x}\,dx = \frac{x^3}{3} - \frac{x^{3/2}}{3/2} + C$

b) $\int (3x-1)^{234}dx = \frac{1}{3}\int (3x-1)^{234}\,3dx = \frac{1}{3}\cdot\frac{(3x-1)^{235}}{235} + C$

c) $\int (2-7t)^{2/3}dt = \frac{-1}{7}\int (2-7t)^{2/3}(-7dt) = \frac{-1}{7}\,\frac{(2-7t)^{5/3}}{5/3} + C$

d) $\int \sqrt{2+5y}\,dy = \frac{1}{5}\int (2+5y)^{1/2}\,5dy = \frac{1}{5}\frac{(2+5y)^{3/2}}{3/2} + C$

e) $\int \frac{dx}{(3x+2)^2} = \frac{1}{3}\int (3x+2)^{-2}\,3dx = \frac{1}{3}\frac{(3x+2)^{-1}}{-1} + C =$

$= \frac{-1}{3(3x+2)} + C = \frac{-1}{9x+6} + C$

f) $\int \frac{3r\,dr}{\sqrt{1-r^2}} = 3\int (1-r^2)^{-1/2}r\,dr = \frac{3}{-2}\int (1-r^2)^{-1/2}(-2r\,dr) = \frac{3}{-2}\,\frac{(1-r^2)^{1/2}}{1/2} + C$

$= -3(1-r^2)^{1/2} + C$

g) $\int \sqrt{2x^2+1}\;x\,dx = \frac{1}{4}\int (2x^2+1)^{1/2}\,4x\,dx = \frac{1}{4}\,\frac{(2x^2+1)^{3/2}}{3/2} + C$

h) $\int t^2(1+2t^3)^{-2/3}dt = \frac{1}{6}\int (1+2t^3)^{-2/3}\,6t^2dt = \frac{1}{6}\,\frac{(1+2t^3)^{1/3}}{1/3} + C$

i) $\int \frac{y\,dy}{\sqrt{2y^2+1}} = \frac{1}{4}\int (2y^2+1)^{-1/2}\,4y\,dy = \frac{1}{4}\,\frac{(2y^2+1)^{1/2}}{1/2} + C = \frac{1}{2}(2y^2+1)^{1/2} + C$

j) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx = \int x^{1/2}dx + \int x^{-1/2}dx = \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C = \frac{2}{3}x^{3/2} + 2x^{1/2} + C$

$\Longrightarrow$ انظر خلف الورقة رقم ٣

K) $\int \frac{(z+1)}{\sqrt[3]{z^2+2z+2}}\,dz = \frac{1}{2}\int (z^2+2z+2)^{-1/3}\, 2(z+1)\,dz$

$$= \frac{1}{2}\,\frac{(z^2+2z+2)^{2/3}}{2/3} + C$$

$$= \frac{3}{4}(z^2+2z+2)^{2/3} + C$$

Applications of indefinite integrations

examples:

1. The velocity, at time t, of a moving body is given by $v = at$, where (a) is a constant. If the body's coordinate is (S_0) at time $t = 0$, find the distance (S) as a function of (t).

$v = \frac{ds}{dt} = at \Rightarrow \therefore ds = at\,dt \Rightarrow s = \int ds = \int at\,dt$

$\therefore s = \frac{1}{2}at^2 + c$

at $t = 0$: the distance becomes:

$S_0 = \frac{1}{2}a(0)^2 + c \Rightarrow \therefore c = S_0$, thus

$\therefore s = \frac{1}{2}at^2 + S_0$

2. Find the curve whose slope at the point (x, y) is $3x^2$ if the curve is also required to pass through the point $(1, -1)$.

We have

$\text{Slope} = \frac{dy}{dx} = 3x^2 \Rightarrow \int dy = 3x^2\,dx \Rightarrow y = x^3 + c$

at the point $(1, -1)$, the curve becomes:

$-1 = (1)^3 + c \Rightarrow \therefore c = -2$ then the curve becomes as

$y = x^3 - 2$

Solved Problems

1. In each of the following, find the position s as a function of t from the given velocity $v = \frac{ds}{dt}$. Evaluate the constant of integration so as to have $s = s_0$ when $t = 0$.

a) $v = (t+1)^2 \Rightarrow \frac{ds}{dt} = (t+1)^2 \Rightarrow ds = (t+1)^2 dt$

$\therefore s = \frac{(t+1)^3}{3} + c$ —— (1)

at $t=0$, eq.(1) becomes $S_0 = \frac{(0+1)^3}{3} + c$

$\Rightarrow \therefore c = S_0 - \frac{1}{3}$ —— (2), put (2) into (1), we get

$$S = \frac{(t+1)^3}{3} + S_0 - \frac{1}{3} \Rightarrow \therefore S = \frac{(t+1)^3}{3} - \frac{1}{3} + S_0$$

b) $v = (t+1)^{-2} \Rightarrow ds = (t+1)^{-2} dt \Rightarrow s = \int (t+1)^{-2} dt$

$s = \frac{(t+1)^{-1}}{-1} + c$ —— (1)

at $t = 0$ eq.(1) becomes: $S_0 = -(0+1)^{-1} + c \Rightarrow \therefore c = S_0 + 1$

then eq(1) becomes: $S = -(t+1)^{-1} + 1 + S_0$

c) $v = (t^2+1)^2 \Rightarrow ds = (t^2+1)^2 dt \Rightarrow s = \int (t^2+1)^2 dt$

$\therefore s = \int (t^4 + 2t^2 + 1) dt \Rightarrow s = \frac{t^5}{5} + \frac{2}{3} t^3 + t + c$ —— (1)

at $t = 0$, eq.(1) becomes $S_0 = \frac{0}{5} + \frac{2}{3} \times 0 + 0 + c \Rightarrow \therefore c = S_0$

$\therefore S = \frac{1}{5} t^5 + \frac{2}{3} t^3 + t + S_0$

d) $v = \sqrt{2gs}$, (g = constant) $\Rightarrow \therefore ds = \sqrt{2g}\, s^{1/2} dt$

$\Rightarrow \int s^{-1/2} ds = \int \sqrt{2g}\, dt \Rightarrow \therefore \frac{s^{1/2}}{1/2} + c_1 = \sqrt{2g}\, t + c_2$

$\Rightarrow 2 s^{1/2} = \sqrt{2g}\, t + (c_2 - c_1) \Rightarrow s^{1/2} = \sqrt{\frac{g}{2}}\, t + \frac{c_2 - c_1}{2}$

squaring both sides, we get

$S = \frac{g}{2} t^2 + 2\left(\frac{c_2 - c_1}{2}\right) \sqrt{\frac{g}{2}}\, t + \left(\frac{c_2 - c_1}{2}\right)^2$

$\Rightarrow S = \frac{g}{2} t^2 + 2\sqrt{c}\sqrt{\frac{g}{2}}\, t + c$, where $c = \left(\frac{c_2 - c_1}{2}\right)^2$

$S = \frac{g}{2} t^2 + \sqrt{2gc}\, t + c$ —— (1)

at $t = 0$, $\therefore S_0 = \frac{g}{2}(0)^2 + \sqrt{2gc}\,(0) + c \Rightarrow \therefore c = S_0$

the eq.(1) becomes: $S = \frac{1}{2} g t^2 + \sqrt{2gS_0}\, t + S_0$

2- In each of the following, find the velocity v and position s as functions of t from the given acceleration $a = \frac{dv}{dt}$. Evaluate the constant of integration so as to have $v = v_0$ and $s = s_0$ when $t = 0$.

a) $a = \sqrt[3]{2t+1} \Rightarrow \frac{dv}{dt} = (2t+1)^{1/3} \Rightarrow \int dv = \frac{1}{2}\int (2t+1)^{1/3}\, 2dt$

$\Rightarrow v = \frac{1}{2}\frac{(2t+1)^{4/3}}{4/3} + C$ ——— (1)

at $t = 0$, eq. (1) becomes $v_0 = \frac{1}{2}\cdot\frac{3}{4} + C \Rightarrow \therefore C = v_0 - \frac{3}{8}$

$\therefore v = \frac{1}{2}\frac{(2t+1)^{4/3}}{4/3} + v_0 - \frac{3}{8} \Rightarrow v = \frac{3}{8}(2t+1)^{4/3} + v_0 - \frac{3}{8}$

Also we have

$v = \frac{ds}{dt} = \frac{3}{8}(2t+1)^{4/3} + v_0 - \frac{3}{8}$

$\therefore \int ds = \int \left[\frac{3}{8}(2t+1)^{4/3} + v_0 - \frac{3}{8}\right] dt$

$\Rightarrow \therefore S = \frac{3}{8}\cdot\frac{1}{2}\int (2t+1)^{4/3}\, 2dt + \left(v_0 - \frac{3}{8}\right)\int dt$

$S = \frac{3}{16}\frac{(2t+1)^{7/3}}{7/3} + \left(v_0 - \frac{3}{8}\right)t + C$

$S = \frac{9}{112}(2t+1)^{7/3} + \left(v_0 - \frac{3}{8}\right)t + C$ ——— (2)

at $t = 0$ eq. (2) will be: $S_0 = \frac{9}{112} + C \Rightarrow \therefore C = S_0 - \frac{9}{112}$

then eq. (2) will be

$S = \frac{9}{112}(2t+1)^{7/3} + \left(v_0 - \frac{3}{8}\right)t + S_0 - \frac{9}{112}$

b) $a = (2t+1)^{-3} \Rightarrow dv = (2t+1)^{-3}dt \Rightarrow v = \frac{1}{2}\int (2t+1)^{-3}\, 2dt$

$v = \frac{1}{2}\frac{(2t+1)^{-2}}{-2} + C$ ——— (1)

at $t = 0$, eq. (1) will be: $v_0 = \frac{1}{2} * \left(-\frac{1}{2}\right) + C \Rightarrow \therefore C = v_0 + \frac{1}{4}$

$\therefore v = -\frac{1}{4}(2t+1)^{-2} + v_0 + \frac{1}{4}$

Also $v = \frac{ds}{dt} \Rightarrow \therefore ds = \left[-\frac{1}{4}(2t+1)^{-2} + v_0 + \frac{1}{4}\right]dt$

$\therefore S = -\frac{1}{4}\cdot\frac{1}{2}\int (2t+1)^{-2}\, 2dt + \int\left(v_0 + \frac{1}{4}\right)dt$

$S = -\frac{1}{8}\frac{(2t+1)^{-1}}{-1} + \left(v_0 + \frac{1}{4}\right)t + C$ ——— (2)

at $t = 0$, eq (1) will be: $S_0 = \frac{1}{8} + C \Rightarrow \therefore C = S_0 - \frac{1}{8}$

$\therefore$ eq (2) will be: $S = \frac{1}{8}(2t+1)^{-1} + \left(v_0 + \frac{1}{4}\right)t + S_0 - \frac{1}{8}$

c) $a = (t^2+1)^2 \Rightarrow \frac{dv}{dt} = (t^2+1)^2 \Rightarrow \int dv = \int (t^4 + 2t^2 + 1)\,dt$

$\therefore v = \frac{1}{5}t^5 + \frac{2}{3}t^3 + t + c$ —(1)

at $t=0$, eq (1) will be $\therefore v_0 = c$

$\therefore v = \frac{1}{5}t^5 + \frac{2}{3}t^3 + t + v_0$, then we can find S.

3. Solve the following differential equations subject to the prescribed initial conditions.

a) $\frac{dy}{dx} = x\sqrt{y}$, $x=0$, $y=1$

$\Rightarrow \frac{dy}{\sqrt{y}} = x\,dx \Rightarrow \int y^{-1/2}dy = \int x\,dx \Rightarrow \frac{y^{1/2}}{1/2} + c_1 = \frac{x^2}{2} + c_2$

$\Rightarrow 2y^{1/2} = \frac{1}{2}x^2 + (c_2 - c_1) \Rightarrow y^{1/2} = \frac{1}{4}x^2 + \frac{1}{2}(c_2 - c_1)$

$\Rightarrow y^{1/2} = \frac{1}{4}x^2 + c$ —(1) where $c = \frac{1}{2}(c_2 - c_1)$

at $x=0$ and $y=1$, eq.(1) will be $(1)^{1/2} = \frac{1}{4}(0)^2 + c$

$\therefore c = 1$. eq.(1) becomes: $[y^{1/2} = 1/4\,x^2 + 1] * 4$

$\Rightarrow 4\sqrt{y} = x^2 + 4$

b) $\frac{dy}{dx} = 2xy^2$, $x=1$, $y=1 \Rightarrow \frac{dy}{y^2} = 2x\,dx \Rightarrow \int y^{-2}dy = \int 2x\,dx$

$\Rightarrow \frac{y^{-1}}{-1} + c_1 = 2\frac{x^2}{2} + c_2 \Rightarrow -y^{-1} = x^2 + \underbrace{(c_2 - c_1)}_{=c}$ —(1)

at $x=1$, $y=1$, eq.(1) will be $-(1)^{-1} = (1)^2 + C \Rightarrow \therefore c = -2$

$\therefore$ eq (1) will be: $-y^{-1} = x^2 - 2$

c) $\frac{dy}{dx} = x\sqrt{1+x^2}$; $x=0$, $y=-3$ $\Rightarrow \int dy = \frac{1}{2}\int (1+x^2)^{1/2}\,2x\,dx \Rightarrow \therefore y = \frac{1}{2}\frac{(1+x^2)^{3/2}}{3/2} + c$ —(1)

at $x=0$ and $y=-3$, eq (1) will be $-3 = \frac{1}{2}\frac{(1+(0)^2)^{3/2}}{3/2} + c \Rightarrow \therefore c = -3 - \frac{1}{3} = -\frac{10}{3}$

$\therefore$ eq.(1) becomes: $y = \frac{1}{3}(1+x^2)^{3/2} - \frac{10}{3} \Rightarrow 3y = (1+x^2)^{3/2} - 10$

$\Rightarrow 3y + 10 = (1+x^2)^{3/2} \Rightarrow (3y+10)^2 = (1+x^2)^3$

d) $\frac{dy}{dx} = \frac{4\sqrt{(1+y^2)^3}}{y}$; $x=0$, $y=1$ $\Rightarrow \int \frac{y\,dy}{\sqrt{(1+y^2)^3}} = \int 4\,dx \Rightarrow \frac{1}{2}\int (1+y^2)^{-3/2}\,2y\,dy = \int 4\,dx$

$\Rightarrow \frac{1}{2}\frac{(1+y^2)^{-1/2}}{-1/2} + c_1 = 4x + c_2 \Rightarrow -(1+y^2)^{-1/2} = 4x + (c_2 - c_1)$

where $c = -(c_2 - c_1)$

$\Rightarrow (1+y^2)^{-1/2} = c - 4x$ —(1)

at $x=0$ and $y=1$, eq (1) becomes $(1+(1)^2)^{-1/2} = c - 4(0) \Rightarrow \therefore c = \frac{1}{\sqrt{2}}$

$\therefore$ eq.(1) will be: $(1+y^2)^{-1/2} = \frac{1}{\sqrt{2}} - 4x$

Differentiation and integration of Sines and Cosines

The differentiation of Sines and Cosines are

1- $\frac{d}{dx}(\sin u) = \cos u \cdot \frac{du}{dx}$

2- $\frac{d}{dx}(\cos u) = -\sin u \cdot \frac{du}{dx}$

examples

Find $\frac{dy}{dx}$ for the following:

1. $y = \cos x^2 \Rightarrow \frac{dy}{dx} = -\sin x^2 \cdot (2x) = -2x \sin x^2$

2. $y = \sin^2(3x) \Rightarrow \frac{dy}{dx} = 2\sin(3x) \cdot [\cos(3x) \cdot (3)]$
$= 6 \sin(3x) \cos(3x)$

3. $y = \sec^2 5x \Rightarrow y = (\cos 5x)^{-2} \Rightarrow \frac{dy}{dx} = -2(\cos 5x)^{-3} \cdot [-\sin 5x \cdot 5]$
$\therefore \frac{dy}{dx} = 10 (\cos 5x)^{-3} \sin 5x = 10 \sec^3 5x \sin 5x$

4. $y = \tan x \Rightarrow y = \frac{\sin x}{\cos x} \Rightarrow \therefore \frac{dy}{dx} = \frac{\cos x \cdot \cos x - \sin x \cdot -\sin x}{\cos^2 x}$
$\Rightarrow \therefore \frac{dy}{dx} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

The integration formulas	The differentiation formulas
1. $\int \cos u \, du = \sin u + c$	1. $d(\sin u) = \cos u \, du$
2. $\int \sin u \, du = -\cos u + c$	2. $d(\cos u) = -\sin u \, du$

examples

Evaluate the following:

1. $\int \cos 2t \, dt = \frac{1}{2}\int \cos 2t \cdot 2dt = \frac{1}{2}\sin 2t + C$

2. $\int \frac{\cos 2x}{\sin^3 2x} dx = \int \sin^{-3} 2x \cos 2x \, dx = \int (\sin 2x)^{-3} \cos 2x \, dx$
$= \frac{1}{2}\int (\sin 2x)^{-3} \underbrace{\cos 2x \cdot 2dx}_{\text{مشتقة داخل القوس}} = \frac{1}{2} \frac{(\sin 2x)^{-2}}{-2} + C$
$= \frac{-1}{4 \sin^2 2x} + C$

Solved Problems

Find $\frac{dy}{dx}$ of the following:

1. $y = \sin(3x+4) \Rightarrow \frac{dy}{dx} = \cos(3x+4)\cdot 3 = 3\cos(3x+4)$

2. $y = x\sin x \Rightarrow \frac{dy}{dx} = x\cdot\cos x + \sin x\cdot(1) = x\cos x + \sin x$

3. $y = \frac{\sin x}{x} \Rightarrow y = x^{-1}\sin x \Rightarrow \frac{dy}{dx} = x^{-1}\cos x + \sin x\cdot(-x^{-2})$

   $\therefore \frac{dy}{dx} = \frac{\cos x}{x} - \frac{\sin x}{x^2} \Rightarrow \frac{dy}{dx} = \frac{x\cos x - \sin x}{x^2}$

4. $y = \cos 5x \Rightarrow \therefore \frac{dy}{dx} = -\sin 5x\cdot(5) = -5\sin 5x$

5. $y = x^2\sin 3x \Rightarrow \frac{dy}{dx} = x^2\cdot\cos 3x\cdot(3) + \sin 3x\cdot 2x$

   $= 3x^2\cos 3x + 2x\sin 3x$

6. $y = \sqrt{2+\cos 2x} \Rightarrow y = (2+\cos 2x)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(2+\cos 2x)^{-\frac{1}{2}}\cdot -\sin 2x\cdot(2)$

   $\therefore \frac{dy}{dx} = -\sin 2x\cdot(2+\cos 2x)^{-\frac{1}{2}}$

7. $y = \sin^2 x + \cos^2 x \Rightarrow y = 1 \Rightarrow \therefore \frac{dy}{dx} = 0$

   OR $\frac{dy}{dx} = 2\sin x\cdot(\cos x) + 2\cos x\cdot(-\sin x) = 0$

8. $y = \frac{2}{\cos 3x} \Rightarrow y = 2\cos^{-1}3x \Rightarrow \frac{dy}{dx} = -2\cos^{-2}3x\cdot -\sin 3x\cdot(3)$

   $\Rightarrow \therefore \frac{dy}{dx} = \frac{6\sin 3x}{\cos^2 3x}$

9. $y = 3\sin 2x - 4\cos 2x \Rightarrow \frac{dy}{dx} = 3\cos 2x\cdot(2) - 4(-\sin 2x\cdot(2))$

   $\therefore \frac{dy}{dx} = 6\cos 2x + 8\sin 2x$

10. $y = 3\cos^2 2x - 3\sin^2 2x$

    $\Rightarrow \frac{dy}{dx} = 3\cdot 2\cos 2x\cdot[-\sin 2x\cdot(2)] - 3\cdot 2\sin 2x\cdot[\cos 2x\cdot(2)]$

    $= -12\cos 2x\sin 2x - 12\sin 2x\cos 2x$

    $= -24\cos 2x\sin 2x$

11. $y = 2\sin x\cos x \Rightarrow \frac{dy}{dx} = 2[\sin x\cdot(-\sin x) + \cos x\cdot(\cos x)]$

    $= 2(\cos^2 x - \sin^2 x)$

12. $y = \frac{1}{\sin x} \Rightarrow y = \sin^{-1}x \Rightarrow \frac{dy}{dx} = -\sin^{-2}x\cdot(\cos x) = \frac{-\cos x}{\sin^2 x}$

13. $y = \cos^2 3x \Rightarrow \frac{dy}{dx} = 2\cos 3x\cdot[-\sin 3x\cdot(3)] = -6\cos 3x\sin 3x$

14. $y = \cot x \Rightarrow y = \frac{\cos x}{\sin x} \Rightarrow \frac{dy}{dx} = \frac{\sin x\cdot(-\sin x) - \cos x\cdot(\cos x)}{\sin^2 x}$

    $= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x}$

15. $x\sin 2y = y\cos 2x$ (Ans: $\frac{\sin 2y + 2y\sin 2x}{\cos 2x - 2x\cos 2y}$)

16. $y^2 = \sin^4 2x + \cos^4 2x$ (Ans: $-\frac{\sin 8x}{y}$)

Evaluate the following integrals: (8)

1. $\int \sin 3x\, dx = \frac{1}{3}\int \sin 3x\, 3dx = -\frac{1}{3}\cos 3x + C$
2. $\int \cos(2x+4)\, dx = \frac{1}{2}\int \cos(2x+4)\, 2dx = \frac{1}{2}\sin(2x+4) + C$
3. $\int x\sin(2x^2)\, dx = \frac{1}{4}\int \sin(2x^2)\, 4x\, dx = -\frac{1}{4}\cos(2x^2) + C$
4. $\int \cos\sqrt{x}\,\frac{dx}{\sqrt{x}} = \int \cos x^{1/2}\cdot x^{-1/2}dx = 2\int \cos x^{1/2}\cdot\frac{1}{2}x^{-1/2}dx = 2\sin x^{1/2} + C$
5. $\int \sin 2t\, dt = \frac{1}{2}\int \sin 2t\, 2dt = -\frac{1}{2}\cos 2t + C$
6. $\int \cos(3\theta-1)\, d\theta = \frac{1}{3}\int \cos(3\theta-1)\cdot 3d\theta = \frac{1}{3}\sin(3\theta-1) + C$
7. $\int 4\cos 3y\, dy = \frac{4}{3}\int \cos 3y\cdot 3dy = \frac{4}{3}\sin 3y + C$
8. $\int 2\sin z\cos z\, dz = \int 2\sin z\cdot(\cos z\, dz) = \sin^2 z + C$
9. $\int \sin^2 x\cos x\, dx = \int \sin^2 x\cdot(\cos x\, dx) = \frac{\sin^3 x}{3} + C$
10. $\int \cos^2 2y\sin 2y\, dy = \frac{1}{-2}\int \cos^2 2y\,(-\sin 2y\cdot 2dy) = -\frac{1}{6}\cos^3 2y + C$
11. $\int (1-\sin^2 3t)\cos 3t\, dt = \int \cos 3t\, dt - \int \sin^2 3t\cos 3t\, dt$

$= \frac{1}{3}\int \cos 3t\, 3dt - \frac{1}{3}\int \sin^2 3t\,(\cos 3t\cdot 3dt)$

$= \frac{1}{3}\sin 3t - \frac{1}{9}\sin^3 3t + C$

12. $\int \frac{\sin x\, dx}{\cos^2 x} = \int \cos^{-2}x\sin x\, dx = \int -\cos^{-2}x\,(-\sin x\, dx) = \cos^{-1}x + C$
13. $\int \frac{\cos x\, dx}{\sin^2 x} = \int \sin^{-2}x\cos x\, dx = \int \sin^{-2}x\,(\cos x\, dx) = -\sin^{-1}x + C$
14. $\int \sqrt{2+\sin 3t}\,\cos 3t\, dt = \frac{1}{3}\int (2+\sin 3t)^{1/2}\cos 3t\, 3dt = \frac{1}{3}\frac{(2+\sin 3t)^{3/2}}{3/2} + C =$

$= \frac{2}{9}(2+\sin 3t)^{3/2} + C$

15. $\int \frac{\sin 2t\, dt}{\sqrt{2-\cos 2t}} = \int \frac{1}{2}(2-\cos 2t)^{-1/2}\cdot(\sin 2t\cdot 2dt) = (2-\cos 2t)^{1/2} + C$
16. $\int \sin^3\frac{y}{2}\cos\frac{y}{2}\, dy = 2\int \sin^3\frac{y}{2}\left(\cos\frac{y}{2}\cdot\frac{dy}{2}\right) = \frac{1}{2}\sin^4\frac{y}{2} + C$
17. $\int \frac{\sin\frac{z-1}{3}\, dz}{\cos^2\frac{z-1}{3}} = -3\int \cos^{-2}\frac{z-1}{3}\left(-\sin\frac{z-1}{3}\cdot\frac{dz}{3}\right) = 3\cos^{-1}\frac{z-1}{3} + C$
18. $\int \cos^2\frac{2x}{3}\sin\frac{2x}{3}\, dx = -\frac{3}{2}\int \cos^2\frac{2x}{3}\left(-\sin\frac{2x}{3}\cdot\frac{2}{3}dx\right) = -\frac{1}{2}\cos^3\frac{2x}{3} + C$
19. $\int (1+\sin 2t)^{3/2}\cos 2t\, dt = \frac{1}{2}\int (1+\sin 2t)^{3/2}\cdot\cos 2t\cdot 2dt = \frac{1}{5}(1+\sin 2t)^{5/2} + C$
20. $\int (3\sin 2x + 4\cos 3x)\, dx = \int 3\sin 2x\, dx + \int 4\cos 3x\, dx$

$= \frac{3}{2}\int \sin 2x\cdot 2dx + \frac{4}{3}\int \cos 3x\cdot 3dx = -\frac{3}{2}\cos 2x + \frac{4}{3}\sin 3x + C$

21. $\int \sin t\cos t(\sin t + \cos t)\, dt = \int \sin^2 t\cos t\, dt + \cos^2 t\sin t\, dt$

$= \frac{1}{3}\int 3\sin^2 t(\cos t\, dt) + \frac{1}{3}\int 3\cos^2 t\,(-\sin t\, dt)$

$= \frac{1}{3}\sin^3 t - \frac{1}{3}\cos^3 t + C$

$= \frac{1}{3}(\sin^3 t - \cos^3 t) + C$

9

1) **The Definite Integration:**

Definition "Area under a curve":

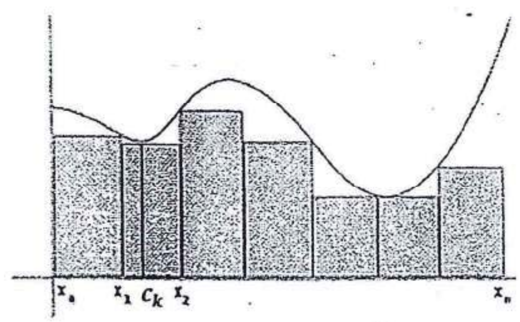
The area under the graph of a nonnegative continuous function f over an interval $[a, b]$ is the limit of the sums of the areas of inscribed rectangles of equal base length as their number n increases without bound. In symbols,

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_K)\, \Delta x$$

Where $f(c_K)$ is the smallest value of f on the interval $[x_{k-1}, x_k]$.

Definition" Definite Integral ":

Let f be a function that is defined on an interval $[a, b]$. The definite integral of f from a to b, denoted by $\int_a^b f(x)dx$ is



$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_K)\, \Delta x$$

Provided the limit exists.

Example: Prove $\int_{-2}^{3} 7dx = 35$ by using the definition.

Sol: let $f(x) = 7$ defined on $[-2, 3]$. Subdividing the interval $[-2, 3]$ into " n" equal parts .

$$\therefore \Delta x = \frac{3-(-2)}{n} = \frac{5}{n}$$

Now,

$$\int_{-2}^{3} 7\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_K)\,\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n} 7\left(\frac{5}{n}\right)$$

$$= 35\left(\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{n}\right)\right) = 35\lim_{n\to\infty}\left(\frac{n}{n}\right) = 35.$$

Properties of Integral:

Let f, g be integrable functions on $[a,b]$ and let c be a constant, then:

1) $\int_a^a f(x)dx = 0.$
2) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$
3) $\int_a^b c\,dx = c(b-a).$
4) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$
5) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$
6) $\int_a^b f(x)\,dx = \int_a^d f(x)dx + \int_d^b f(x)dx;\quad a \le d \le b.$
7) If $f(x) \ge 0\quad \forall x \in [a,b]$, then $\int_a^b f(x)\,dx \ge 0.$
8) If $f(x) \ge g(x)\quad \forall x \in [a,b]$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx.$

Example:

1) $\int_1^1 (x^2+3)\,dx = 0.$
2) $\int_{-5}^1 8\,dx = 8(1+5) = 48.$

Definition "Antiderivatives":

A function F is an antiderivative of f on an interval I, if $F'(x) = f(x)$ for all x in I