Chapter Four

The natural numbers and integers

Principle of mathematical induction

مبدأ الاستقراء الرياضي

Let S be a subset of $\mathbb{N}$, the natural numbers, with the following properties:-

1) $1 \in S$ 2) $n \in S$ implies $n+1 \in S$

Then S is the set of natural numbers (i.e. $S = \mathbb{N}$)

The way of using principle of mathematical induction:-

*) suppose that $p(n)$ is a statement depend on natural number n.

*) suppose that S be a solution set of natural numbers n such that the statement is true i.e $S = \{n \in \mathbb{N} ; p(n) \text{ is true}\}$

We shall prove that

1) $1 \in S$ (i.e $p(1)$ is true)

2) Suppose that $p(K)$ is true. (i.e $K \in S$)

3) $p(K+1)$ is true. (i.e $K+1 \in S$)

So that, $S = \mathbb{N}$

i.e $p(n)$ is true $\forall n$

Example prove that $1+2+3+\cdots+n=\frac{n(n+1)}{2}$ $\forall\, n \geq 1$

Solution suppose that $p(n)$ is $1+2+3+\cdots+n=\frac{n(n+1)}{2}$

and $S=\{n\in\mathbb{N} : p(n) \text{ is true}\}$

1) since $\frac{1(1+1)}{2}=\frac{2}{2}=1$

hence $p(1)$ is true (i.e $1\in S$) ✓

2) suppose that $p(K)$ is true

i.e $1+2+\cdots+K=\frac{K(K+1)}{2}$ ✓

we have to show that $1+2+3+\cdots+K+(K+1)=\frac{(K+1)[(K+1)+1]}{2}$

$1+2+3+\cdots+K+(K+1)=\frac{K(K+1)}{2}+(K+1)$ (بإضافة $(K+1)$ للطرفين)

$$=\frac{K(K+1)+2(K+1)}{2}=\frac{(K+1)(K+2)}{2}$$

$$=\frac{(K+1)[(K+1)+1]}{2}$$

So that $p(K+1)$ is true (i.e $K+1\in S$)

hence $p(n)$ is true $\forall\, n\geq 1$

ample rove that $\sum_{K=1}^{n} (2K-1)^2 = \frac{n(2n-1)(2n+1)}{3} \quad \forall n \in \mathbb{N}$

Solution: Suppose that $p(n)$ is $\sum_{K=1}^{n} (2K-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

i.e $(1)^2 + (3)^2 + (5)^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

Since $(1)^2 = \frac{1(2-1)(2+1)}{3} = 1$, hence $p(1)$ is true. ✓

Suppose that $p(K)$ is true

i.e $1^2 + 3^2 + 5^2 + \cdots + (2K-1)^2 = \frac{K(2K-1)(2K+1)}{3}$ ---(*)

Now, we prove that $p(K+1)$ is true

i.e $1^2 + 3^2 + \cdots + (2K-1)^2 + (2(K+1)-1)^2 = \frac{(K+1)(2(K+1)-1)(2(K+1)+1)}{3}$

باضافة $[2(K+1)-1]^2 = (2K+1)^2$ لطرفي (*) نحصل على

$$1^2 + 3^2 + \cdots + (2K-1)^2 + (2(K+1)-1)^2 = \frac{K(2K-1)(2K+1)}{3} + (2K+1)^2$$

$$= \frac{K(2K-1)(2K+1) + 3(2K+1)^2}{3}$$

$$= \frac{(2K+1)\left[K(2K-1) + 3(2K+1)\right]}{3}$$

$$= \frac{(2K+1)[2K^2+5K+3]}{3}$$

$$= \frac{(2K+1)[(K+1)(2K+3)]}{3} = \frac{(K+1)(2K+1)(2K+3)}{3}$$

$$= \frac{(K+1)(2(K+1)-1)(2(K+1)+1)}{3}$$

So that, $p(K+1)$ is true.

Example prove that $2^{3n}-1$ divide by 7 $\forall n \geq 1$

Solution suppose that $p(n)$ is $2^{3n}-1$ divide by 7

1) $p(1)$ is true. Since $2^{3(1)}-1=7$ divide by 7

2) suppose that $p(k)$ is true. i.e $2^{3k}-1$ divide by 7

that is mean $\exists c \in \mathbb{Z}$ s.t $2^{3k}-1=7c$

Now, we have to show that, $\exists d \in \mathbb{Z}$ s.t $2^{3(k+1)}-1=7d$

$$2^{3(k+1)}-1 = 2^{3k+3}-1 = 2^3 2^{3k}-1$$

$$= 2^3(7c+1)-1 \qquad \text{since } 2^{3k}-1=7c$$

$$= 2^3 7c + 8 - 1$$

$$= 2^3 \cdot 7c + 7$$

$$= 7(2^3 c + 1)$$

$$= 7d \quad ; \quad d = 2^3 c + 1 \in \mathbb{Z}$$

So that $2^{3(k+1)}-1$ divide by 7

hence $p(n)$ is true $\forall n$

Peano's axioms

فرضيات بيانو

أن مجموعه الاعداد الطبيعيه هي مجموعه يرمز لها بالرمز $\mathbb{N}$ تحقق الفرضيات الاتيه :-

① $\mathbb{N}$ مجموعه غيرخاليه

② لكل عنصر $n \in \mathbb{N}$ يوجد عنصر اخر يرمز له بالرمز n^+ يسمى تابع العنصر n وهذا التابع وحيد فضلاً عن هذا اذا كان $n^+ = m^+$ فأن $n = m$

③ يوجد عنصر واحد فقط في $\mathbb{N}$ ليس تابعاً لاي عنصر اخر يرمز له بالرمز 1 ويسمى العدد الاول في $\mathbb{N}$

Theorem

1- For each $n, m \in \mathbb{N}$, there exist unique element $n+m \in \mathbb{N}$ is called summation n, m such that

a) $n+1 = n^+$ b) $m+1 = m^+$ c) $n+m = m+n$ (Commutative Law)

d) $\forall\, n, m, k \in \mathbb{N}$, $(n+m)+k = n+(m+k)$ (associative Law)

e) If $m+n_1 = m+n_2$ then $n_1 = n_2$ (cancellation Law)

2- For each $n, m \in \mathbb{N}$, there exist unique element $m \cdot n \in \mathbb{N}$ is called multiplication elements n, m such that

a) $m \cdot 1 = 1 \cdot m = m$ b) $n \cdot m = m \cdot n$ c) $(m \cdot n) \cdot k = m \cdot (n \cdot k)$

d) $k \cdot (m+n) = k \cdot m + k \cdot n$

e) If $m \cdot n_1 = m \cdot n_2$ then $n_1 = n_2$

Theorem For each $n \in \mathbb{N}$ then $n^+ \neq n$ (i.e $n+1 \neq n$)

proof By using principle of mathematical induction

Let $S = \{n \in \mathbb{N} \,;\, n+1 \neq n\}$

1) $1 \in S$.

if $1 \notin S \longrightarrow 1+1 = 1$ C!

معناه (1) هو تابع وهذا تناقض مع فرضية بيانو (1 ليس تابع لعدد)

2) suppose that $K \in S$

We have to show that $K^+ = K+1 \in S$

suppose that $K^+ \notin S$

$\rightarrow K^+ + 1 = K^+$ (def. of S)

$\rightarrow (K^+)^+ = K^+$

$\rightarrow K^+ = K$ (if $n^+ = m^+ \longrightarrow n = m$)

$\rightarrow K \notin S$ C! (with 2)

so that $K^+ = K+1 \in S$

By mathematical induction we get $S = \mathbb{N}$

Definition Let $n, m \in \mathbb{N}$ then $m < n$ iff there exists a unique $k \in \mathbb{N}$ such that $m + k = n$.

Remark For each $n \in \mathbb{N}$, $n < n+1$.

Proposition For each $n \in \mathbb{N}$, $n \neq 1$ then $n > 1$

(i.e 1 is smallest element in $\mathbb{N}$)

proof Since $n \neq 1$ then n is successor for another element say m

hence $n = m+1$

$\rightarrow m+1 > 1$

$\rightarrow n > 1$

Proposition Let $n, m \in \mathbb{N}$, if $m < n$ then $m^+ \leq n$

proof Since $m < n \longrightarrow \exists\, k \in \mathbb{N}$ s.t $m + k = n$, $k \geq 1$

1) If $k = 1 \longrightarrow m+1 = n \longrightarrow m^+ = n$

2) If $k > 1 \longrightarrow \exists\, t \in \mathbb{N}$ s.t $k = t+1$

$\therefore\ n = m + k = m + (t+1) = (m+1) + t$

$= m^+ + t$

$\longrightarrow m^+ < n$

from 1 & 2 we get $m^+ \leq n$

Proposition For each $n \in \mathbb{N}$, then $\nexists$ $k \in \mathbb{N}$ s.t $n < k < n+1$.

proof suppose that there exists $k \in \mathbb{N}$ s.t $n < k < n+1$

Now,

$n < k \rightarrow \exists\, k_1 \in \mathbb{N}$ s.t $n + k_1 = k$

$k < n+1 \rightarrow \exists\, k_2 \in \mathbb{N}$ s.t $k + k_2 = n+1$

$(n + k_1) + k_2 = n+1 \longrightarrow \cancel{n} + (k_1 + k_2) = \cancel{n} + 1$

$\rightarrow k_1 + k_2 = 1$

$\rightarrow k_1 < 1$ C! since $k_1 \in \mathbb{N}$

Proposition Let $m, n, k \in \mathbb{N}$. Then

1) Either $m = n$ or $m < n$ or $n < m$

2) If $m < n$ and $k < m$ then $k < n$

3) If $m < n$ then $m + t < n + t$, $t \in \mathbb{N}$

4) If $m < n$ then $mt < nt$

5) If $m + t < n + t$ then $m < n$

6) If $mt < nt$ then. $m < n$

proof

1) If $n=1$ or $m=1$ then by preceding proposition the proof is clear

[Suppose that $n \neq 1$ and $m=1 \xrightarrow{\text{pro.}} n>1=m$]

Thus suppose that $n\neq 1$ and $m \neq 1$, we use mathematical induction

Let $S_1=\{n\}$, $S_2=\{x\in \mathbb{N};\ x>n\}$, $S_3=\{x\in\mathbb{N};\ x<n\}$

Let $S=S_1\cup S_2\cup S_3$

1) $1\in S$. since $\forall n\in\mathbb{N}$ & $n\neq 1 \rightarrow n>1$

$\rightarrow 1\in S_3 \longrightarrow 1\in S$

2) Let $k\in S \longrightarrow k\in S_1$ or $k\in S_2$ or $k\in S_3$

Case ① if $k\in S_1 \longrightarrow k=n \longrightarrow k^+>n \longrightarrow k^+\in S_2 \longrightarrow k^+\in S$

Case ② if $k\in S_2 \longrightarrow k>n \rightarrow \exists t\in\mathbb{N}$ s.t

$k=n+t \longrightarrow k^+=(t+n)^+$

$=(t+n)+1 = n+(t+1)$

$\longrightarrow k^+>n \longrightarrow k^+\in S_2 \longrightarrow k^+\in S$

Case ③ if $k\in S_3 \longrightarrow k<n \longrightarrow k^+\leq n$

if $k^+=n \longrightarrow k^+\in S_1 \longrightarrow k^+\in S$

if $k^+<n \longrightarrow k^+\in S_3 \longrightarrow k^+\in S$

By ①, ② & ③ $\forall k\in S \longrightarrow k^+\in S$, thus

by mathematical induction $S=\mathbb{N}$

, $m<n \rightarrow \exists\, t_1 \in \mathbb{N}$ s.t $n = m + t_1$

$k<m \rightarrow \exists\, t_2 \in \mathbb{N}$ s.t $m = k + t_2$

$n = m + t_1 = (k + t_2) + t_1 = k + (t_1 + t_2)$

$\rightarrow n = k + t$; where $t = t_1 + t_2 \in \mathbb{N}$

$\rightarrow k < n$

3) $m<n \longrightarrow \exists\, k \in \mathbb{N}$ s.t $n = m + k$

$n + t = m + k + t$ (أضافة t)

$= (m + t) + k$

$\rightarrow m + t < n + t$

4) $m<n \longrightarrow \exists\, k \in \mathbb{N}$ s.t $m + k = n$

$nt = (m + k) \cdot t$ (ضرب بـ t)

$= mt + kt$

$\rightarrow mt < nt$

5) $m + t < n + t$

By (1) either $m = n$ or $m < n$ or $n < m$

if $m = n \longrightarrow m + t = n + t$ C!

if $n < m \longrightarrow n + t < m + t$ by (3) C!

[Since $m + t < n + t$]

so that $m < n$

Construction of the integer numbers

Let $\mathbb{N}$ be the natural numbers, and Let $\mathbb{N}\times\mathbb{N} = \{(m,n) ; m,n \in \mathbb{N}\}$

First, we define relation $(\sim)$ on the set $\mathbb{N}\times\mathbb{N}$ as follows

$$(m,n) \sim (p,q) \quad \text{iff} \quad m+q = n+p$$

claim that $(\sim)$ is equivalence relation

1) reflexive

Let $(m,n) \in \mathbb{N}\times\mathbb{N}$

we known that $m+n = n+m \longrightarrow (m,n) \sim (m,n)$

2) symmetric

Let $(m,n), (p,q) \in \mathbb{N}\times\mathbb{N}$

we have to show that if $(m,n)\sim(p,q)$ then $(p,q)\sim(m,n)$

$(m,n)\sim(p,q) \longrightarrow m+q = n+p \longrightarrow p+n = q+m$

So that, $(p,q)\sim(m,n)$

3) transitive

Let $(m,n), (p,q), (r,s) \in \mathbb{N}\times\mathbb{N}$

we have to show that

if $(m,n)\sim(p,q)$

and $(p,q)\sim(r,s)$ $\Longrightarrow (m,n)\sim(r,s)$

$m+q=n+p$

$p+s=q+r$ جمع

$m+\cancel{q}+\cancel{p}+s=n+\cancel{p}+\cancel{q}+r \longrightarrow m+s=n+r$

So that $(m,n)\sim(r,s)$

Hence $(\sim)$ is equivalence relation on $\mathbb{N}\times\mathbb{N}$

Second,

Since $(\sim)$ is the equivalence relation on $\mathbb{N}\times\mathbb{N}$, then the equivalence classes of (m,n) is called integer number and denoted by $\overline{(m,n)}$.

(m,n) is called the representation of the integer number. i.e

$\overline{(m,n)}=\{(p,q)\in\mathbb{N}\times\mathbb{N};\ (p,q)\sim(m,n)\}$

The set of all equivalence classes is called "set of integer numbers" and is denoted by $\mathbb{Z}$.

Remark

$x=\overline{(m,n)}$ denoted by $x=m-n$

ينظر الى صنف التكافؤ $\overline{(m,n)}$ على انه العدد $m-n$

xample

1) $0 = \overline{(m,m)}$

$= \{(p,q) \in \mathbb{N}\times\mathbb{N} ; (p,q) \sim (m,m)\}$

$= \{(p,q) \in \mathbb{N}\times\mathbb{N} ; p+\cancel{m} = q+\cancel{m}\}$

$= \{(p,q) \in \mathbb{N}\times\mathbb{N} ; p = q\}$

$= \{(1,1),(2,2),(3,3),\ldots\}$

2) $\overline{(2,1)} = \{(p,q) \in \mathbb{N}\times\mathbb{N} ; (p,q) \sim (2,1)\}$

$= \{(p,q) ; p+1 = q+2\}$

$= \{(p,q) ; p = q+1\}$

$= \{(2,1),(3,2),(4,3),\ldots\}$

3) $\overline{(1,2)} = \{(p,q) \in \mathbb{N}\times\mathbb{N} ; (p,q) \sim (1,2)\}$

$= \{(p,q) ; p+2 = q+1\}$

$= \{(p,q) ; p+1 = q\}$

$= \{(1,2),(2,3),(3,4),\ldots\}$

4) what is the integer number $(+7)$

$+7 = \{(8,1),(9,2),(10,3),\ldots\}$

Summation and multiplication on $\mathbb{Z}$

Definition Let $x=\overline{(m,n)}$ and $y=\overline{(r,s)}$ are two integer numbers. Then

1) $x+y=\overline{(m+r,n+s)}$

2) $x\cdot y=\overline{(m\cdot r+n\cdot s,m\cdot s+n\cdot r)}$

Example Find

1) $(-3)+7$ 2) $(-3)\cdot 7$

Solution

$-3=\overline{(1,4)}$, $7=\overline{(8,1)}$

1) $(-3)+7=\overline{(1,4)}+\overline{(8,1)}=\overline{(1+8,4+1)}=\overline{(9,5)}=4$

2) $(-3)\cdot 7=\overline{(1,4)}\cdot\overline{(8,1)}=\overline{(8+4,1+32)}=\overline{(12,33)}=-21$

Proposition Let $x,y,z\in\mathbb{Z}$. Then

1) $x+y=y+x$, $x\cdot y=y\cdot x$

2) $(x+y)+z=x+(y+z)$. , $(x\cdot y)\cdot z=x\cdot(y\cdot z)$

3) $x+0=0+x=x$, $x\cdot 1=1\cdot x=x$

4) $x\cdot(y+z)=x\cdot y+x\cdot z$

5) $x+(-x)=0$, $(-x)$ is called inverse of x

Proof

1) Let $x=\overline{(m,n)}$, $y=\overline{(r,s)}$

$$x+y=\overline{(m,n)}+\overline{(r,s)}=\overline{(m+r,n+s)}=\overline{(r+m,s+n)}=\overline{(r,s)}+\overline{(m,n)}=y+x$$

$$x.y=\overline{(m,n)}.\overline{(r,s)}=\overline{(mr+ns,ms+nr)}=\overline{(rm+sn,sm+rn)}=\overline{(r,s)}.\overline{(m,n)}=y.x$$

3) Let $x=\overline{(r,s)}$, $0=\overline{(m,m)}$

$$x+0=\overline{(r,s)}+\overline{(m,m)}=\overline{(r+m,s+m)}=\overline{(r,s)}=x$$

Since $r+m+s=s+m+r \rightarrow (r+m,s+m)\sim(r,s) \rightarrow \overline{(r+m,s+m)}=\overline{(r,s)}$

Similarly $0+x=x$

Let $1=\overline{(m+1,m)}$

$$\therefore x.1=\overline{(r,s)}.\overline{(m+1,m)}=\overline{(r.(m+1)+sm,rm+s(m+1))}$$

$$=\overline{(rm+r+sm,rm+sm+s)}=\overline{(r,s)}=x$$

Since

$rm+r+sm+s=rm+sm+s+r$

$\rightarrow (rm+r+sm,rm+sm+s)\sim(r,s)$

$\rightarrow \overline{(rm+r+sm,rm+sm+s)}=\overline{(r,s)}$

5) Let $x=\overline{(m,n)}$, put $-x=\overline{(n,m)}$

$$x+(-x)=\overline{(m,n)}+\overline{(n,m)}=\overline{(m+n,n+m)}=\overline{(m,m)}=0$$

Since

$m+n+m=n+m+m \rightarrow (m+n,n+m)\sim(m,m)$

$\rightarrow \overline{(m+n,n+m)}=\overline{(m,m)}$

Definition Let $x, y \in \mathbb{Z}$, the difference between x and y denoted by

$$x - y = x + (-y)$$

Definition Let $x = \overline{(m,n)}$, then x is said to be positive integer number if $m > n$ and x is called negative integer number if $n > m$, where " $>$ " is order relation on $\mathbb{N}$.

Proposition Let $x, y \in \mathbb{Z}$

1- If each of x, y are positive then $x+y$, $x \cdot y$ are positive.

2- If each of x, y are negative then $x+y$ is negative and $x \cdot y$ is positive.

3- If x is positive and y is negative then $x \cdot y$ is negative.

Proof

1) Let $x = \overline{(m,n)}$, $y = \overline{(r,s)}$

Since each of x, y are positive

So that $m > n$

$r > s$

$\rightarrow m+r > n+s$

Hence $x + y = (m+r, n+s)$ is positive

Remark

1- For each $x, y \in \mathbb{Z}$ then either $x = y$ or $x > y$ or $x < y$

2- $x \geq y$ that is mean $x > y$ or $x = y$

Proposition Let $x, y, z \in \mathbb{Z}$

1- $x \geq x$

2- If $x \geq y$ and $y \geq x$ then $x = y$

3- If $x \geq y$ and $y \geq z$ then $x \geq z$

proof

1- $x \geq x \longrightarrow x > x$ or $x = x$ ✓

2- $x \geq y \longrightarrow x > y$ or $x = y$

$x \leq y \longrightarrow x < y$ or $x = y$

$x \geq y$ and $x \leq y \rightarrow (x > y \vee x = y) \wedge (x < y \vee x = y)$

$\rightarrow (x = y \vee x > y) \wedge (x = y \vee x < y)$

$\rightarrow (x = y) \vee \underbrace{(x > y \wedge x < y)}_{C!}$

$\rightarrow x = y$

$\cdot\ x \geq y \to x > y$ or $x = y$

$y \geq z \to y > z$ or $y = z$

$(x > y \vee x = y) \wedge (y > z \vee y = z)$

$$\left.\begin{array}{l} x > y \wedge y > z \longrightarrow x > z \\ x > y \wedge y = z \longrightarrow x > z \\ x = y \wedge y > z \longrightarrow x > z \\ x = y \wedge y = z \longrightarrow x = z \end{array}\right\} \Rightarrow x > z \vee x = z \to x \geq z$$

Since $\left.\begin{array}{l} x > y \to x - y > 0 \\ y > z \to y - z > 0 \end{array}\right\} \to (x-y)+(y-z) > 0 \to x - z > 0 \to x > z$

Theorem [Division Algorithm] خوارزمية القسمة

Let a, b are positive integer numbers and $a > b$, there exist positive integer number q and nonnegative integer number r such that

$$a = bq + r \quad , \quad 0 \leq r < b$$

In addition, q and r are unique elements satifies this condition.

r is called Remainder and q is called quotient

for example

1) $a = 3$, $b = 2 \longrightarrow 3 = 2 \cdot 1 + 1$; $q = 1$, $r = 1$, $0 \leq r \leq 2$

2) $a = 7$, $b = 3 \longrightarrow 7 = 3 \cdot 2 + 1$; $q = 2$, $r = 1$, $0 \leq r \leq 3$

3) $a = 8$, $b = 4 \longrightarrow 8 = 4 \cdot 2 + 0$; $q = 2$, $r = 0$

xample Let a be positive number, if $b=2$ then either

$$a=2q \quad \text{or} \quad a=2q+1 \; ; \quad 0\leq r<2$$

(i.e either a is even number or a is odd number)

Solution By division algorithm, $\exists\, q, r$ s.t

$$a=2q+r \; ; \; 0\leq r<2$$

so that

if $r=0 \rightarrow a=2q$, hence a is even number

if $r=1 \rightarrow a=2q+1$, hence a is odd number

Example prove that each odd integer number can be written as

$$4K+1 \quad \text{or} \quad 4K+3 \; ; \; K\in\mathbb{Z}$$

Solution

We take a is odd number and take $b=4$, hence by division algorithm $a=4K+r$ where $0\leq r<4$

if $r=0 \rightarrow a=4K$ C! since a is odd while $4K$ is even

if $r=1 \rightarrow a=4k+1$

if $r=2 \rightarrow a=2k+2$ C! " " " " " "

if $r=3 \rightarrow a=4K+3$

So that, any odd number is write as $4k+1$ or $4k+3$

Chapter Five

Rational and Real numbers

The construction of rational numbers

. Let $\mathbb{Z}$ be a set of integer numbers and Let $A = \mathbb{Z}/\{0\}$

$\mathbb{Z} \times A = \{(m,n) ; m \in \mathbb{Z}, n \in A \text{ i.e } n \neq 0\}$

Let $\sim$ be a relation defined on $\mathbb{Z} \times A$ as follows

$(m,n), (p,q) \in \mathbb{Z} \times A$

$(m,n) \sim (p,q)$ iff $mq = np$

prove this is equivalence relation

Definition The set of all equivalence classes

$$\overline{(m,n)} = \{(p,q) \in \mathbb{Z} \times A ; (p,q) \sim (m,n)\}$$

is called the rational numbers and denoted by $\mathbb{Q}$

for example

$$\overline{(0,1)} = \{(p,q) \in \mathbb{Z} \times A ; (p,q) \sim (0,1)\} = \{(p,q) ; p \cdot 1 = 0 \cdot q\}$$
$$= \{(p,q) ; p = 0\} = \{(0,1), (0,2), (0,3), ----\}$$

Remark $\overline{(m,n)} \equiv \frac{m}{n}$

ينظر الى العدد النسبي $\overline{(m,n)}$ على أنه $\frac{m}{n}$