FUZZY SET THEORY

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Chapter 1

The Crisp Sets

1.1 The Crisp Sets

Definition 1.1.1. Let $\mathcal{X}$ be a nonempty set, called the **universe set**, consisting of all the possible elements of concern in a particular context. Each of these elements is called a **member**, or an **element**, of $\mathcal{X}$.

$x \in \mathcal{X}$ means x is an element of $\mathcal{X}$.

$x \notin \mathcal{X}$ means x is not an element of $\mathcal{X}$.

Definition 1.1.2. A union of several (finite or infinite) members of $\mathcal{X}$ is called a **subset of** $\mathcal{X}$, which is denoted by $\mathcal{A} \subset \mathcal{X}$.

There are two cases of subset:

1. **Proper subset** ($\mathcal{A} \subset \mathcal{X}$) means $\exists x \in \mathcal{X}$ but $x \notin \mathcal{A}$
2. **Subset** ($\mathcal{A} \subseteq \mathcal{X}$) means $\mathcal{A} \subset \mathcal{X}$ or $\mathcal{X} \subset \mathcal{A}$

Remark 1.1.1. Two sets $\mathcal{A}$ and $\mathcal{B}$ is **equal** if $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A}$. Thus Subset ($\mathcal{A} \subseteq \mathcal{X}$) means $\mathcal{A} \subset \mathcal{X}$ or $\mathcal{A} = \mathcal{X}$

Definition 1.1.3. The empty set is denoted by $\emptyset$.

Example 1.1.1. Let $\mathbb{R}^2$ be the universe set and $\mathbb{R}$ is a subset of $\mathbb{R}^2$ ($\mathbb{R} \subset \mathbb{R}^2$).

Set - Theoretic Opeations

Let $\mathcal{A}, \mathcal{B}$ be two subsets of the universe $\mathcal{X}$

1. **The Difference of two subsets:**

$$\mathcal{A} - \mathcal{B} := \{x \in \mathcal{X} | x \in \mathcal{A} \text{ but } x \notin \mathcal{B}\}$$

2. **The Complement of a subset:**

$$\mathcal{A}^c := \mathcal{X} - \mathcal{A} := \{x \in \mathcal{X} | x \notin \mathcal{A}\}$$

Remarks 1.1.1.

(a) $(\mathcal{A}^c)^c = \mathcal{A}$

(b) $\mathcal{X}^c = \emptyset$

(c) $\emptyset^c = \mathcal{X}$

3. **The Union of two subsets:**

$$\mathcal{A} \cup \mathcal{B} := \{x | x \in \ A \text{ or } x \in \mathcal{B}\} = \mathcal{B} \cup \mathcal{A}.$$

Remarks 1.1.2.

(a) $\mathcal{A} \cup \mathcal{X} = \mathcal{X}$.

(b) $\mathcal{A} \cup \emptyset = \mathcal{A}$.

(c) $\mathcal{A} \cup \mathcal{A}^c = \mathcal{X}$.

4. **The Intersection of two subsets:**

$$\mathcal{A} \cap \mathcal{B} := \{x | x \in \ A \text{ and } x \in \mathcal{B}\} = \mathcal{B} \cap \mathcal{A}.$$

Remarks 1.1.3.

(a) $\mathcal{A} \cap \mathcal{X} = \mathcal{A}$.

(b) $\mathcal{A} \cap \emptyset = \emptyset$.

(c) $\mathcal{A} \cap \mathcal{A}^c = \emptyset$.

Definition 1.1.4. Two subsets $\mathcal{A}$ and $\mathcal{B}$ are said to be **disjoint** if $\mathcal{A} \cap \mathcal{B} = \emptyset$.

Definition 1.1.5. Let $\mathcal{A}$ be a set, **a partion of** $\mathcal{A}$ which is denoted by $\pi(\mathcal{A})$ is

$$\pi(\mathcal{A}) := \{\mathcal{A}_i | i \in I; \mathcal{A}_i \subseteq \mathcal{A}\}$$

satisfied

(a) $\mathcal{A}_i \neq \emptyset$ for all $i \in I$.

(b) $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for all $i \neq j$.

(c) $\cup \mathcal{A}_i = \mathcal{A}$.

If condition (b) does not satisfy, then $\pi(\mathcal{A})$ becomes a **cover or covering of the set** $\mathcal{A}$.

5. **The Multiplication of a real number** r **and a subset** $\mathcal{A}$ **of** $\mathbb{R}$**:**

$$r\mathcal{A} := \{ra | a \in \mathcal{A}\}.$$

Properties of Classical Set Operations

Involutive law

$$(\mathcal{A}^c)^c = A$$

Commutative law

$$\begin{aligned} A \cup B &= B \cup A \\ A \cap B &= B \cap A \end{aligned}$$

Associative law

$$\begin{aligned} (A \cup B) \cup C &= A \cup (B \cup C) \\ (A \cap B) \cap C &= A \cap (B \cap C) \end{aligned}$$

Distributive law

$$
\begin{aligned}
A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\
A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\
A \cup A &= A \\
A \cap A &= A \\
A \cup (A \cap B) &= A \\
A \cap (A \cup B) &= A \\
A \cup (\mathcal{A}^c \cap B) &= A \cup B \\
A \cap (\mathcal{A}^c \cup B) &= A \cap B \\
A \cup \mathcal{X} &= \mathcal{X} \\
A \cap \emptyset &= \emptyset \\
A \cup \emptyset &= A \\
A \cap \mathcal{X} &= A \\
\mathcal{A}^c \cap A &= \emptyset \\
\mathcal{A}^c \cup A &= \mathcal{X}
\end{aligned}
$$

DeMorgans law

$$
\begin{aligned}
(A \cap B)^c &= \mathcal{A}^c \cup \mathcal{B}^c \\
(A \cup B)^c &= \mathcal{A}^c \cap \mathcal{B}^c
\end{aligned}
$$

Definition 1.1.6. The number of elements in a set $\mathcal{A}$ is denoted by the **cardinality** $|\mathcal{A}|$.

Definition 1.1.7. A **power set** $P(\mathcal{A})$ is a family set containing the subsets of set $\mathcal{A}$. Therefore the number of elements in the power set $P(\mathcal{A})$ is represented by

$$|P(\mathcal{A})| = 2^{|\mathcal{A}|}.$$

Example 1.1.2. If $\mathcal{A} = \{a, b, c\}$, then

$$|\mathcal{A}| = 3$$

$$P(\mathcal{A}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

$$|P(\mathcal{A})| = 2^3 = 8.$$

Definition 1.1.8. A subset $\mathcal{A} \subseteq \mathbb{R}^n$ that is said to be **convex** if for each $x, y \in \mathcal{A}$,

$$\lambda x + (1 - \lambda)y \in \mathcal{A}, \text{ for each } \lambda \in [0, 1]$$

i.e. every point on the line connected between two points $x, y \in \mathcal{A}$ is also in $\mathcal{A}$.

Definition 1.1.9. Let $\mathcal{X}$ be the universe. **Membership** in a crisp subset $\mathcal{A}$ of $\mathcal{X}$ is often viewed as a characteristic function,

$$\mu_{\mathcal{A}} : \mathcal{X} \longrightarrow \{0, 1\}$$

defined as

$$\mu_{\mathcal{A}}(x) = \begin{cases} 1, & x \in \mathcal{A}; \\ 0, & x \notin \mathcal{A}. \end{cases}$$

Remark 1.1.2. By using the membership function, the union and the intersection of two sets A, B will be,

For all $x \in \mathcal{X}$

$$\begin{aligned} \mu_{A \cup B}(x) &= \max\{\mu_A(x), \mu_B(x)\} \\ \mu_{A \cap B}(x) &= \min\{\mu_A(x), \mu_B(x)\} \end{aligned}$$

Chapter 2

FUZZY SET THEORY

2.1 Definitions

Definition 2.1.1. Let $\mathcal{X}$ be the universal set, **The fuzzy set** $\mathcal{A}$ in $\mathcal{X}$ is a set of ordered pairs;

$$\mathcal{A} := \{(x, \mu_{\mathcal{A}}(x)) : x \in \mathcal{X}\}$$

where,

$$\mu_{\mathcal{A}} : \mathcal{X} \longrightarrow [0, 1]$$

is called **the membership function**,

and each $x \in \mathcal{X}$, the value of $\mu_{\mathcal{A}}(x)$ is called **the grade of membership of** x **in** $\mathcal{A}$.

Figure 2.1: The fuzzy set

Notation 2.1.1.

1. When $\mathcal{X}$ is a finite set $\{x_1, x_2, \cdots, x_n\}$, a fuzzy set $\mathcal{A}$ on $\mathcal{X}$ is expressed as

$$\mathcal{A} = \mu_{\mathcal{A}}(x_1)/x_1 + \mu_{\mathcal{A}}(x_2)/x_2 + \cdots + \mu_{\mathcal{A}}(x_n)/x_n = \sum_{i=1}^{n} \mu_{\mathcal{A}}(x_i)/x_i$$

where the term $\mu_{\mathcal{A}}(x_i)$, $i = 1, \cdots, n$ signifies that μ_i is the grade of membership of x_i in $\mathcal{A}$ and the plus sign represents the union.

2. When $\mathcal{X}$ is not finite, we write,

$$\mathcal{A} = \int_{\mathcal{X}} \mu_{\mathcal{A}}(x)/x$$

Definition 2.1.2. Let $x \in \mathcal{X}$, then x is called

Not include in the fuzzy set if $\mu_{\mathcal{A}}(x) = 0$.

Partial include if $0 < \mu_{\mathcal{A}}(x) < 1$.

Full include if $\mu_{\mathcal{A}}(x) = 1$.

Definition 2.1.3. A fuzzy set is **empty** if and only if its membership function is zero on $\mathcal{X}$.

Definition 2.1.4. Two fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ are **equal**, written as $\mathcal{A} = \mathcal{B}$, if and only if $\mu_{\mathcal{A}}(x) = \mu_{\mathcal{B}}(x)$ for all x in $\mathcal{X}$.

Example 2.1.1.

1. A realtor wants to classify the house he offers to his clients. One indicator of comfort of these houses is the number of bedrooms in it. Let $\mathcal{X} = \{1, 2, 3, 4, ..., 10\}$ be the set of available types of houses described by $x =$ number of bedrooms in a house. Then the fuzzy set "comfortable type of house for a four-person family" may be described as

$$\mathcal{A} = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 1), (5, 0.7), (6, 0.3)\}$$

   or

$$\mathcal{A} = 0.2/1 + 0.5/2 + 0.8/3 + 1/4 + 0.7/5 + 0.3/6$$

2. the universe set $\mathcal{X}$ is the set of people. $\mathcal{B}$ fuzzy subset YOUNG is also defined, which answers the question "to what degree is person x young?" To each person in the universe set, we have to assign a degree of membership in the fuzzy subset YOUNG. The easiest way to do this is with a membership function based on the person's age.

$$\mu_{\mathcal{B}}(x) = \begin{cases} 1, & age(x) \leq 20 \\ (30 - age(x))/10, & 20 \leq age(x) \leq 30 \\ 0, & age(x) > 30 \end{cases}$$

Thus $\mathcal{B} = \int_{\mathcal{B}} \mu_{\mathcal{B}}(x)/x$

3. $\mathcal{A} =$ "real numbers close to 10"

$$\mathcal{A} = \left\{(x, \mu_{\mathcal{A}}(x)) : \mu_{\mathcal{A}}(x) = (1 + (x - 10^2))^{-1}\right\}$$

Thus

$$\mathcal{A} = \int_{\mathcal{A}} (1 + (x - 10)^2)^{-1}/x$$

2.2 Expanding Concepts of Fuzzy Set

Definition 2.2.1. The support of a fuzzy set $\mathcal{A}$, $supp(\mathcal{A})$, is the crisp set of all $x \in \mathcal{X}$ such that $\mu_{\mathcal{A}}(x) > 0$. i.e.

$$supp(\mathcal{A}) := \{x \in \mathcal{X} : \mu_{\mathcal{A}}(x) > 0\}.$$

Example 2.2.1. Let $\mathcal{X} := \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be the universe set, and $\mathcal{A} = 0.2/1 + 0.5/2 + 0.8/3 + 1/4 + 0.7/5 + 0.3/6$, then the support of $supp(\mathcal{A}) = \{1, 2, 3, 4, 5, 6\}$.

The elements $\{7, 8, 9, 10\}$ are not part of the support of $\mathcal{A}$.

Definition 2.2.2. A fuzzy subset $\mathcal{A}$ of the universal set $\mathcal{X}$ is called **normal** if there exists an $x \in \mathcal{X}$ such that $\mu_{\mathcal{A}}(x) = 1$. Otherwise $\mathcal{A}$ is **subnormal**.

Example 2.2.2. Let $\mathcal{A} = 0.2/1 + 0.5/2 + 0.8/3 + 1/4 + 0.7/5 + 0.3/6$. Since $\mu_{\mathcal{A}}(4) = 1$, then this fuzzy set is normal.

While the fuzzy set $\mathcal{A} = 0.2/1 + 0.5/2 + 0.8/3 + 0.7/5 + 0.3/6$ is subnormal.

Definition 2.2.3. The maximum value of the membership is called **height**

Example 2.2.3. Let $\mathcal{A} = 0.2/1 + 0.5/2 + 0.8/3 + 0.3/6$. the height of this fuzzy set is 0.8.

Definition 2.2.4. The (crisp) set of elements that belong to the fuzzy set $\mathcal{A}$ at least to the degree α is called **the α-cut**:

$$\mathcal{A}_\alpha := \{x \in \mathcal{X} : \mu_{\mathcal{A}}(x) \geq \alpha\}$$

and

$$\tilde{\mathcal{A}}_\alpha := \{x \in \mathcal{X} : \mu_{\mathcal{A}}(x) > \alpha\}$$

is called **strong α-cut**.

Figure 2.2: Examples of α-cuts

Example 2.2.4. Let $\mathcal{A} = 0.2/1 + 0.5/2 + 0.8/3 + 1/4 + 0.7/5 + 0.3/6$, then list possible α-cut sets:

$$\begin{aligned} \mathcal{A}_{0.2} &= \{1, 2, 3, 4, 5, 6\} \\ \mathcal{A}_{0.5} &= \{2, 3, 4, 5\} \\ \mathcal{A}_{0.8} &= \{3, 4\} \\ \mathcal{A}_{1} &= \{4\} \end{aligned}$$

Definition 2.2.5. The value α which explicitly shows the value of the membership function, is in the range of $[0, 1]$. The **level set** is obtained by the α's. That is, $\Lambda_{\mathcal{A}} := \{\alpha : \mu_{\mathcal{A}}(x) = \alpha, \alpha \geq 0, x \in \mathcal{X}\}$

Example 2.2.5. Let $\mathcal{A} = 0.2/1 + 0.5/2 + 0.8/3 + 1/4 + 0.7/5 + 0.3/6$, then the level set is

$$\Lambda_{\mathcal{A}} := \{0.2, 0.3, 0.5, 0.7, 0.8, 1\}$$

Homework 2.2.1. Consider a universal set $\mathcal{X}$ which is defined on the age domain. $\mathcal{X} := \{5, 15, 25, 35, 45, 55, 65, 75, 85\}$

age(element)	infant	young	adult	senior
5	0	0	0	0
15	0	0.2	0.1	0
25	0	1	0.9	0
35	0	0.8	1	0
45	0	0.4	1	0.1
55	0	0.1	1	0.2
65	0	0	1	0.6
75	0	0	1	1
85	0	0	1	1

Answer the following

1. Find fuzzy sets such as infant, young, adult and senior in $\mathcal{X}$
2. Find the support set of each fuzzy set.
3. Find the α-cut set is derived from fuzzy set young.
4. Is the fuzzy set "adult" is normal or subnormal?
5. What is the hight of fuzzy set "senior"?

2.3 Basic Set-Theoretic Operations for Fuzzy Sets

Definition 2.3.1. Let $\mathcal{X}$ be the universe set, a fuzzy set $\mathcal{A}$ is a **subset** of a fuzzy set $\mathcal{B}$, if and only if $\mu_{\mathcal{A}}(x) \leq \mu_{\mathcal{B}}(x)$ for all $x \in \mathcal{X}$, which is denoted by $\mathcal{A} \subseteq \mathcal{B}$.

Definition 2.3.2. Let $\mathcal{X}$ be the universe set, a fuzzy set $\mathcal{A}$ is a **proper subset** of a fuzzy set $\mathcal{B}$, if and only if $\mu_{\mathcal{A}}(x) < \mu_{\mathcal{B}}(x)$ for all $x \in \mathcal{X}$, which is denoted by $\mathcal{A} \subset \mathcal{B}$.

Figure 2.3: fuzzy subset

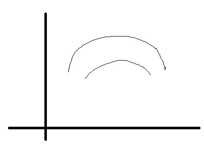

Figure 2.4: Proper fuzzy subset

Remarks 2.3.1.

1. Every fuzzy subset is included in itself.

2. Empty fuzzy subset is included in every fuzzy subset.

Definition 2.3.3. The complement of a fuzzy set $\mathcal{A}$ is denoted by $\widetilde{\mathcal{A}}$ and is defined by

$$\mu_{\widetilde{\mathcal{A}}}(x) = 1 - \mu_{\mathcal{A}}(x)$$

Figure 2.5: The complement of a fuzzy subset

Example 2.3.1. Let $\mathcal{A} = 0/3 + 0.4/7 + 1/8$, then

$$\begin{aligned} \mu_{\widetilde{\mathcal{A}}}(3) &= 1 - \mu_{\mathcal{A}}(3) = 1 - 0 = 1 \\ \mu_{\widetilde{\mathcal{A}}}(7) &= 1 - \mu_{\mathcal{A}}(7) = 1 - 0.4 = 0.6 \\ \mu_{\widetilde{\mathcal{A}}}(8) &= 1 - \mu_{\mathcal{A}}(8) = 1 - 1 = 0 \end{aligned}$$

thus, $\widetilde{\mathcal{A}} = 1/3 + 0.6/7 + 0/8$

Definition 2.3.4. The union of two fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ with respective membership functions $\mu_{\mathcal{A}}(x)$ and $\mu_{\mathcal{B}}(x)$ is a fuzzy set $\mathcal{C}$, written as $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$, whose membership function is related to those of $\mathcal{A}$ and $\mathcal{B}$ by

$$\mu_{\mathcal{C}}(x) := \max\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)\}$$

Figure 2.6: The union of a fuzzy subset

Example 2.3.2. Let $\mathcal{A} := \{(5, 1), (15, 0.9), (25, 0.1)\}$ and $\mathcal{B} := \{(5, 0.1), (10, 0.7), (25, 0.8)\}$, then

$$\begin{aligned}
\mu_{\mathcal{C}}(5) &= \max\{\mu_{\mathcal{A}}(5), \mu_{\mathcal{B}}(5)\} = \max\{1, 0.1\} = 1 \\
\mu_{\mathcal{C}}(10) &= \max\{\mu_{\mathcal{A}}(10), \mu_{\mathcal{B}}(10)\} = \max\{0, 0.7\} = 0.7 \\
\mu_{\mathcal{C}}(15) &= \max\{\mu_{\mathcal{A}}(15), \mu_{\mathcal{B}}(15)\} = \max\{0.9, 0\} = 0.9 \\
\mu_{\mathcal{C}}(25) &= \max\{\mu_{\mathcal{A}}(25), \mu_{\mathcal{B}}(25)\} = \max\{0.1, 0.8\} = 0.8
\end{aligned}$$

Thus, $\mathcal{C} = \{(5, 1), (10, 0.7), (15, 0.9), (25, 0.8)\}$

Proposition 2.3.1. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be fuzzy sets, then*

$$\mathcal{A} \cup (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C}$$

Proof. Without lose of the generality, we can assume that

$$\mu_{\mathcal{A}}(x) \leq \mu_{\mathcal{B}}(x) \leq \mu_{\mathcal{C}}(x) \text{ for all } x \in \mathcal{X}$$

then

$$\begin{aligned}
\mu_{\mathcal{A}\cup(\mathcal{B}\cup\mathcal{C})}(x) &= \max\{\mu_{\mathcal{A}}(x), \max\{\mu_{\mathcal{B}}(x), \mu_{\mathcal{C}}(x)\}\} \\
&= \max\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{C}}(x)\} \\
&= \mu_{\mathcal{C}}(x) \qquad (2.3.1)
\end{aligned}$$

On the other hand,

$$\begin{aligned}\mu_{(\mathcal{A}\cup\mathcal{B})\cup\mathcal{C}}(x) &= \max\{\max\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)\}, \mu_{\mathcal{C}}(x)\} \\ &= \max\{\mu_{\mathcal{B}}(x), \mu_{\mathcal{C}}(x)\} \\ &= \mu_{\mathcal{C}}(x) \qquad (2.3.2)\end{aligned}$$

Thus from (2.3.1) and (2.3.2) we get,

$$\mathcal{A}\cup(\mathcal{B}\cup\mathcal{C}) = (\mathcal{A}\cup\mathcal{B})\cup\mathcal{C}$$

□

Theorem 2.3.2. *If D is any fuzzy set contains both $\mathcal{A}$ and $\mathcal{B}$, then it also contains the union of $\mathcal{A}$ and $\mathcal{B}$.*

Proof. Let $x \in \mathcal{X}$, and $\mathcal{C} = \mathcal{A}\cup\mathcal{B}$.

$\because \mathcal{A} \subseteq D$ and $\mathcal{B} \subseteq D$ for all $x \in \mathcal{X}$

$\therefore \mu_D(x) \geq \mu_{\mathcal{A}}(x)$ and $\mu_D(x) \geq \mu_{\mathcal{B}}(x)$ for all $x \in \mathcal{X}$

$\therefore \mu_D(x) \geq \max\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)\} = \mu_{\mathcal{C}}(x)$ for all $x \in \mathcal{X}$

$\therefore \mathcal{C} = \mathcal{A}\cup\mathcal{B} \subseteq D$.

□

Remark 2.3.1. The union of $\mathcal{A}$ and $\mathcal{B}$ is the smallest fuzzy set containing both $\mathcal{A}$ and $\mathcal{B}$.

Definition 2.3.5. The intersection of two fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ with respective membership functions $\mu_{\mathcal{A}}(x)$ and $\mu_{\mathcal{B}}(x)$ is a fuzzy set $\mathcal{C}$, written as $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$, whose membership function is related to those of $\mathcal{A}$ and $\mathcal{B}$ by

$$\mu_{\mathcal{C}}(x) := \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)\}$$

Figure 2.7: The intersection of a fuzzy subset

Remark 2.3.2. the intersection of $\mathcal{A}$ and $\mathcal{B}$ is the largest fuzzy set which is contained in both $\mathcal{A}$ and $\mathcal{B}$.

Theorem 2.3.3 (De Morgan's laws). *Let $\mathcal{A}$ and $\mathcal{B}$ be two fuzzy sets, then*

1. $(\widetilde{\mathcal{A} \cup \mathcal{B}}) = \widetilde{\mathcal{A}} \cap \widetilde{\mathcal{B}}$
2. $(\widetilde{\mathcal{A} \cap \mathcal{B}}) = \widetilde{\mathcal{A}} \cup \widetilde{\mathcal{B}}$

Proof. 1. Without lose of the generality, assume that

$$\mu_{\mathcal{A}}(x) < \mu_{\mathcal{B}}(x) \text{ for all } x \in \mathcal{X}$$

$\therefore 1 - \mu_{\mathcal{A}}(x) > 1 - \mu_{\mathcal{B}}(x)$

Thus

$$\begin{aligned} \mu_{\widetilde{\mathcal{A} \cup \mathcal{B}}}(x) &= 1 - \mu_{\mathcal{A} \cup \mathcal{B}}(x) \\ &= 1 - \max\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)\} \\ &= 1 - \mu_{\mathcal{B}}(x) \qquad (2.3.3) \end{aligned}$$

On the other hand,

$$\begin{aligned} \mu_{\widetilde{\mathcal{A}} \cap \widetilde{\mathcal{B}}}(x) &= \min\left\{\mu_{\widetilde{\mathcal{A}}}(x), \mu_{\widetilde{\mathcal{B}}}(x)\right\} \\ &= \min\{1 - \mu_{\mathcal{A}}(x), 1 - \mu_{\mathcal{B}}(x)\} \\ &= 1 - \mu_{\mathcal{B}}(x) \qquad (2.3.4) \end{aligned}$$

Hence form (2.3.3) and (2.3.4) we get

$$\mu_{\widetilde{\mathcal{A} \cup \mathcal{B}}}(x) = \mu_{\widetilde{\mathcal{A}}}(x) \cap \mu_{\widetilde{\mathcal{B}}}(x)$$

Therefore, $(\widetilde{\mathcal{A} \cup \mathcal{B}}) = \widetilde{\mathcal{A}} \cap \widetilde{\mathcal{B}}$.

2. HW.

□

Theorem 2.3.4 (Distributive Laws)**.** *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be fuzzy sets, then*

1. $\mathcal{C} \cap (\mathcal{A} \cup \mathcal{B}) = (\mathcal{C} \cap \mathcal{A}) \cup (\mathcal{C} \cap \mathcal{B})$
2. $\mathcal{C} \cup (\mathcal{A} \cap \mathcal{B}) = (\mathcal{C} \cup \mathcal{A}) \cap (\mathcal{C} \cup \mathcal{B})$

Proof.

1. Without lose of the generality, we assume that

$$\mu_{\mathcal{A}}(x) < \mu_{\mathcal{B}}(x) < \mu_{\mathcal{C}}(x) \text{ for all } x \in \mathcal{X}$$

Therefore,

$$\begin{aligned} \mu_{\mathcal{C}\cap(\mathcal{A}\cup\mathcal{B})}(x) &= \min\{\mu_{\mathcal{C}}(x), \max\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)\}\} \\ &= \min\{\mu_{\mathcal{C}}(x), \mu_{\mathcal{B}}(x)\} \\ &= \mu_{\mathcal{B}}(x) \qquad (2.3.5) \end{aligned}$$

On the other hand,

$$\begin{aligned} \mu_{(\mathcal{C}\cap\mathcal{A})\cup(\mathcal{C}\cap\mathcal{B})}(x) &= \max\{\min\{\mu_{\mathcal{C}}(x), \mu_{\mathcal{A}}(x)\}, \min\{\mu_{\mathcal{C}}(x), \mu_{\mathcal{B}}(x)\}\} \\ &= \max\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)\} \\ &= \mu_{\mathcal{B}}(x) \qquad (2.3.6) \end{aligned}$$

Hence from (2.3.5) and (2.3.6) we get,

$$\mu_{\mathcal{C}\cap(\mathcal{A}\cup\mathcal{B})}(x) = \mu_{(\mathcal{C}\cap\mathcal{A})\cup(\mathcal{C}\cap\mathcal{B})}(x) \text{ for all } x \in \mathcal{X}$$

Therefore

$$\mathcal{C} \cap (\mathcal{A} \cup \mathcal{B}) = (\mathcal{C} \cap \mathcal{A}) \cup (\mathcal{C} \cap \mathcal{B}).$$

2. HW.

□

2.4 Convex Fuzzy Subsets and The Cardinality

Definition 2.4.1. A fuzzy set $\mathcal{A}$ is **convex** if and only if its α-cuts are convex.

Theorem 2.4.1. *$\mathcal{A}$ is **convex** if and only if for all $x, y \in \mathcal{X}$,*

$$\mu_{\mathcal{A}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y)\} \text{ for all } \lambda \in [0,1]. \quad (2.4.1)$$

Proof. $\Rightarrow$) Let $x, y \in \mathcal{X}$, assume that $\alpha = \mu_{\mathcal{A}}(x) \leq \mu_{\mathcal{A}}(y)$.

$\because \mathcal{A}_\alpha = \{z \in \mathcal{X} : \mu_{\mathcal{A}}(z) \geq \alpha\} = \{z \in \mathcal{X} : \mu_{\mathcal{A}}(z) \geq \mu_{\mathcal{A}}(x)\}$

$\therefore x, y \in \mathcal{A}_\alpha$

$\because \mathcal{A}_\alpha$ is convex set

$\therefore \lambda x + (1-\lambda)y \in \mathcal{A}_\alpha$

Hence,

$$\mu_{\mathcal{A}}(\lambda x + (1-\lambda)y) \geq \mu_{\mathcal{A}}(x) \quad (2.4.2)$$

Similarly, if $\alpha = \mu_{\mathcal{A}}(y) \leq \mu_{\mathcal{A}}(x)$, then

$$\mu_{\mathcal{A}}(\lambda x + (1-\lambda)y) \geq \mu_{\mathcal{A}}(y) \tag{2.4.3}$$

Therefore, form (2.4.2) and (2.4.3) we get

$$\mu_{\mathcal{A}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y)\}.$$

$\Leftarrow$) Let $x \in \mathcal{X}$ and $\alpha = \mu_{\mathcal{A}}(x)$

Claim: $\mathcal{A}_\alpha$ is convex set:

Let $u, v \in \mathcal{A}_\alpha$ and $\lambda \in [0,1]$

$\therefore \mu_{\mathcal{A}}(u) \geq \mu_{\mathcal{A}}(x)$ and $\mu_{\mathcal{A}}(v) \geq \mu_{\mathcal{A}}(x)$

By (2.4.1),

$$\begin{aligned} \mu_{\mathcal{A}}(\lambda u + (1-\lambda)v) &\geq \min\{\mu_{\mathcal{A}}(u), \mu_{\mathcal{A}}(v)\} \\ &\geq \mu_{\mathcal{A}}(x) = \alpha \end{aligned}$$

$\therefore \lambda u + (1-\lambda)v \in \mathcal{A}_\alpha$

$\therefore \mathcal{A}_\alpha$ is convex set. $\square$

Example 2.4.1.

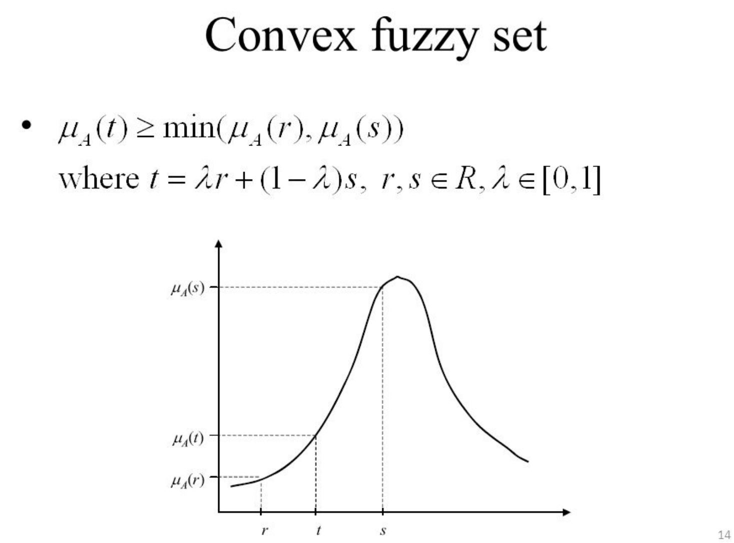

Figure 2.8: convex fuzzy set

Figure 2.9: non-convex fuzzy set

Theorem 2.4.2. *If $\mathcal{A}$ and $\mathcal{B}$ are convex sets, so is their intersection.*

Proof. Let $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are convex fuzzy sets.

Let $x, y \in \mathcal{X}$ and $\lambda \in [0, 1]$, then

$$\begin{aligned}
\mu_{\mathcal{C}}(\lambda x + (1-\lambda)y) &= \min\{\mu_{\mathcal{A}}(\lambda x + (1-\lambda)y), \mu_{\mathcal{B}}(\lambda x + (1-\lambda)y)\} \\
&\geq \min\{\min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y)\}, \min\{\mu_{\mathcal{B}}(x), \mu_{\mathcal{B}}(y)\}\} \\
&= \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y), \mu_{\mathcal{B}}(x), \mu_{\mathcal{B}}(y)\} \\
&= \min\{\min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)\}, \min\{\mu_{\mathcal{A}}(y), \mu_{\mathcal{B}}(y)\}\} \\
&= \min\{\mu_{\mathcal{C}}(x), \mu_{\mathcal{C}}(y)\}
\end{aligned}$$

$\therefore \mu_{\mathcal{C}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\mathcal{C}}(x), \mu_{\mathcal{C}}(y)\}$

Hence by theorem (2.4.1), $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$ is convex fuzzy set □

Definition 2.4.2. Let $\mathcal{X}$ be the universe set, and $\mathcal{A}$ be a fuzzy set,

Scalar Cardinality ($|\mathcal{A}|$): the scalar is defined as the sum of the grade of the membership of finite fuzzy set $\mathcal{A}$. That is:

$$|\mathcal{A}| := \sum_{x \in \mathcal{X}} \mu_{\mathcal{A}}(x).$$

Remarks 2.4.1. For any fuzzy sets $\mathcal{A}$ and $\mathcal{B}$,

1. If for all k, $\mu_{\mathcal{B}}(x_k) \leq \mu_{\mathcal{A}}(x_k)$, then $|\mathcal{B}| \leq |\mathcal{A}|$.

   Proof. $|\mathcal{B}| = \sum_k \mu_{\mathcal{B}}(x_k) \leq \sum_k \mu_{\mathcal{A}}(x_k) = |\mathcal{A}|$ □

2. $\left|\widetilde{\mathcal{A}}\right| = |\mathcal{X}| - |\mathcal{A}|$.

Proof.

$$\begin{aligned}\left|\widetilde{\mathcal{A}}\right| &= \sum_{k=1}^{n} \mu_{\widetilde{\mathcal{A}}}(x_k) = \sum_{k=1}^{n}(1 - \mu_{\mathcal{A}}(x_k)) \\ &= n - \sum_{k=1}^{n} \mu_{\mathcal{A}}(x_k) = |\mathcal{X}| - |\mathcal{A}|\end{aligned}$$

□

3. $|\mathcal{A} \cup \mathcal{B}| + |\mathcal{A} \cap \mathcal{B}| = |\mathcal{A}| + |\mathcal{B}|$.

Relative Cardinality ($\|\mathcal{A}\|$):

$$\|\mathcal{A}\| := \frac{|\mathcal{A}|}{|\mathcal{X}|}$$

Remarks 2.4.2. For any fuzzy set $\mathcal{A}$,

1. $0 \leq \|\mathcal{A}\| \leq 1$.

2. $\|\mathcal{A}\| = 0$ if $\mu_{\mathcal{A}}(x_k) = 0$ for all k. Since,

$$\|\mathcal{A}\| := \frac{|\mathcal{A}|}{|\mathcal{X}|} = \frac{0}{|\mathcal{X}|} = 0$$

3. $\|\mathcal{A}\| = 1$ if $\mu_{\mathcal{A}}(x_k) = 1$ for all k. Since,

$$\|\mathcal{A}\| := \frac{|\mathcal{A}|}{|\mathcal{X}|} = \frac{\sum_{k=1}^{n} \mu_{\mathcal{A}}(x_k)}{|\mathcal{X}|} = \frac{n}{n} = 1$$

Fuzzy Cardinality ($|\mathcal{A}|_F$):

$$|\mathcal{A}|_F := \{(|\mathcal{A}_\alpha|, \alpha) : \alpha \in [0,1]\}$$

Example 2.4.2. Let $\mathcal{X} = \{1,2,3,4,5,6,7,8,9,10\}$

$\mathcal{A} = 0.2/1 + 0.5/2 + 0.8/3 + 1/4 + 0.7/5 + 0.3/6,$

$\mathcal{B} = 0.3/1 + 0.9/3 + 0.8/6 + 0.1/7 + 0.4/8 + 0.6/9 + 1/10,$

then

Scalar Cardinality ($|\mathcal{A}|$):

$$\begin{aligned} |\mathcal{A}| &= \sum_{x\in\mathcal{X}} \mu_{\mathcal{A}}(x) = 0.2 + 0.5 + 0.8 + 1 + 0.7 + 0.3 = 3.5 \\ |\mathcal{B}| &= \sum_{x\in\mathcal{X}} \mu_{\mathcal{B}}(x) = 0.3 + 0.9 + 0.8 + 0.1 + 0.4 + 0.6 + 1 = 4.1 \end{aligned}$$

While,

$$\begin{aligned} |\mathcal{A}\cup\mathcal{B}| &= 0.3 + 0.5 + 0.9 + 1 + 0.7 + 0.8 + 0.1 + 0.4 + 0.6 + 1 = 6.3 \\ |\mathcal{A}\cap\mathcal{B}| &= 0.2 + 0.8 + 0.3 = 1.3 \end{aligned}$$

Thus,

$$|\mathcal{A}\cup\mathcal{B}| + |\mathcal{A}\cap\mathcal{B}| = 7.6 = |\mathcal{A}| + |\mathcal{B}|$$

Relative Cardinality ($\|\mathcal{A}\|$):

$$\|\mathcal{A}\| := \frac{|\mathcal{A}|}{|\mathcal{X}|} = \frac{3.5}{10} = 0.35$$

Fuzzy Cardinality ($|\mathcal{A}|_F$):

$$\begin{aligned}
\mathcal{A}_{0.2} &= \{1, 2, 3, 4, 5, 6\} \\
\mathcal{A}_{0.3} &= \{2, 3, 4, 5, 6\} \\
\mathcal{A}_{0.5} &= \{2, 3, 4, 5\} \\
\mathcal{A}_{0.7} &= \{3, 4, 5\} \\
\mathcal{A}_{0.8} &= \{3, 4\} \\
\mathcal{A}_{1} &= \{4\}
\end{aligned}$$

Thus,

$$\begin{aligned}
|\mathcal{A}_{0.2}| &= 6 \\
|\mathcal{A}_{0.3}| &= 5 \\
|\mathcal{A}_{0.5}| &= 4 \\
|\mathcal{A}_{0.7}| &= 3 \\
|\mathcal{A}_{0.8}| &= 2 \\
|\mathcal{A}_{1}| &= 1
\end{aligned}$$

$$\begin{aligned}
|\mathcal{A}|_F &= \{(|\mathcal{A}_\alpha|, \alpha) : \alpha \in [0, 1]\} \\
&= \{(6, 0.2), (5, 0.3), (4, 0.5), (3, 0.7), (2, 0.8), (1, 1)\}
\end{aligned}$$

Homework 2.4.1.

1. Let $\mathcal{X} = [1, 5]$ and $\mathcal{A} := \int_{\mathcal{X}} \frac{1}{x}/x$. Is $\mathcal{A}$ convex?

2. According to homework (2.2.1). Calculate the scalar cardinality of young fuzzy set, the relative cardinality to adult fuzzy set, and the fuzzy cardinality of senior fuzzy set.

3. Compute the relative cardinality of $\mathcal{A} \cup \mathcal{B}$ and the scalar cardinality of $\widetilde{\mathcal{A} \cap \mathcal{B}}$ where,

$$A = \{(x, 0.4), (y, 0.5), (z, 0.9), (w, 1)\}$$

$$B = 0.5/u + 0.8/v + 0.9/w + 0.1/x$$

2.5 Expansion of Fuzzy Set

Definition 2.5.1. If the value of membership function is given by a fuzzy set, it is a type-2 fuzzy set. This concept can be extended up to Typen fuzzy set.

Example 2.5.1. Consider set $\mathcal{A}$ = adult. The membership function of this set maps whole age to youth, manhood and senior. For instance, for any person x, y, and z,

$$\begin{aligned} \mu_{\mathcal{A}}(x) &= \text{youth} \\ \mu_{\mathcal{A}}(y) &= \text{manhood} \\ \mu_{\mathcal{A}}(z) &= \emptyset \end{aligned}$$

The values of membership for youth and manhood are also fuzzy sets, and thus the set adult is a type-2 fuzzy set. The sets youth and manhood are type-1 fuzzy sets.

Remark 2.5.1. In the same manner, if the values of membership function of youth and manhood are type-2, the set adult is type-3.

Definition 2.5.2. consider a fuzzy set satisfying $\mathcal{A} \neq \emptyset$ and $\mathcal{A} \neq \mathcal{X}$. The pair $(\mathcal{A}, \widetilde{\mathcal{A}})$ is defined as **fuzzy partition**. More generally, if there are m subsets defined in $\mathcal{X}$, $(\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n)$ holding the following conditions is called a **fuzzy partition**.

1. $\forall i$, $\mathcal{A}_i \neq \emptyset$.

2. $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for all $i \neq j$.

3. $\forall x \in \mathcal{X}$, $\sum_{i=1}^{m} \mu_{\mathcal{A}_i}(x) = 1$.

Chapter 3

FUZZY RELATION AND COMPOSITION

3.1 Fuzzy Relation on sets

Definition 3.1.1. If a **crisp relation** R represents that of from sets A to B, for $x \in A$ and $y \in B$, its membership function $\mu_R((x,y))$ is,

$$\mu_R(x,y) = \begin{cases} 1, & (x,y) \in R \\ 0, & (x,y) \notin R \end{cases}$$

This membership function maps $A \times B$ to set $\{0,1\}$.

$$\mu_R : A \times B \to \{0,1\}$$

Remarks 3.1.1. In general,

1. A relation among crisp sets $A_1, A_2, \cdots, A_n$ is a subset of the Cartesian product. It is denoted by R.

$$R \subseteq A_1 \times A_2 \times \cdots \times A_n$$

2. Using the membership function defines the crisp relation R :

$$\mu_R(x_1, x_2, \cdots, x_n) = \begin{cases} 1, & (x_1, x_2, \cdots, x_n) \in R \\ 0, & o.w. \end{cases}$$

Where $x_1 \in A_1$, $x_2 \in A_2$, $\cdots$, $x_n \in A_n$.

Definition 3.1.2. Let $\mathcal{X}$, $\mathcal{Y}$ be crisp sets, then **Fuzzy relation on** $\mathcal{X} \times \mathcal{Y}$ has degree of membership whose value lies in $[0, 1]$.

$$\mu_{\mathcal{R}} : \mathcal{X} \times \mathcal{Y} \to [0, 1]$$

$$\mathcal{R} := \{((x, y), \mu_{\mathcal{R}}(x, y)) : x \in \mathcal{X}, y \in \mathcal{Y}\}$$

Remarks 3.1.2.

1. Here $\mu_{\mathcal{R}}(x, y)$ is interpreted as strength of relation between x and y. When $\mu_{\mathcal{R}}(x, y) \geq \mu_{\mathcal{R}}(x', y')$, (x, y) is more strongly related than (x', y').

2. When a fuzzy relation $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{Y}$ is given, this relation $\mathcal{R}$ can be thought as a fuzzy set in the space $\mathcal{X} \times \mathcal{Y}$.

Figure 3.1: crisp and fuzzy relation

Example 3.1.1. crisp relation R in the figure (a) reflects a relation in $A \times A$. Expressing this by membership function,

$$\mu_R(a,c) = 1, \mu_R(b,a) = 1, \mu_R(c,b) = 1, \mu_R(c,d) = 1.$$

If this relation is given as the value between 0 and 1 as in the figure (b), this relation becomes a fuzzy relation. Expressing this fuzzy relation by membership function yields,

$$\mu_R(a,c) = 0.8, \mu_R(b,a) = 1.0, \mu_R(c,b) = 0.9, \mu_R(c,d) = 1.0$$

Figure 3.2: Crisp and fuzzy Relation

Remark 3.1.1. When crisp relation R represents the relation from crisp sets A to B, its domain and range can be defined as,

$$\begin{aligned} dom(R) &:= \{x \in A : x \in A, y \in B, \mu_R(x,y) = 1\} \\ ran(R) &:= \{y \in B : x \in A, y \in B, \mu_R(x,y) = 1\} \end{aligned}$$

Definition 3.1.3. When fuzzy relation $\mathcal{R}$ is defined in crisp sets A and B, the domain and range of this relation are defined as :

$$\begin{aligned} \mu_{dom(\mathcal{R})}(x) &= \max_{y \in B} \mu_R(x,y) \\ \mu_{ran(\mathcal{R})}(y) &= \max_{x \in A} \mu_R(x,y) \end{aligned}$$

Set A becomes the support of $dom(\mathcal{R})$ and $dom(\mathcal{R}) \subseteq A$. Set B is the support of $ran(\mathcal{R})$ and $ran(\mathcal{R}) \subseteq B$.

Example 3.1.2. Let $A = \{x_1, x_2\}$ and $B = \{y_1, y_2\}$ such that

$$\begin{aligned} \mu_{\mathcal{R}}(x_1, y_1) &= 0.4 \text{ and } \mu_{\mathcal{R}}(x_1, y_2) = 0.7 \\ \mu_{\mathcal{R}}(x_2, y_1) &= 0.6 \text{ and } \mu_{\mathcal{R}}(x_2, y_2) = 0.3 \end{aligned}$$

then

$$
\begin{aligned}
\mu_{dom(\mathcal{R})}(x_1) &= \max_{y \in B} \mu_R(x_1, y) \\
&= \max\{\mu_R(x_1, y_1), \mu_R(x_1, y_2)\} = \max\{0.4, 0.7\} = 0.7 \\
\mu_{dom(\mathcal{R})}(x_2) &= \max_{y \in B} \mu_R(x_2, y) \\
&= \max\{\mu_R(x_2, y_1), \mu_R(x_2, y_2)\} = \max\{0.6, 0.3\} = 0.6 \\
\mu_{ran(\mathcal{R})}(y_1) &= \max_{x \in A} \mu_R(x, y_1) \\
&= \max\{\mu_R(x_1, y_1), \mu_R(x_2, y_1)\} = \max\{0.4, 0.6\} = 0.6 \\
\mu_{ran(\mathcal{R})}(y_2) &= \max_{x \in A} \mu_R(x, y_2) \\
&= \max\{\mu_R(x_1, y_2), \mu_R(x_2, y_2)\} = \max\{0.7, 0.3\} = 0.7
\end{aligned}
$$

3.2 Fuzzy Matrix

Definition 3.2.1. if an element of the vector has its value between 0 and 1, we call this vector a **fuzzy vector**.

Example 3.2.1.

$$V = \begin{bmatrix} 0.1 \\ 0.005 \\ 0.77 \\ 0 \\ 1 \end{bmatrix}$$

Definition 3.2.2. A fuzzy matrix is a matrix which has its elements from the interval $[0, 1]$, called the **unit fuzzy interval**.

Example 3.2.2.

$$P = \begin{pmatrix} 0.2 & 0.33 \\ 0.5 & 1 \\ 0 & 0.24 \end{pmatrix}_{3\times 2}$$

Definition 3.2.3. An $m \times n$ fuzzy matrix for which $m = n$ (i.e the number of rows is equal to the number of columns) and whose elements belong to the unit interval $[0, 1]$ is called a **fuzzy square matrix of order** n.

Example 3.2.3.

$$P = \begin{pmatrix} 0.2 & 0.3 \\ 0 & 1 \end{pmatrix}_{2\times 2}$$

3.2.1 operations on fuzzy matrices.

Given a fuzzy matrix $A = (a_{ij})$ and $B = (b_{ij})$, we can perform operations on these fuzzy matrices.

- Sum: $A + B = \max\{a_{ij}, b_{ij}\}$

Example 3.2.4. Let

$$A = \begin{pmatrix} 0.1 & 0.4 \\ 0.5 & 0.9 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0.7 & 0.2 \\ 0.6 & 0.4 \end{pmatrix}$$

Then

$$A + B = \begin{pmatrix} 0.7 & 0.4 \\ 0.6 & 0.9 \end{pmatrix}$$

- Max product: $C = A \bullet B = AB = \max\{\min(a_{ik}, b_{kj})\}$

Example 3.2.5. Let

$$A = \begin{pmatrix} 0.2 & 0.5 & 0.0 \\ 0.4 & 1.0 & 0.1 \\ 0.0 & 1.0 & 0.0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1.0 & 0.1 & 0.0 \\ 0.0 & 0.0 & 0.5 \\ 0.0 & 1.0 & 0.1 \end{pmatrix}$$

Note that,

$$\begin{array}{rllll} & 0.2 & 0.5 & 0.0 & :(a_{1k}) \\ \min\Downarrow & 1.0 & 0.0 & 0.0 & :(b_{k1}) \\ \hline & 0.2 & 0.0 & 0.0 & \overset{\Longrightarrow}{\max} \quad 0.2 = c_{11} \end{array} \qquad \begin{array}{rllll} & 0.2 & 0.5 & 0.0 & :(a_{1k}) \\ \min\Downarrow & 0.1 & 0.0 & 1.0 & :(b_{k2}) \\ \hline & 0.1 & 0.0 & 0.0 & \overset{\Longrightarrow}{\max} \quad 0.1 = c_{12} \end{array}$$

$$\begin{array}{rccccll} & 0.2 & 0.5 & 0.0 & :(a_{1k}) & & \\ \min \Downarrow & 0.0 & 0.5 & 1.0 & :(b_{k3}) & & \\ \hline & 0.0 & 0.5 & 0.0 & \overrightarrow{\max} & 0.5 = c_{13} \end{array} \qquad \begin{array}{rccccll} & 0.4 & 1.0 & 0.1 & :(a_{2k}) & & \\ \min \Downarrow & 1.0 & 0.0 & 0.0 & :(b_{k1}) & & \\ \hline & 0.4 & 0.0 & 0.0 & \overrightarrow{\max} & 0.4 = c_{21} \end{array}$$

$$\begin{array}{rccccll} & 0.4 & 1.0 & 0.1 & :(a_{2k}) & & \\ \min \Downarrow & 0.1 & 0.0 & 1.0 & :(b_{k2}) & & \\ \hline & 0.1 & 0.0 & 0.1 & \overrightarrow{\max} & 0.1 = c_{22} \end{array} \qquad \begin{array}{rccccll} & 0.4 & 1.0 & 0.1 & :(a_{2k}) & & \\ \min \Downarrow & 0.0 & 0.5 & 1.0 & :(b_{k3}) & & \\ \hline & 0.0 & 0.5 & 0.1 & \overrightarrow{\max} & 0.5 = c_{23} \end{array}$$

$$\begin{array}{rccccll} & 0.0 & 1.0 & 0.0 & :(a_{3k}) & & \\ \min \Downarrow & 1.0 & 0.0 & 0.0 & :(b_{k1}) & & \\ \hline & 0.0 & 0.0 & 0.0 & \overrightarrow{\max} & 0.0 = c_{31} \end{array} \qquad \begin{array}{rccccll} & 0.0 & 1.0 & 0.0 & :(a_{3k}) & & \\ \min \Downarrow & 0.1 & 0.0 & 1.0 & :(b_{k2}) & & \\ \hline & 0.0 & 0.0 & 0.0 & \overrightarrow{\max} & 0.0 = c_{32} \end{array}$$

$$\begin{array}{rccccll} & 0.0 & 1.0 & 0.0 & :(a_{3k}) & & \\ \min \Downarrow & 0.0 & 0.5 & 1.0 & :(b_{k3}) & & \\ \hline & 0.0 & 0.5 & 0.0 & \overrightarrow{\max} & 0.5 = c_{33} \end{array}$$

Thus,

$$C = A \bullet B = \begin{pmatrix} 0.2 & 0.1 & 0.5 \\ 0.4 & 0.1 & 0.5 \\ 0.0 & 0.0 & 0.5 \end{pmatrix}$$

-Scalar product: λA where $0 \leq \lambda \leq 1$

Example 3.2.6.

$$A = \begin{pmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.7 & 1.0 \end{pmatrix}$$

Then,

$$0.5A = \begin{pmatrix} 0.2 & 0.25 & 0.05 \\ 0.15 & 0.35 & 0.5 \end{pmatrix}$$

3.3 Fuzzy relation matrix

Definition 3.3.1.

If a fuzzy relation $\mathcal{R}$ is given in the form of fuzzy matrix, its elements represent the membership values of this relation. That is, if the matrix is denoted by $\mathcal{M}_{\mathcal{R}}$, and membership values by $\mu_{\mathcal{R}}(i, j)$, then $\mathcal{M}_{\mathcal{R}} = (\mu_{\mathcal{R}}(i, j))$.

Example 3.3.1.

Its corresponding fuzzy matrix is as follows.

$\mathcal{M}_\mathcal{R}$	a	b	c	d
a	0.0	0.0	0.8	0.0
b	1.0	0.0	0.0	0.0
c	0.0	0.9	0.0	1.0
d	0.0	0.0	0.0	0.0

3.3.1 Operation of Fuzzy Relation

Let $\mathcal{R} \subseteq A \times B$, and $\mathcal{S} \subseteq A \times B$

-Union relation: Union of two relations $\mathcal{R}$ and § is defined as follows:

$$\mu_{\mathcal{R}\cup\mathcal{S}}(x,y) = \max\{\mu_\mathcal{R}(x,y), \mu_\mathcal{S}(x,y)\} = M_\mathcal{R} + M_\mathcal{S}$$

For more general,

$$\mu_{\mathcal{R}_1\cup\mathcal{R}_2\cup\cdots\cup\mathcal{R}_n}(x,y) = \max_{\mathcal{R}_i}\{\mu_{\mathcal{R}_i}(x,y)\} = \sum_i M_{\mathcal{R}_i}$$

Example 3.3.2.

$\mathcal{M}_\mathcal{R}$	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.0	1.0	0.0

$\mathcal{M}_\mathcal{S}$	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.6	0.9	0.3

Thus,

$\mathcal{M}_{\mathcal{R}\cup\mathcal{S}}$	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.6	1.0	0.3

-Intersection relation: The intersection relation $\mathcal{R}\cap\mathcal{S}$ of sets A and B is defined by the following membership function.

$$\mu_{\mathcal{R}\cap\mathcal{S}}(x,y)=\min\{\mu_{\mathcal{R}}(x,y),\mu_{\mathcal{S}}(x,y)\}$$

For more general,

$$\mu_{\mathcal{R}_1\cap\mathcal{R}_2\cap\cdots\cap\mathcal{R}_n}(x,y)=\min_{\mathcal{R}_i}\{\mu_{\mathcal{R}_i}(x,y)\}$$

Example 3.3.3.

$\mathcal{M}_{\mathcal{R}}$	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.0	1.0	0.0

$\mathcal{M}_{\mathcal{S}}$	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.6	0.9	0.3

Thus,

$\mathcal{M}_{\mathcal{R}\cap\mathcal{S}}$	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.0	0.9	0.0

-Complement relation: Complement relation $\tilde{\mathcal{R}}$ for fuzzy relation $\mathcal{R}$ shall be defined by the following membership function.

For all $(x, y) \subseteq A \times B$; $\mu_{\tilde{\mathcal{R}}}(x, y) = 1 - \mu_{\mathcal{R}}(x, y)$

Example 3.3.4.

$\mathcal{M}_{\mathcal{R}}$	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.0	1.0	0.0

$\mathcal{M}_{\tilde{\mathcal{R}}}$	a	b	c
1	0.7	0.8	0.0
2	0.2	0.0	0.0
3	1.0	0.0	1.0

-Inverse relation: When a fuzzy relation $\mathcal{R} \subseteq A \times B$ is given, the inverse relation; $\mathcal{R}^{-1}$, of $\mathcal{R}$ is defined by the following membership function.

For all $(x, y) \subseteq A \times B$; $\mu_{\mathcal{R}^{-1}}(y, x) = \mu_{\mathcal{R}}(x, y)$

Example 3.3.5.

$\mathcal{M}_{\mathcal{R}}$	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.0	1.0	0.0

$\mathcal{M}_{\mathcal{R}^{-1}}$	1	2	3
a	0.3	0.8	0.0
b	0.2	1.0	1.0
c	1.0	1.0	0.0

3.4 Composition of Fuzzy Relation

Definition 3.4.1. Two fuzzy relations $\mathcal{R}$ and $\mathcal{S}$ are defined on sets A, B and C. That is, $\mathcal{R} \subseteq A \times B$, $\mathcal{S} \subseteq B \times C$. **The composition** $\mathcal{S} \circ \mathcal{R} = \mathcal{S}\mathcal{R}$ of two relations $\mathcal{R}$ and $\mathcal{S}$ is expressed by the relation from A to C, and this composition is defined by the following.

For $(x, y) \in A \times B$, $(y, z) \in B \times C$,

$$\mu_{\mathcal{S}\circ\mathcal{R}}(x, z) = \max\left\{\min_{y}\{\mu_{\mathcal{R}}(x, y), \mu_{\mathcal{S}}(y, z)\}\right\}$$

Remarks 3.4.1.

1. $\mathcal{S} \circ \mathcal{R} \subseteq A \times C$.

2. If the relations $\mathcal{R}$ and $\mathcal{S}$ are represented by matrices $M_{\mathcal{R}}$ and $M_{\mathcal{S}}$, the matrix $M_{\mathcal{S}\bullet\mathcal{R}}$ corresponding to $\mathcal{S} \circ \mathcal{R}$ is obtained from the

product of $M_{\mathcal{R}}$ and $M_{\mathcal{S}}$.

$$M_{\mathcal{S}\circ\mathcal{R}} = M_{\mathcal{R}} \bullet M_{\mathcal{S}}.$$

Example 3.4.1. Let $A := \{1, 2, 3\}$, $B := \{a, b, c, d\}$, and $C := \{\alpha, \beta, \gamma\}$ are sets. Consider fuzzy relations $\mathcal{R} \subseteq A \times B$, $\mathcal{S} \subseteq B \times C$, such that:

$\mathcal{R}$	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

$\mathcal{S}$	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

Then,

$\mathcal{S} \circ \mathcal{R}$	α	β	γ
1	0.4	0.2	0.3
2	0.3	0.3	0.3
3	0.8	0.9	0.8

Figure 3.3: Composition of fuzzy relation

3.4.1 α-cut of Fuzzy Relation

Definition 3.4.2. We can obtain **α-cut relation** from a fuzzy relation by taking the pairs which have membership degrees no less than α. Assume $\mathcal{R} \subseteq A \times B$, and $\mathcal{R}_\alpha$ is a α -cut relation. Then

$$\mathcal{R}_\alpha := \{(x, y) : \mu_{\mathcal{R}}(x, y) \geq \alpha, x \in A, y \in B\}$$

Remark 3.4.1. Note that $\mathcal{R}_\alpha$ is a crisp relation.

Definition 3.4.3. The **level set** is all the degrees of membership function.

Example 3.4.2. Let $\mathcal{R}$ be a fuzzy relation such that

$$M_{\mathcal{R}} = \begin{array}{|ccc} \hline 0.9 & 0.4 & 0.0 \\ 0.2 & 1.0 & 0.4 \\ 0.0 & 0.7 & 1.0 \\ 0.4 & 0.2 & 0.0 \end{array}$$

Now the level set with degrees of membership function is,

$$\Lambda = \{0, 0.2, 0.4, 0.7, 0.9, 1.0\}$$

then we can have some α - cut relations in the following

$$M_{\mathcal{R}_{0.2}} = \begin{array}{|ccc} \hline 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{array} \quad M_{\mathcal{R}_{0.4}} = \begin{array}{|ccc} \hline 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{array} \quad M_{\mathcal{R}_{0.7}} = \begin{array}{|ccc} \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}$$

$$M_{\mathcal{R}_{0.9}} = \begin{array}{|ccc} \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \quad M_{\mathcal{R}_{1}} = \begin{array}{|ccc} \hline 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$$

Homework 3.4.1.

1. Draw the following relation

$\mathcal{R}$	x	y	z	w
a	1.0	0.6	0.3	0.0
b	0.0	0.8	0.4	0.9
c	0.1	0.0	1.0	0.0
d	0.2	0.4	0.0	1.0

2. Compute the complements, intersection and union of the following fuzzy relations $\mathcal{R}$ and $\mathcal{S}$.

$\mathcal{R}$	a	b	c	d
a	1.0	0.2	0.4	0.0
b	0.0	0.1	0.0	0.9
c	0.1	0.0	1.0	0.0
d	0.0	0.4	0.0	1.0

$\mathcal{S}$	a	b	c	d
a	1.0	0.0	0.0	0.4
b	0.0	0.0	0.4	0.9
c	0.4	0.0	0.1	0.0
d	0.5	0.1	0.0	0.0

3. Determine the composition relation $\mathcal{S} \circ \mathcal{R} \subseteq A \times C$ where $\mathcal{R} \subseteq A \times B$ and $\mathcal{S} \subseteq B \times C$ are defined as follows

$\mathcal{R}$	a	b	c	d
1	0.4	0.0	0.0	1.0
2	0.5	0.4	0.9	0.0
3	0.2	0.1	1.0	0.4
4	0.0	0.2	0.0	1.0

$\mathcal{S}$	a	b	c
a	0.4	0.1	0.0
b	0.2	0.0	0.9
c	0.2	0.0	0.5
d	0.1	0.0	0.9

Find the level set of $\mathcal{R}$ and α-cut of $\mathcal{S}$.

3.5 Properties of fuzzy relations

-Reflexive Relation: Let $\mathcal{R}$ be a fuzzy relation in $\mathcal{X} \times \mathcal{X}$ then $\mathcal{R}$ is called **reflexive**, if

$$\mu_{\mathcal{R}}(x, x) = 1 \text{ for all } x \in \mathcal{X}.$$

Example 3.5.1. Let $\mathcal{X} = \{a, b, c\}$

$\mathcal{R}$	a	b	c
a	1.0	0.1	0.0
b	0.2	1.0	0.9
c	0.2	0.0	1.0

is reflexive relation.

- Antireflexive relations: Fuzzy relation $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{X}$ is **antireflexive** if

$$\mu_{\mathcal{R}}(x, x) = 0 \text{ for all } x \in \mathcal{X}.$$

Example 3.5.2. Let $\mathcal{X} = \{a, b, c\}$

$\mathcal{R}$	a	b	c
a	0.0	0.1	0.0
b	0.2	0.0	0.9
c	0.2	0.0	0.0

is antireflexive relation.

-Symmetric Relation: A fuzzy relation $\mathcal{R}$ is called **symmetric** if,

$$\mu_{\mathcal{R}}(x,y) = \mu_{\mathcal{R}}(y,x) \text{ for all } y, x \in \mathcal{X}.$$

Example 3.5.3. Let $\mathcal{X} = \{a, b, c\}$

$\mathcal{R}$	a	b	c
a	0.5	0.1	0.3
b	0.1	0.0	0.9
c	0.3	0.9	0.6

is a symmetric relation.

- Antisymmetric Relation: Fuzzy relation $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{X}$ is **antisymmetric** iff

$$\text{if } \mu_{\mathcal{R}}(x,y) > 0 \text{ then } \mu_{\mathcal{R}}(y,x) = 0 \text{ for all } x, y \in \mathcal{X}, x \neq y$$

Example 3.5.4. Let $\mathcal{X} = \{a, b, c\}$

$\mathcal{R}$	a	b	c
a	0.4	0.0	0.7
b	0.1	0.0	0.0
c	0.0	0.2	0.6

is a antisymmetric relation.

-Transitive Relation: Fuzzy relation $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{X}$ is **transitive** iff

$\mathcal{R}^2 = \mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$ means $\mu_{\mathcal{R}^2}(x, y) \leq \mu_{\mathcal{R}}(x, y)$.

Example 3.5.5. Let $\mathcal{X} = \{x_1, x_2, x_3\}$

$\mathcal{R}$	x_1	x_2	x_3
x_1	0.7	0.9	**0.4**
x_2	**0.1**	0.3	0.5
x_3	0.2	**0.1**	**0.0**

$\mathcal{R}^2$	x_1	x_2	x_3
x_1	0.7	0.7	**0.5**
x_2	**0.2**	0.3	0.3
x_3	0.2	**0.2**	**0.2**

Since $\mu_{\mathcal{R}^2}(x_i, x_j)$ is not always less than or equal to $\mu_{\mathcal{R}}(x_i, x_j)$, hence $\mathcal{R}$ is not transitive.

Example 3.5.6. Let $\mathcal{X} = \{x_1, x_2\}$

$\mathcal{R}$	x_1	x_2
x_1	0.4	0.2
x_2	0.7	0.3

$\mathcal{R}^2$	x_1	x_2
x_1	0.4	0.2
x_2	0.4	0.3

Since $\mu_{\mathcal{R}^2}(x_i, x_j)$ is always less than or equal to $\mu_{\mathcal{R}}(x_i, x_j)$, hence $\mathcal{R}$ is transitive.

3.5.1 Fuzzy Equivalence Relation

If a fuzzy relation $\mathcal{R} \subseteq A \times A$ satisfies the following conditions:

1. Reflexive relation.
2. Symmetric relation.
3. Transitive relation.

we call it a **fuzzy equivalence relation** or **similarity relation**.

Example 3.5.7. Lets consider a fuzzy relation expressed in the following matrix.

$\mathcal{R}$	a	b	c	d
a	1.0	0.8	0.7	1.0
b	0.8	1.0	0.7	0.8
c	0.7	0.7	1.0	0.7
d	1.0	0.8	0.7	1.0

Since this relation is reflexive, symmetric and transitive, then it is a fuzzy equivalence relation

Figure 3.4: fuzzy equivalence relation

Definition 3.5.1. The fuzzy set A is done **partition** into subsets $A_1, A_2, \cdots$ by the equivalence relation.

Example 3.5.8. A partition of A by the given relation $\mathcal{R}$. At this point, fuzzy equivalence relation holds in class A_1 and A_2, but not between A_1 and A_2.

$\mathcal{R}$	a	b	c	d	e
a	1.0	0.5	1.0	0.0	0.0
b	0.5	1.0	0.5	0.0	0.0
c	1.0	0.5	1.0	0.0	0.0
d	0.0	0.0	0.0	1.0	0.5
e	0.0	0.0	0.0	0.5	1.0

Figure 3.5: Partition by fuzzy equivalence relation

Definition 3.5.2. if a partition is done on set A into subsets $A_1, A_2, A_3, \cdots$, the similarity among elements in A_i is no less than α. **The α-cut equivalence relation** $\mathcal{R}_\alpha$ is defined by

$$\mu_{\mathcal{R}}(x, y) = \begin{cases} 1, & \mu_{\mathcal{R}}(x, y) \geq \alpha, \forall x, y \in A_i \\ 0, & \text{o.w} \end{cases}$$

If we apply α - cut according to α_1 in level set $\{\alpha_1, \alpha_2, \cdots\}$, the partition by this procedure is denoted by $\pi(\mathcal{R}_{\alpha_1})$. In the same manner, we get $\pi(\mathcal{R}_{\alpha_1})$ by the procedure of α_2-cut. Then, we know

if $\alpha_1 \geq \alpha_2$, $\mathcal{R}_{\alpha_1} \subseteq \mathcal{R}_{\alpha_2}$ and we can say that $\pi(\mathcal{R}_{\alpha_1})$ is more refined than $\pi(\mathcal{R}_{\alpha_2})$.

Example 3.5.9.

$\mathcal{R}$	a	b	c	d	e	f
a	1.0	0.8	0.0	0.4	0.0	0.0
b	0.8	1.0	0.0	0.4	0.0	0.0
c	0.0	0.0	1.0	0.0	1.0	0.5
d	0.4	0.4	0.0	1.0	0.0	0.0
e	0.0	0.0	1.0	0.0	1.0	0.5
f	0.0	0.0	0.5	0.0	0.5	1.0

Figure 3.6: Partition tree

Definition 3.5.3. Let X, Y be the universal sets, and

$$\begin{aligned} \mathcal{A} &= \{(x, \mu_{\mathcal{A}}(x)) : x \in \mathcal{X}\} \\ \mathcal{B} &= \{(y, \mu_{\mathcal{B}}(y)) : y \in \mathcal{Y}\} \end{aligned}$$

Then $\mathcal{R} = \{((x, y), \mu_{\mathcal{A}}(x, y)) : (x, y) \in \mathcal{X} \times \mathcal{Y}\}$ is **fuzzy relation on $\mathcal{A}$ and $\mathcal{B}$** if

$$\mu_{\mathcal{R}}(x, y) \leq \mu_{\mathcal{A}}(x), \forall (x, y) \in \mathcal{X} \times \mathcal{Y}$$

and

$$\mu_{\mathcal{R}}(x, y) \leq \mu_{\mathcal{B}}(y), \forall (x, y) \in \mathcal{X} \times \mathcal{Y}$$

Remark 3.5.1. Fuzzy relation are obviously fuzzy sets in product spaces. Therefore set - theoretic and algebraic operations can be defined for them in analogy to the above definitions.

Homework 3.5.1.

1. Determine whether the following fuzzy relation is an equivalence relation. Determine the partition tree of the set A

(a)

$\mathcal{R}$	α	β	γ
α	1.0	0.8	0.7
β	0.8	1.0	0.7
γ	0.7	0.7	1.0

(b)

$\mathcal{R}$	a	b	c	d
a	1.0	0.8	0.4	0.1
b	0.8	1.0	0.0	0.0
c	0.4	0.0	1.0	0.5
d	0.1	0.0	0.5	1.0

2. Let the two fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ be defined as

$$\mathcal{A} = \{(0, 0.2), (1, 0.3), (2, 0.4), (3, 0.5)\}$$
$$\mathcal{B} = \{(0, 0.5), (1, 0.4), (2, 0.3), (3, 0.0)\}$$

Is the following set a fuzzy relation on $\mathcal{A}$ and $\mathcal{B}$?

$$\{(0, 0), 0.2), (0, 2), 0.2), (2, 0), 0.2)\}$$

Give an example of a fuzzy relation on $\mathcal{A}$ and $\mathcal{B}$.

Chapter 4

Kinds of Fuzzy Function

4.1

Definition 4.1.1. Let X and Y be crisp sets, and f be a crisp function. $\mathcal{A}$ and $\mathcal{B}$ are fuzzy sets defined on universal sets X and Y respectively. Then the function satisfying the condition

$$\mu_{\mathcal{A}}(x) \leq \mu_{\mathcal{B}}(f(x)) \ \ \forall x \in \mathcal{X}$$

is called a **function with constraints** on fuzzy domain $\mathcal{A}$ and fuzzy range $\mathcal{B}$.

Example 4.1.1. Consider two fuzzy sets,

$$\mathcal{A} := \{(1, 0.5), (2, 0.8)\}, \mathcal{B} := \{(2, 0.7), (4, 0.9)\}$$

and a function

$$y = f(x) = 2x, \text{ for } x \in \mathcal{A}, y \in \mathcal{B}$$

We see the function f satisfies the condition, $\mu_{\mathcal{A}}(x) \leq \mu_{\mathcal{B}}(y)$.

Definition 4.1.2. Consider a function satisfying fuzzy constraint $f : \mathcal{A} \to \mathcal{B}$, $g : \mathcal{B} \to \mathcal{C}$ ($\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ denote fuzzy sets defined on X, Y and Z). **The composition of these two functions** yields fuzzy function with fuzzy constraint.

$$g \circ f : \mathcal{A} \to \mathcal{C}.$$

That is due to the conditions

$$\mu_{\mathcal{A}}(x) \leq \mu_{\mathcal{B}}(f(x)) \;\; and \; \mu_{\mathcal{B}}(y) \leq \mu_{\mathcal{C}}(g(y))$$

Where $y = f(x), z = g(y)$. The following holds.

$$\mu_{\mathcal{A}}(x) \leq \mu_{\mathcal{C}}(g(f(x))) \,.$$

Definition 4.1.3. Let X and Y be universes and $\tilde{P}(Y)$ the set of all fuzzy sets in Y (power set). $\tilde{f} : X \to \tilde{P}(Y)$ is a mapping, $\tilde{f}$ is a **fuzzy function** if and only if

$$\mu_{\tilde{f}(x)}(y) = \mu_{\tilde{R}}(x, y) \;\; \forall (x, y) \in X \times Y$$

where $\mu_{\tilde{R}}(x, y)$ is the membership function of a fuzzy relation.

Example 4.1.2. Let $A = \{2, 3, 4\}$ and $B = \{2, 3, 4, 6, 8, 9, 12\}$ be two crisp sets. And suppose also the fuzzy function defined from the elements of A to the power set $\tilde{P}(B)$, established by:

$$\tilde{f}(2) = B_1, \tilde{f}(3) = B_2, \tilde{f}(4) = B_3$$

with:

$$\tilde{P}(B) = \{B_1, B_2, B_3\} = \left\{\tilde{f}(2), \tilde{f}(3), \tilde{f}(4)\right\}$$

being:

$$B_1 = \{(2, 0.5), (4, 1), (6, 0.5)\}, B_2 = \{(3, 0.5), (6, 1), (9, 0.5)\},$$

$$B_3 = \{(4, 0.5), (8, 1), (12, 0.5)\}$$

So, for instance:

$$\begin{aligned} \tilde{f} &: 2 \in A \to B_1 \\ \tilde{f} &: 3 \in A \to B_2 \\ \tilde{f} &: 4 \in A \to B_3 \end{aligned}$$

And also it is possible to introduce the α-cut operation on the fuzzy function:

$$\tilde{f} \ : \ 2 \in A \rightarrow \begin{cases} \{2,4,6\}, & \text{for } \alpha = 0.5 \\ \{4\}, & \text{for } \alpha = 1.0 \end{cases}$$
$$\tilde{f} \ : \ 3 \in A \rightarrow \begin{cases} \{3,6,9\}, & \text{for } \alpha = 0.5 \\ \{6\}, & \text{for } \alpha = 1.0 \end{cases}$$
$$\tilde{f} \ : \ 4 \in A \rightarrow \begin{cases} \{4,8,12\}, & \text{for } \alpha = 0.5 \\ \{8\}, & \text{for } \alpha = 1.0 \end{cases}$$

corresponding to B_1, B_2 and B_3, respectively.

Definition 4.1.4. Suppose defined $f : U \rightarrow V$, crisp function between universal sets. The **fuzzy extension function** propagates the fuzziness, or ambiguity, from the independent variable to dependent variable. Such extension function defines the image, $\mathcal{B} = f(\mathcal{A})$, of fuzzy set $\mathcal{A}$.

Example 4.1.3. There is a crisp function, $f(x) = 3x + 1$ where its domain is $\mathcal{A} := \{(0, 0.9), (1, 0.8), (2, 0.7), (3, 0.6), (4, 0.5)\}$ and its range is $B = [0, 20]$. We can obtain a fuzzy set $\mathcal{B}$ in B

$$\mathcal{B} := \{(1, 0.9), (4, 0.8), (7, 0.7), (10, 0.6), (13, 0.5)\}.$$

Definition 4.1.5. Fuzzy Bunch of Functions of crisp functions from X to Y is defined with fuzzy set of crisp function f_i $(i = 1, \cdots, n)$ with

$$f_i : X \rightarrow Y$$

and it is denoted as

$$\tilde{f} = \sum \mu_{\tilde{f}}(f_i) / f_i$$

This function produces fuzzy set as its outcome.

Example 4.1.4. In the case of crisp sets f_1, f_2 and f_3, the bunch will be, for example, $X = \{1, 2, 3\}$

$$\tilde{f} = \{(f_1, 0.4), (f_2, 0.7), (f_3, 0.5)\}$$

where,

$$f_1(x) = x, f_2(x) = x^2, f_3(x) = 1 - x$$

By, f_1 we get $\tilde{f_1} = \{(1, 0.4), (2, 0.4), (3, 0.4)\}$
By, f_2 we get $\tilde{f_2} = \{(1, 0.7), (4, 0.7), (9, 0.7)\}$
By, f_3 we get $\tilde{f_3} = \{(0, 0.5), (-1, 0.5), (-2, 0.5)\}$
then, we can summarize the outputs as follows :

$$\begin{aligned}
\tilde{f}(1) &= \{(1, 0.4), (1, 0.7), (0, 0.5)\} = \{(0, 0.5), (1, 0.7)\} \\
\tilde{f}(2) &= \{(2, 0.4), (4, 0.7), (-1, 0.5)\} = \{(-1, 0.5), (2, 0.4), (4, 0.7)\} \\
\tilde{f}(3) &= \{(3, 0.4), (9, 0.7), (-2, 0.5)\} = \{(-2, 0.5), (3, 0.4), (9, 0.7)\}
\end{aligned}$$

We can see that the fuzzy function maps 2 to 2 with possibility 0.4 through f_1, to 4 with 0.7 through f_2 and to -1 with 0.5 through f_3. This result is represented by the above $\tilde{f_2}(2)$.

Homework 4.1.1.

1. Show the following function satisfies the conditions.
   Condition : $\mu_A(x) \leq \mu_B(y)$
   Function : $y = f(x) = 3x^2$, $x \in A$, $y \in B$ where

$$\begin{aligned}
A &= \{(2, 0.5), (3, 0.4)\} \\
B &= \{(4, 0.4), (12, 0.5), (27, 0.5)\}
\end{aligned}$$

2. Show the following function is a fuzzy function.

$$\tilde{f} : A \to \tilde{P}(B)$$

$$\tilde{f}(1) = B_1$$
$$\tilde{f}(2) = B_2$$
$$\tilde{f}(3) = B_3$$

where $A = \{1, 2, 3\}, B = \{1, 2, 3, 4, 6\}$
$\tilde{P}B) = \{(\} B_1, B_2, B_3)$

$$B_1 = \{(1, 0.9), (2, 0.5)\}$$
$$B_2 = \{(2, 0.5), (4, 0.9)\}$$
$$B_3 = \{(3, 1.0), (6, 0.5)\}$$

3. There in a fuzzy bunch of function, $X = \{2, 3, 4\}$

$$f_1(x) = x + 1, f_2(x) = x^2, f_3(x) = x^2 + 1$$

$$\tilde{f} = \{(f_1, 0.4), (f_2, 0.5), (f_3, 0.9)\}$$

Find $\tilde{f}(2), \tilde{f}(3)$ and $\tilde{f}(4)$.