Chapter One

The set and Logic theory

The set theory :- The set is mathematical concept which is undefined since any try to define the set implies to use the word set say "collection", "ensemble" Sometimes called group, family, system. Although these words are usually reserved for more special types of collection.

Remark :- symbole of the sets usually capital Letter say A, B, X, ... on the other hand the symoble of elements of a set small Letter say a, b, x, ...

If a is an element in the set A then we write $a \in A$ and read "a belongs to A" and if a is not in A then we write $a \notin A$ and real "a not belongs to A"

This set is written by two ways:-

Type (1) by mention its elements between two brackets separated by comma's this method is usually used when the elements are Known and their number is small

Examples:-

1) The set of the numbers 130527 is $\{1,3,0,5,2,7\}$

2) The set of the natural numbers $\mathbb{N}=\{0,1,2,\dots\}$

3) The set of integer numbers $\mathbb{Z}=\{\dots,-2,-1,0,1,2,\dots\}$

Type ② by mention characteristic features of their elements and using brackets

Examples:-

$E=\{x\in\mathbb{N};\ x=2n,\ n\in\mathbb{N}\}$ even numbers

$O=\{x\in\mathbb{N};\ x=2n+1,\ n\in\mathbb{N}\}$ odd numbers

Definition :- The set which is contains no any element called the empty set and denoted by ϕ.

Example :- $A=\{x\in\mathbb{N};\ 2<x<3\}=\phi$

Definition :- The set A is a subset of the set B denoted by $A\subseteq B$, if each element in A belongs to B.

Example :- $\mathbb{N}\subseteq\mathbb{Z}\subseteq\mathbb{Q}$; $\mathbb{Q}$ is rational numbers

Remark :- If $A\subseteq B$ and if there exists an element $b\in B$ and $b\notin A$ then we say $A\subset B$ (proper subset)

For example, $\mathbb{N}\subset\mathbb{Z}\subset\mathbb{Q}$

Definition:- A set A is called equal to a set B if $A \subseteq B$ and $B \subseteq A$

i.e

$$A = B \text{ iff } A \subseteq B \text{ and } B \subseteq A.$$

Example:- $A = \{2,4,6\}$, $B = \{x ; x = 2n , n = 1,2,3\}$, $C = \{2,6,4\}$

then $A = B = C$.

Proposition:- Let A, B, C are sets, then

1) $A \subseteq A$

2) if $A \subseteq B$, $B \subseteq C$ then $A \subseteq C$.

3) if $A \subset B$, $B \subset C$ then $A \subset C$.

Proof:-

1) Let $x \in A \rightarrow x \in A \rightarrow A \subseteq A$

2) Let $x \in A$, since $A \subseteq B \rightarrow x \in B$

since $B \subseteq C \rightarrow x \in C \longrightarrow A \subseteq C$

3) Let $x \in A$, since $A \subset B \longrightarrow x \in B$ and since $B \subset C \rightarrow x \in C$

hence $A \subseteq C$

Now,

since $A \subset B \rightarrow$ there exists $w \in B$ & $w \notin A$

but $B \subset C \longrightarrow w \in C$

So that $w \notin A$ & $w \in C \longrightarrow A \subset C$

$\because$

Definition:- The Universal set اَلْمَجْمُوعَة الشَّامِلَة

ALL set which deal with are subsets from "Big" set or another set then this set is called universal set.

Example:- $A=\{1,3,5\}$, $B=\{2,4,5\}$, $C=\{2,9,10\}$

so that $U=\{1,2,3,4,5,6,9,10\}$

Definition:- Power set مجموعة القوى

Let A be any set, the set of all subsets of the set A is called the power set and it is denoted by $P(A)$ or 2^A.

i.e

$$P(A)=\{B;B\subseteq A\}\ ,\ B\in P(A)\longleftrightarrow B\subseteq A$$

Example:- ① $A=\{1,2,3\}$, find $P(A)$?

$$P(A)=\{\phi,\{1,2,3\},\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$$

② $A=\{a,b\}$, then

$$P(A)=2^A=\{\phi,\{a,b\},\{a\},\{b\}\}$$

Operations on sets العمليات على المجموعات

<u>Definition</u>:- [the intersection of set التقاطع]

The intersection of a set A and a set B is the following set

$A \cap B = \{ x ; x \in A \text{ and } x \in B \}$

<u>Remark</u>:- Let A, B are two sets, then

1) $A \cap B \subseteq A$, $A \cap B \subseteq B$

2) $A \subseteq B$ iff $A \cap B = A$

<u>Proof</u>:-

1) Let $x \in A \cap B \rightarrow x \in A \wedge x \in B$

So that $A \cap B \subseteq A$, $A \cap B \subseteq B$

2) $\Rightarrow$ suppose that $A \subseteq B$

from 1 we have $A \cap B \subseteq B$, $A \cap B \subseteq A$ --- 1

Now we have to show that $A \subseteq A \cap B$

Let $x \in A$, since $A \subseteq B \rightarrow x \in B$

So that $x \in A \wedge x \in B \rightarrow x \in A \cap B$ --- 2

from 1 & 2 we get $A \cap B = A$

$\Leftarrow$ suppose $A \cap B = A$

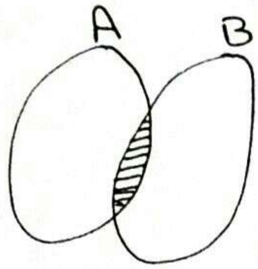
Let $y \in A \rightarrow y \in A \cap B \rightarrow y \in A \wedge y \in B \rightarrow y \in B$

hence $A \subseteq B$

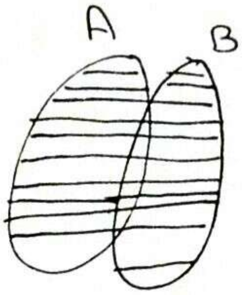
Definition:- [The Union of set الاتحاد]

The union of a set A and B is the set

$A \cup B = \{x : x \in A \text{ or } x \in B\}$



$A \cap B$



$A \cup B$

Example:- ① Let $A = \{a, b, c, d\}$, $B = \{f, b, d, g\}$

then $A \cap B = \{b, d\}$, $A \cup B = \{a, b, c, d, f, g\}$

② Let E = even numbers & O = odd numbers

then $E \cap O = \phi$, $E \cup O = \mathbb{N}$

Proposition:- Let A, B and C are sets then

1- $A \cup A = A$, $A \cap A = A$

2- $A \cap B = B \cap A$, $A \cup B = B \cup A$ [commutative]

3- $A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$ [associative]

Proof:- ③

$\Rightarrow$) Let $x \in A \cup (B \cup C)$

$\rightarrow x \in A \vee x \in (B \cup C)$

$\rightarrow x \in A \vee (x \in B \vee x \in C)$

$\rightarrow (x \in A \vee x \in B) \vee x \in C$

$\rightarrow x \in (A \cup B) \vee x \in C$

$\rightarrow x \in (A \cup B) \cup C$ ---- ①

$\Leftarrow$) Let $y \in (A \cup B) \cup C$

$\rightarrow y \in (A \cup B) \vee y \in C$

$\rightarrow (y \in A \vee y \in B) \vee y \in C$

$\rightarrow y \in A \vee (y \in B \vee y \in C)$

$\rightarrow y \in A \vee y \in (B \cup C)$

$\rightarrow y \in A \cup (B \cup C)$ --- ②

from ① & ② we get $(A \cup B) \cup C = A \cup (B \cup C)$

problem :- Let A, B are two sets, then

1- $A \subseteq A \cup B$, $B \subseteq A \cup B$

2- $A \subseteq B$ iff $A \cup B = B$

3- $A \cup \phi = A$, $A \cap \phi = \phi$

$A \cup U = U$, $A \cap U = A$

Definition :- The Disjoin set الأوصه المنفصلة

Let A, B are two sets, then A, B is called disjoin set if $A \cap B = \phi$

for example, E & O are disjoin set since $E \cap O = \phi$

Definition :- The Difference الفرق

The difference between the sets A, B is the set

$A - B = \{x ; x \in A \wedge x \notin B\}$

for example, Let $A = \{a, b, c, d\}$, $B = \{f, b, d, g\}$

then $A - B = \{a, c\}$

Definition :- Complement المتممة

The complement of the set A is the set which contains elements not belong to A, denoted by A^c i.e

$A^c = \{x ; x \notin A\} = U - A$

for example, Let E = even numbers & $U = \mathbb{N}$

then $E^c = U - E = O \leftarrow$ is odd numbers

Proposition: Let A, B are subsets of U, then

1- $A - A = \phi$, $A - \phi = A$, $\phi^c = U$, $U^c = \phi$

2- $A \cup A^c = U$, $A \cap A^c = \phi$

3- $(A^c)^c = A$

4- If $A \subseteq B$ then $B^c \subseteq A^c$

5- $(A-B) \cap (B-A) = \phi$

6- $A - B = A - (A \cap B)$

7- $A - B = A \cap B^c$

8- $A - B = B^c - A^c$

Proof: 3- Let $x \in (A^c)^c \longrightarrow x \notin A^c \longrightarrow x \in A$

$\therefore (A^c)^c \subseteq A$ --- ①

Let $x \in A \longrightarrow x \notin A^c \longrightarrow x \in (A^c)^c$

$\therefore A \subseteq (A^c)^c$ --- ②

from ① & ② we get $A = (A^c)^c$

4- Let $x \in B^c \longrightarrow x \notin B \xrightarrow[A \subseteq B]{\text{Since}} x \notin A$

$\longrightarrow x \in A^c$

So that, $B^c \subseteq A^c$

5- Suppose that $(A-B)\cap(B-A)\neq\phi$

$\rightarrow$ there exists $x\in(A-B)\cap(B-A)$

$\rightarrow x\in(A-B)$ and $x\in(B-A)$

$\rightarrow (x\in A$ and $x\notin B)$ and $(x\in B$ and $x\notin A)$ C!

hence $(A-B)\cap(B-A)=\phi$

7- $A-B=A\cap B^c$

Let $x\in A-B \leftrightarrow x\in A \wedge x\notin B$

$\leftrightarrow x\in A \wedge x\in B^c$

$\leftrightarrow x\in A\cap B^c$

8- $A-B=B^c-A^c$

Let $x\in A-B \longleftrightarrow x\in A \wedge x\notin B$

$\longleftrightarrow x\notin A^c \wedge x\in B^c$

$\leftrightarrow x\in B^c \wedge x\notin A^c$

$\leftrightarrow x\in B^c-A^c$

Problem :-

Let A, B are two sets; proof that

1- $A \subseteq B$ iff $p(A) \subseteq p(B)$

2- $p(A \cap B) = p(A) \cap p(B)$

3- $p(A) \cup p(B) \subseteq p(A \cup B)$. Is the inverse is true ? why ?

Proof :-

1- $\Rightarrow$) Let $c \in p(A)$, for every set c

$$\rightarrow c \subseteq A \rightarrow c \subseteq B \rightarrow c \in p(B)$$

So that, $p(A) \subseteq p(B)$

$\Leftarrow$) Let $x \in A$

$$\rightarrow \{x\} \subseteq A \rightarrow \{x\} \in p(A) \rightarrow \{x\} \in p(B)$$

$$\rightarrow \{x\} \subseteq B \rightarrow x \in B$$

$\therefore A \subseteq B$

2- Let $E \in p(A \cap B) \leftrightarrow E \subseteq A \cap B$

$$\leftrightarrow E \subseteq A \wedge E \subseteq B$$

$$\leftrightarrow E \in p(A) \wedge E \in p(B)$$

$$\leftrightarrow E \in p(A) \cap p(B)$$

• Distributive Laws قوانين التوزيع

Let A, B and C are sets, then

1- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

2- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof:-

1- Let $x \in A \cap (B \cup C)$

$\to x \in A \wedge x \in (B \cup C)$

$\to x \in A \wedge (x \in B \vee x \in C)$

$\to (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$

$\to x \in (A \cap B) \vee x \in (A \cap C)$

$\to x \in (A \cap B) \cup (A \cap C)$

$\therefore A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ --- ①

Now,

Let $y \in (A \cap B) \cup (A \cap C)$

$\to y \in (A \cap B) \vee y \in (A \cap C)$

$\to (y \in A \wedge y \in B) \vee (y \in A \wedge y \in C)$

$\to y \in A \wedge (y \in B \vee y \in C)$

$\to y \in A \wedge y \in (B \cup C)$

$\to y \in A \cap (B \cup C)$

$\therefore (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ -- ②

from ① & ② we get $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

De. Morgan's Laws قوانين دي مورگان

Let A, B be any two sets, then

1- $(A\cup B)^c = A^c \cap B^c$

2- $(A\cap B)^c = A^c \cup B^c$

Proof:- 1- Let $x \in (A\cup B)^c$

$\rightarrow x \notin A\cup B \rightarrow x \notin A \wedge x \notin B$

$\rightarrow x \in A^c \wedge x \in B^c \rightarrow x \in A^c \cap B^c$

$\therefore (A\cup B)^c \subseteq A^c \cap B^c$

Now, Let $y \in A^c \cap B^c$

$\rightarrow y \in A^c \wedge y \in B^c \longrightarrow y \notin A \wedge y \notin B$

$\rightarrow \cancel{y \notin A\cup B}$

$\rightarrow y \notin A\cup B \longrightarrow y \in (A\cup B)^c$

$\therefore A^c \cap B^c \subseteq (A\cup B)^c$.

So that $(A\cup B)^c = A^c \cap B^c$

2- $x \in (A\cap B)^c$

$\leftrightarrow x \notin A\cap B$

$\leftrightarrow x \notin A \vee x \notin B$

$\leftrightarrow x \in A^c \vee x \in B^c$

$\leftrightarrow x \in A^c \cup B^c$

Generalization for union and the intersection of sets

It can be define the union and the intersection of sets for more than two sets finite or infinite.

Let $A_1, A_2, ---, A_n$ be sets then

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup --- \cup A_n = \{x ; x \in A_i \text{ for some } 1 \leq i \leq n\}$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap --- \cap A_n = \{x ; x \in A_i \text{ for all } 1 \leq i \leq n\}$$

for example,

$A_1 = \{2, 3\}$, $A_2 = \{3, 5, 7\}$, $A_3 = \{1, 2, 3\}$

$$\bigcup_{i=1}^{3} A_i = A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 5, 7\}$$

$$\bigcap_{i=1}^{3} A_i = A_1 \cap A_2 \cap A_3 = \{3\}$$

More general if we have a family of sets $\{A_i\}_{i \in J}$ then

$$\bigcup_{i \in J} A_i = \{x ; x \in A_i \text{ for some } i \in J\}$$

$$\bigcap_{i \in J} A_i = \{x ; x \in A_i \text{ for all } i \in J\}$$

Generalization for De Morgan's Laws

Let $\{A_i\}_{i\in J}$ be a family of sets then

1- $\left(\bigcup_{i\in J} A_i\right)^c = \bigcap_{i\in J} A_i^c$

2- $\left(\bigcap_{i\in J} A_i\right)^c = \bigcup_{i\in J} A_i^c$

Proof:- 1. $x \in \left(\bigcup_{i\in J} A_i\right)^c \longleftrightarrow x \notin \bigcup_{i\in J} A_i$

$\longleftrightarrow x \notin A_i$, for all $i \in J$

$\longleftrightarrow x \in A_i^c$, for all $i \in J$

$\longleftrightarrow x \in \bigcap_{i\in J} A_i^c$

$\therefore \left(\bigcup_{i\in J} A_i\right)^c = \bigcap_{i\in J} A_i^c$

2- $x \in \left(\bigcap_{i\in J} A_i\right)^c \longleftrightarrow x \notin \bigcap_{i\in J} A_i$ for some $i \in J$

$\longleftrightarrow x \notin A_i$ for some $i \in J$

$\longleftrightarrow x \in A_i^c$ for some $i \in J$

$\longleftrightarrow x \in \bigcup_{i\in J} A_i^c$

$\therefore \left(\bigcap_{i\in J} A_i\right)^c = \bigcup_{i\in J} A_i^c$

Example :-

1) prove that $(A \cup B) \cap (A \cup B^c) = A$.

Solution :- By Distributive Law (2)

$$(A \cup B) \cap (A \cup B^c) = A \cup (B \cap B^c)$$
$$= A \cup \phi$$
$$= A$$

2) prove that $A \cup (A \cup B^c)^c = A \cup B$.

Solution :- By De Morgan's Law

$$A \cup (A \cup B^c)^c = A \cup (A^c \cap (B^c)^c)$$
$$= A \cup (A^c \cap B) \quad \text{by Distributive Law}$$
$$= (A \cup A^c) \cap (A \cup B)$$
$$= \mathcal{U} \cap (A \cup B)$$
$$= A \cup B$$

3) prove that $B - (B - A) = A \cap B$

$$B - (B - A) = B \cap (B - A)^c \quad \text{by } A - B = A \cap B^c$$
$$= B \cap (B \cap A^c)^c$$
$$= B \cap (B^c \cup (A^c)^c) \quad \text{De Morgan's Law}$$
$$= B \cap (A \cup B^c)$$
$$= (B \cap A) \cup (B \cap B^c) \quad \text{Distributive Law}$$
$$= (B \cap A) \cup \phi$$
$$= B \cap A$$
$$= A \cap B$$

Prove that

1- $(A \cap B) \cup (A - B) = A$

2- $A \cap (B - C) = (A \cap B) - (A \cap C)$

- $A \cap (A^c \cup B) = A \cap B$

- $A \cup (A \cup B^c)^c = A \cup B$

- $A - (B \cap C) = (A - B) \cup (A - C)$

- $(A \cup B) - C = (A - C) \cup (B - C)$

- $A \cap (B - C) = (A \cap B) - C$

- $A \subseteq B$ iff $A = B - (B - A)$

Statements العبارات

statements is a verbal sentence helpful will be denoted by the letters
$p, q, r, \ldots$

The fundamental property of a statement is that it is either true or false, but not both. The truthfulness or falsity of a statement is called its truth value.

Some statements are composite, that is, composed of substatements and various connectives which will be discussed subsequently.

Example:-

1- "Huba is a nice and a clever girl" is a composite statement with substatements "Huba is a nice" and "Huba is a clever girl".

- "where are you going?" is not a statement since it is neither true nor false

- "John is sick or old" is a composite statement with substatements "John is sick" or "John is old".

A fundamental property of a composite statement is that its truth value is completely determined by the truth value of each of its substatements and the way they are connected to form the composite statement.

Conjunction

الوصل

Any two statemets can be combined by the word "and" to form a composit statement which is called the conjunction of the original statements. Symbolically, the conjunction of the two statements p and q is denoted by $p \wedge q$

Example:- Let p be "It is raining", and Let q be "The sun is shining." Then $p \wedge q$ denotes the statement "It is raining and the sun is shining"

The truth value of the composite statement $p \wedge q$ satisfies the following property:-

If p is true and q is true, then $p \wedge q$ is true; otherwise $p \wedge q$ is false. In other words, the conjunction of two statements is true only if each component is true.

Example:- consider the following four statements

1) Paris is in France and $2+2=5$.

2) Paris is in England and $2+2=4$.

3) Paris is in England and $2+2=5$.

4) Paris is in France and $2+2=4$.

It's clear that only (4) is true. Each of the other statements is false since at least one of its [illegible]

Truth table of "$p \wedge q$" can be written in the form

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

الفصل

Any two statements can be combined by the word "or" to form a new statement which is called the disjunction of the original two statements. Symbolically, the disjunction of statements p and q is denoted by

$$p \vee q$$

Example :- Let p be "He studied French at the university", and Let q be "He lived in France".

Then $p \vee q$ is the statement "He studied French at the university or he lived in France.

The truth value of the composite statement $p \vee q$ ~~st~~ satisfies the following property:- If p is true or q is true or both p and q are true, then $p \vee q$ is true; otherwise, $p \vee q$ is false. In other words, the disjunction of two statements is false only if ...

The truth table of "$p \vee q$" can be written in the form

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example:-

consider the following four statements

1) Paris is in France or $2+2=5$.

2) Paris is in England or $2+2=4$.

3) Paris is in France or $2+2=4$.

4) Paris is in England or $2+2=5$.

only (4) is false. Each of the other statements is true since at least one of its components is true.

Vegation

النفي

Given any statement p, another statement, called the negation of p, can be formed by writing "It is false that ---" before p or, if possible, by inserting in p the word "not".

Symbolically, the negation of p is denoted by $\sim p$.

Example ①:-

consider the following three statements:

1) Paris is in France.

2) It is false that Paris is in France.

3) Paris is not in France.

Then (2) and (3) are each the negation of (1).

Example ②:-

consider the following statements:

1) $2+2=5$

2) It is false that $2+2=5$

3) $2+2\neq 5$

Then (2) & (3) are each the negation of (1).

The truth value of the negation of a statement satisfies the following property:

If p is true, then $\sim p$ is false; if p is false, then $\sim p$ is true.

Example:- consider the statement, in Example (1). Notice that (1) is true while (2) & (3) its negations, are false.

Example:- consider the statements in Example (2). Notice that (1) is false while (2) & (3) are true.

P	$\sim P$
T	F
F	T

Conditional

الشرط

Many statements, especially in mathematics, are of the form "If p then q". Such statements are called conditional statements and are denoted by

$$p \longrightarrow q$$

The truth value of the conditional statement $p \longrightarrow q$ satisfies the following property :-

The conditional $p \to q$ is true unless p is true and q is false.

The truth table of "$p \longrightarrow q$" can be written in the form

p	q	$p \longrightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Remark :- consider the conditional proposition $p \longrightarrow q$ and other simple conditional propositions which contain p and q, i.e

$q \longrightarrow p$, $\sim p \longrightarrow \sim q$, and $\sim q \longrightarrow \sim p$, called, respectively, the converse معكوس inverse, and contrapositive propositions.

نظير

ضد معاكس

The truth tables of these four propositions are as follows:-

p	q	~p	~q	p → q	q → p	~p → ~q	~q → ~p
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Example:- Let p:- Noor at home.

q:- Noor answer to the phone.

p→q :- if Noor at home then she will answer to the phon

q→p :- if Noor answer to the phone then she is at home.

~p→~q :- if Noor is not at home then she is not answer to the phone.

~q→~p :- if Noor is not answer to the phone then she is not at home.

Biconditional

ثنائية الشرط

Another common statement is of the form "p if and only if q" or, simply, "p iff q". such statements are called biconditional statements and are denoted by $p \longleftrightarrow q$

The truth value of the biconditional statement $p \longleftrightarrow q$ satisfies property:
If p and q have the same truth value, then $p \longleftrightarrow q$ is true;
if p and q have opposite truth values, then $p \longleftrightarrow q$ is false.

Example:- consider the following statements

1) Paris is in France iff $2+2=5$.
2) paris is in England iff $2+2=4$.
3) paris is in France iff $2+2=4$.
4) paris is in England iff $2+2=5$.

According, (3) and (4) are true while (1) and (2) are false.

The truth table written as follows

p	q	$p \longleftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

-ogical Equivalence التكافؤ المنطقي

Two statements are said to be logically equivalent if their truth tables are dentical. We denote the logical equivalent of p and q by " $p \equiv q$ ".

Example :- The truth tables of $(p\to q)\wedge(q\to p)$ and $p\longleftrightarrow q$ are as follows

p	q	$p\to q$	$q\to p$	$(p\to q)\wedge(q\to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

p	q	$p\leftrightarrow$
T	T	T
T	F	F
F	T	F
F	F	T

Hence $(p\to q)\wedge(q\to p) \equiv p\longleftrightarrow q$

Example :- The truth tables below ~~stotha~~ show that $p\to q$ and $\sim p\vee q$ are logically equivalent, i.e., $p\to q \equiv \sim p\vee q$

p	q	$p\to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\sim p$	$\sim p\vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Theorem:- The following propositions are logically equivalent

- $p \vee p \equiv p$ / $p \wedge p \equiv p$
- $(p \vee q) \vee s \equiv p \vee (q \vee s)$ / $(p \wedge q) \wedge s \equiv p \wedge (q \wedge s)$
- $p \vee q \equiv q \vee p$ / $p \wedge q \equiv q \wedge p$
- $p \vee (q \wedge s) \equiv (p \vee q) \wedge (p \vee s)$ / $p \wedge (q \vee s) \equiv (p \wedge q) \vee (p \wedge s)$
- $\sim(\sim p) \equiv p$ / $(p \to q) \wedge (q \to p) \equiv p \leftrightarrow q$
- $\sim(p \vee q) \equiv \sim p \wedge \sim q$ / $\sim(p \wedge q) \equiv \sim p \vee \sim q$
- $p \to q \equiv \sim q \to \sim p$ / $q \to p \equiv \sim p \to \sim q$
- $p \to q \equiv \sim p \vee q$ / $p \to \sim q \equiv \sim p \vee \sim q$
- $\sim p \to q \equiv p \vee q$ / $\sim q \to p \equiv p \vee q$
- $\sim(p \to q) \equiv p \wedge \sim q$
- $(p \vee s) \to q \equiv (p \to q) \wedge (s \to q)$

$p \to (q \wedge s) \equiv (p \to q) \wedge (p \to s)$

- $p \to (q \vee s) \equiv (p \to q) \vee (p \to s)$

Tautologies and Contradiction   التحصيل حاصل والتناقض

Some propositions $P(p,q,\ldots)$ contain only T in the last column of their truth tables. In other words the proposition $P(p,q,\ldots)$ will always become a true statement no matter which statements $p_0, q_0, \ldots$, true or false, are substituted for the variables. Such propositions are called tautologies.

Definition:- A proposition $P(p,q,\ldots)$ is a tautology if $P(p_0,q_0,\ldots)$ is true for any statements $p_0, q_0, \ldots$

Definition:- A proposition $P(p,q,\ldots)$ is a contradication if $P(p_0,q_0,\ldots)$ is fals for any statements $p_0, q_0, \ldots$ In other words, a contradiction will contain only F in the last column of its truth table.

Example:- The proposition "p or not p", i.e. $p \vee \sim p$, is a tautology. while the proposition "p and not p", i.e $p \wedge \sim p$, is a contradiction.

This fact is verified by the following table

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F

Arguments

مجادلة (محاججة)

A statement that a set of assumptions $S_1, S_2, \ldots, S_n$ yields another assumption S, (S is called result), denoted by $S_1 \wedge S_2 \wedge \cdots \wedge S_n \longrightarrow S$ is called a argument.

Remark:- An argument on propositions $S_1 \wedge S_2 \cdots \wedge S_n \longrightarrow S$ is said to be valid if S is true.

i.e

$S_1 \wedge S_2 \cdots \wedge S_n \longrightarrow S$ is a tautology تحصيل حاصل

Example:- consider the argument,

S_1: If a man is not married, he is unhappy.

S_2: If a man is unhappy, he dies young.

S: A man is not married die young.

Solution:-

Let p:- he is not married.

q:- he is unhappy.

r:- he dies young

Then $S_1, S_2 \longrightarrow S$ can be written

$\therefore$

$p \rightarrow q,\ q \rightarrow r \longrightarrow (p \rightarrow r)$

q	r	$p \to q$	$q \to r$	$(p\to q)\wedge(q\to r)$	$p \to r$	Argument
T	T	T	T	T	T	T
T	F	T	F	F	F	T
F	T	F	T	F	T	T
F	F	F	T	F	F	T
T	T	T	T	T	T	T
T	F	T	F	F	T	T
F	T	T	T	T	T	T
F	F	T	T	T	T	T

So that, the given argument is valid.

xample:- If the sky is not rain then Ahmed will feel very well. But it is rain, so that Ahmed won't feel very well.

Solution:-

Let p:- The sky is not rain

q:- Ahmed will feel very well

Then the above argument can be written in the form

$$(p \longrightarrow q)\wedge \sim p \longrightarrow \sim q$$

OR

Let P_1:- The sky is rain

P_2:- Ahmed will feel very well

So that $\therefore$

$$(\sim P_1 \longrightarrow P_2)\wedge P_1 \longrightarrow \sim P_2$$